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Research Article

Domination chains in graphs

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Abstract: In the study of domination in graphs the following Domination Chain of inequalities is well known and well studied: $ir(G) \leq \gamma(G) \leq i(G) \leq \gamma(G) \leq \alpha(G) \leq \Gamma(G) \leq IR(G)$, where $ir(G)$ and $IR(G)$ denote the irredundance number and upper irredundance number, $\gamma(G)$ and $\Gamma(G)$ denote the domination number and upper domination number, and $i(G)$ and $\alpha(G)$ denote the independent domination number and vertex independence number of a graph G. The Domination Chain is a consequence of the facts that (i) every maximal independent set is a minimal dominating set and every minimal dominating set is a maximal irredundant set and (ii) the property of being an independent set is hereditary (every subset of an independent set is also an independent set), the property of being a dominating set is superhereditary (every superset of a dominating set is also a dominating set), and the property of being an irredundant set is hereditary. In this paper we consider several other hereditary properties and superhereditary properties which give rise to similar domination chains.

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1. Introduction

In a graph $G = (V, E)$, the *open neighborhood* $N(v) = \{u \in V \mid uv \in E\}$ of a vertex $v \in V$ is the set of neighbors u of v, and the *degree* of v in G is $\deg(v) = |N(v)|$.

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The *closed neighborhood* of a vertex v is the set $N[v] = N(v) \cup \{v\}$, and the *open neighborhood* of a set $S \subset V$ is the set $N(S) = \bigcup_{v \in S} N(v)$.
Throughout this paper we will use the following notation: $\overline{S} = V \setminus S$ denotes the vertices in V but not in S, called the *complement* of S in G, $\deg_S(v) = |N(v) \cap S|$, $G[S]$ is the *subgraph* of G induced by S, $\delta(G) = \min\{\deg(v) \mid v \in V\}$ is the *minimum degree* of a vertex in V, and $\Delta(G) = \max\{\deg(v) \mid v \in V\}$ is the *maximum degree* of a vertex in V.
A set of vertices $S \subset V$ is called *independent* if no two vertices in S are adjacent, that is, for every $u, v \in S$, $u \notin N(v)$ and $v \notin N(u)$. Let $i(G)$ and $\alpha(G)$ equal the minimum and maximum cardinalities of a maximal independent set in G.
A set of vertices $S \subset V$ is called a *dominating set* if for every vertex $v \in V$, either $v \in S$ or $v \in N(S)$. Let $\gamma(G)$ and $\Gamma(G)$ equal the minimum and maximum cardinalities of a minimal dominating set in G.
A property $\mathcal{P}$ of a set S of vertices is called *hereditary* (resp. *superhereditary*) if every subset $S' \subset S$ of S (resp. every superset S' of S, $S \subset S'$) also has property $\mathcal{P}$. For instance, the property of being an independent set is hereditary since every subset of an independent set is also independent, while the property of being a dominating set is superhereditary, since every superset of a dominating set is also a dominating set.
The following well known result was established in [3].

Proposition 1. *Let S be a set of vertices having some hereditary or superhereitary property $\mathcal{P}$.*

1. *If $\mathcal{P}$ is hereditary, then S is maximal with respect to $\mathcal{P}$ if and only if for every vertex $w \in \overline{S}$, $S \cup \{w\}$ does not have property $\mathcal{P}$.*

2. *If $\mathcal{P}$ is superhereditary, then S is minimal with respect to $\mathcal{P}$ if and only if for every vertex $v \in S$, $S - \{v\}$ does not have property $\mathcal{P}$.*

Discussions of hereditary and superhereditary properties related to dominating sets in graphs can be found in the book on domination in graphs by Haynes, Hedetniemi and Slater in 1998 [9], and papers by Cockayne et al. in 1997 [3], and Cockayne et al. in 1997 [2].
A *domination chain* expresses relationships that exist among independent sets, dominating sets, and irredundant sets in graphs. Irredundance is the concept that describes the minimality of a dominating set. If a dominating set S is *minimal*, then for every vertex $u \in S$ the set $S - \{u\}$ is no longer a dominating set. This means that vertex u dominates some vertex, which could be itself, that no other vertex in S dominates.
Given a vertex set $S \subseteq V$ and a vertex $v \in S$, we make the following definitions:

1. vertex v is a *self private neighbor* if v has no neighbors in S, that is, $N[v] \cap S = \{v\}$.

2. vertex v has an *S-external private neighbor* if there exists a vertex $w \in \overline{S}$ such that $N(w) \cap S = \{v\}$.

3. vertex v has an *S-internal private neighbor* if there exists a vertex $w \in S$ such that $N(w) \cap S = \{v\}$.

A nonempty set S is *irredundant* if and only if every vertex $v \in S$ either is a self-private neighbor or has an S-external private neighbor. The *irredundance numbers*, $ir(G)$ and $IR(G)$, equal the minimum and maximum cardinalities, respectively, of a maximal irredundant set. For a comprehensive treatment of irredundance in graphs the reader is referred to two recent 2021 chapters on this subject by Mynhardt and Roux [14] and Hedetniemi, McRae, and Mohan [11].
From the foregoing, it is easy to see that in every graph G, every maximal independent set is a minimal dominating set in G, and every minimal dominating set in G is a maximal irredundant set in G, leading to the domination chain:

Theorem 1 (The Domination Chain). *For every graph G,*

$$ir(G) \leq \gamma(G) \leq i(G) \leq \alpha(G) \leq \Gamma(G) \leq IR(G).$$

Since its introduction by Cockayne, Hedetniemi, and Miller in 1978 [4], the Domination Chain of Theorem 1 has become one of the major focal points in the study of domination in graphs, resulting in several hundred papers. The general framework for developing the Domination Chain, starting with the *seed property* of an independent set, can be used to obtain inequality chains similar to the domination chain starting from other hereditary or superhereditary seed properties. This is largely what motivates our study, since almost any property of subsets could be considered as a seed property.
In this paper, we will define a series of possible hereditary seed properties of graphs, and for each, we will attempt to determine the existence of a corresponding domination chain.

2. Total hereditary properties

A property $\mathcal{P}$ of vertex sets $S \subset V$ is considered to be a *total hereditary property* if it holds for every vertex $v \in V$ with respect to a given set S, rather than for every vertex $v \in S$. In this section we present two total hereditary properties $\mathcal{P}$, one of which apparently has not been studied.

2.1. Total Irredundant Sets

Definition 1. A set $S \subset V$ is called *total irredundant* if for every vertex $v \in V$, $N[v]$ contains at least one vertex that is not contained in $N[S - \{v\}]$. Let $ir_t(G)$ and $IR_t(G)$ equal the minimum and maximum cardinalities of a maximal total irredundant set in G.

Total irredundance was first introduced by Hedetniemi, Hedetniemi and Jacobs in 1993 [10] but was only studied in terms of its algorithmic complexity and NP-completeness.

Proposition 2. *The property of being a total irredundant set is hereditary.*

Proof. Let $S \subset V$ be a total irredundant set and let $S' \subset S$. We must show that S' is also a total irredundant set. That is, we must show that for every vertex $v \in V$, $N[v]$ contains at least one vertex that is not contained in $N[S' - \{v\}]$. Consider three cases:
Case 1. $v \in V - S$. Since S is a total irredundant set, let $x \in N[v]$, where $x \notin N[S - \{v\}]$. But if x is not in $N[S - \{v\}]$, then it is not in the subset $N[S' - \{v\}]$ of $N[S - \{v\}]$.
Case 2. $v \in S - S'$. There are two cases: (i) v is a self-private neighbor in S, meaning it is not adjacent to any vertex in $S - \{v\}$. But if v is not adjacent to any vertex in $S - \{v\}$, then is also not adjacent to any vertex in the subset S' of $S - \{v\}$. (ii) v has an external private neighbor in $V - S$, say w. Thus, $N(w) \cap S = \{v\}$. Thus, w is not adjacent to any vertices in S', and therefore w is a private neighbor of v with respect to the set S'.
Case 3. $v \in S'$. There are two cases: (i) v is a self-private neighbor in S, meaning that it is not adjacent to any vertices in S. But if v is not adjacent to any vertices in S, then it is not adjacent to any vertices in the subset S' of S. (ii) v has an external private neighbor with respect to the set S, that is there is a vertex $w \in V - S$, such that $N(w) \cap S = \{v\}$. But if $N(w) \cap S = \{v\}$, then $N(w) \cap S' = \{v\}$. Thus, v has an external private neighbor with respect to S'. □
It follows from Proposition 2 that a total irredundant set S is a maximal total irredundant set if and only if for every vertex $w \in V - S$, the set $S \cup \{w\}$ is not a total irredundant set. This means that one of the following three conditions holds:
(i) there is a vertex $x \in V - S$ whose only private neighbor with respect to S is vertex $w \in V - S$, so that x has no private neighbors with respect to $S \cup \{w\}$.
(ii) there is a vertex $v \in S$ whose only private neighbor with respect to S is the vertex $w \in V - S$, so that v has no private neighbors with respect to $S \cup \{w\}$.
(iii) there is a vertex $v \in S$ whose only private neighbor with respect to S is v itself, but since v is adjacent to $w \in V - S$, v has no private neighbor with respect to $S \cup \{w\}$.
These three conditions define a new type of set.

Definition 2. A set S is called total redundant if for every vertex $w \in V - S$, one of the following three conditions holds:
(i) there is a vertex $x \in V - S$ whose only private neighbor with respect to S is vertex $w \in V - S$, so that x has no private neighbors with respect to $S \cup \{w\}$.
(ii) there is a vertex $v \in S$ whose only private neighbor with respect to S is the vertex $w \in V - S$, so that v has no private neighbors with respect to $S \cup \{w\}$.
(iii) there is a vertex $v \in S$ whose only private neighbor with respect to S is v itself, but since v is adjacent to $w \in V - S$, v has no private neighbor with respect to $S \cup \{w\}$.
Let $tr(G)$ and $TR(G)$ equal the minimum and maximum cardinalities of a total redundant set in G.

Proposition 3. *For any graph G, a set S is a maximal total irredundant set if and only if S is a minimal total redundant set.*

Proof. Let S be a maximal total irredundant set. Since S is maximal, it meets the definition of being a total redundant set. All that remains is to show that S is a minimal total redundant set.
Suppose, therefore, that S is not a minimal total redundant set. Let $S' \subset S$ be a total redundant set. Consider any vertex $v \in (S - S')$.
Since S' is total redundant, then vertex v must meet one of the three specified conditions: (i) it is the only private neighbor of a vertex $x \in V - S'$ with respect to S'. But this is a contradiction since S is total irredundant and x must have a private neighbor with respect to S.
(ii) it is the only private neighbor of a vertex $w \in S'$ with respect to S'. But this is a contradiction since S is total irredundant and v cannot be a private neighbor of w with respect to S since both v and w are in S.
(iii) the only private neighbor of a vertex $w \in S'$ with respect to S' is itself but vertex v is adjacent to w. This means that w can't be a self- private neighbor with respect to the larger set S since w is adjacent to v and both w and v are in S. □

Corollary 1. *For any graph G, $tr(G) \leq ir_t(G) \leq IR_t(G) \leq TR(G)$.*

2.2. Totally k-dependent Sets

In [8], Fink and Jacobson introduced the concept of k-dependent sets as a generalisation of independent sets. For an integer $k \geq 1$, a set $S \subset V$ is called *k-dependent* if for every vertex $v \in S$, $|N(v) \cap S| < k$. A domination chain corresponding to the hereditary property of being a k-dependent set was developped in 2009 by Chellali and Favaron [1] by introducing the concept of k-star forming sets.
In this subsection we consider *totally k-independent sets* S where every vertex $v \in V$ has less than k neighbors in S. Let $i_{tk}(G)$ and $\alpha_{tk}(G)$ equal the minimum and maximum cardinalities of a maximal totally k-dependent set in G.
Clearly, the property of being totally k-dependent is hereditary. What then is the maximality condition for being totally k-dependent?
Every vertex $w \in V - S$ has a neighbor x, either in $V - S$ or in S, whose neighborhood in $S \cup \{w\}$ has at least k vertices, that is $|N_{S\cup\{w\}}(x)| \geq k$. This partially coincides with the concept of k-star forming sets defined in [1], which leads us to give the following definition.

Definition 3. For an integer $k \geq 1$, a subset S of vertices of G is a *weak k-star forming set* if every vertex $v \in V - S$ has a neighbor u in V such that $\deg_S(u) \geq k - 1$. Let $\phi_{wkf}(G)$ and $\Phi_{wkf}(G)$ equal the minimum and maximum cardinalities of a minimal weak k-star forming set.

Since the property of being a weak k-star forming set is superhereditary, a weak k-star forming set is minimal if and only if for every vertex $w \in S$, $S - \{w\}$ is no longer a weak k-star forming set.

Proposition 4. *Every maximal totally k-dependent set is a minimal weak k-star forming set.*

Proof. Let S be a maximal totally k-dependent set. Since S is maximal, it meets the definition of being a weak k-star forming set. All that remains is to show that S is a minimal weak k-star forming set.
Assume therefore, that S is not a minimal weak k-star forming set. Then there exists a vertex $v \in S$ such that $S - \{v\}$ is a weak k-star forming set. In this case vertex $v \in V - (S - \{v\})$ must have a neighbor x in V such that $\deg_{S-\{v\}}(x) \geq k - 1$. Since $v \in S$, we deduce that $\deg_S(x) \geq k$. But this contradicts the assumption that S is a totally k-dependent set. Thus, S must be a minimal weak k-star forming set. □

Corollary 2. *For every graph G,*

$$\phi_{wkf}(G) \leq i_{tk}(G) \leq \alpha_{tk}(G) \leq \Phi_{wkf}(G).$$

3. Hereditary properties

In this section, we consider five hereditary properties. To avoid unnecessary redundancy, most of the proofs are omitted in this section, as they can be constructed using the same methods as in the previous section.

3.1. Open Irredundant Sets

Definition 4. A set $S \subset V$ is called *open irredundant* if for every vertex $v \in S$, there exists a vertex $w \in V - S$ for which $N(w) \cap S = \{v\}$, that is, every vertex v in S has at least one external private neighbor w. The minimum and maximum cardinalities of a maximal open irredundant set are denoted $oir(G)$ and $OIR(G)$, respectively.

The concept of irredundant sets in graphs was first introduced in 1978 by Cockayne, Hedetniemi, and Miller [4]. Open irredundance was first studied by Farley and Shachum in 1983 [6] and by Farley and Proskurowski in 1984 [5]; see also Favaron in 1988 [7].
The property of being an open irredundant set is hereditary since any vertex in a subset $S' \subset S$ of an open irredundant set S has the same external private neighbor in S' that it has in S. Thus, an open irredundant set S is a maximal open irredundant set if and only if for every vertex $w \in V - S$ the set $S \cup \{w\}$ is not open irredundant. This condition is equivalent to saying that for every vertex $w \in V - S$, either (i) w has no external private neighbor in the set $S \cup \{w\}$, or (ii) there exists a vertex $v \in S$ which has an external private neighbor in $V - S$ but does not have an external private

neighbor in $V-(S\cup\{w\})$, that is, the only external private neighbor that $v\in S$ has is w.

Definition 5. A set S is called *open redundant* if for every vertex $w\in V-S$, either (i) w has no external private neighbor in the set $S\cup\{w\}$, or (ii) there exists a vertex $v\in S$ which has an external private neighbor in $V-S$ but does not have an external private neighbor in $V-(S\cup\{w\})$, that is, the only external private neighbor that $v\in S$ has is w. Let $or(G)$ and $OR(G)$ equal the minimum and maximum cardinalities of a minimal open redundant set in G.

Proposition 5. *If $S\subseteq V$ is a maximal open irredundant set, then S is a minimal open redundant set.*

Proof. Let S be a maximal open irredundant set. By definition S is also an open redundant set. All that remains is to show that S is a minimal open redundant set. Suppose not. Then there exists an open redundant set $S'\subset S$. But this means that for every vertex $w\in V-S'$, either (i) w has no external private neighbor in $V-S'$ or (ii) there exists a vertex $v\in V-S'$ which has an external private neighbor in $V-S'$ but does not have an external private neighbor in $V-(S'\cup\{w\})$.
In particular let this vertex $w\in S-S'\subset V-S'$. Since S is maximal open irredundant, w has an external private neighbor in $V-S$, and therefore it also has the same private neighbor in $V-S'$, which is a contradiction. $\square$

Corollary 3. *For any graph G, $or(G)\le oir(G)\le OIR(G)\le OR(G)$.*

3.2. 2-packing Sets

Definition 6. A set $S\subset V$ is called a 2-*packing* if the vertices in S are pairwise at distance at least three apart in G. The *lower* 2-*packing number* $\rho_2(G)$ and *upper* 2-*packing number* $\rho_2^+(G)$ equal the minimum and maximum cardinalities of a maximal 2-packing in G.

The study of 2-packings in graphs is very well established in graph theory, indeed, more than 5,000 papers have been published related to packings in graphs. In this section we develop a domination chain corresponding to 2-packings in graphs.
Note that the property of being a 2-packing is hereditary, and thus a 2-packing is maximal if and only if for every vertex $w\in V-S$, the set $S\cup\{w\}$ is not a 2-packing. This means that vertex w is within distance-2 of at least one vertex in S, leading to the following, well-known definition.

Definition 7. A set $S\subset V$ is called a *distance*-2 *dominating set* if every vertex in $V-S$ is within distance-2 of at least one vertex in S. The minimum cardinality of a distance-2 dominating set is called the *distance*-2 *domination number* and is denoted $\gamma_{\le 2}(G)$. Similarly the *upper distance*-2 *domination number* $\Gamma_{\le 2}(G)$ equals the maximum cardinality of a minimal distance-2 dominating set in G.

Clearly, the property of being a distance-2 dominating set is superhereditary, and hence from the preceding definitions, we have the following result.

Proposition 6. *If S is a maximal 2-packing set, then S is a minimal distance-2 dominating set.*

Corollary 4. *For any graph G, $\gamma_{\leq 2}(G) \leq \rho_2(G) \leq \rho_2^+(G) \leq \Gamma_{\leq 2}(G)$.*

Since a set S is a minimal distance-2 dominating set if and only if for every vertex $v \in S$, the set $S - \{v\}$ is not a distance-2 dominating set, this, in turn, means, as is discussed by Henning in [12], that for every vertex $v \in S$, either (i) no vertex in $S - \{v\}$ is within distance-2 of v, or (ii) there exists a vertex $w \in V - S$ that is within distance-2 of v but is not within distance-2 of any vertex in $S - \{v\}$.
These two conditions (i) and (ii) suggest the following definitions.

Definition 8. A set $S \subseteq V$ is called a *distance-2 irredundant set* if for every vertex $v \in S$, either (i) no vertex in $S - \{v\}$ is within distance-2 of v, or (ii) there exists a vertex $w \in V - S$ that is within distance-2 of v but is not within distance-2 of any vertex in $S - \{v\}$. Let $ir_2(G)$ and $IR_2(G)$ equal the minimum and maximum cardinalities of a maximal distance-2 irredundant set.

In a distance-2 irredundant set S, if a vertex $v \in S$ meets condition (i), that no vertex $w \in S - \{v\}$ is within distance-2 of v, then v is said to be a *self distance-2 private neighbor*. If a vertex $v \in S$ meets condition (ii), that there exists a vertex $w \in V - S$ that is within distance-2 of v but is not within distance-2 of any vertex in $S - \{v\}$, then w is said to be an *external distance-2 private neighbor* of v.
Thus, an equivalent definition of a distance-2 irredundant set S is a set having the property that for every vertex $v \in S$, either v is a self distance-2 private neighbor or v has an external distance-2 private neighbor $w \in V - S$.
Note that the property of being a distance-2 irredundant set is hereditary. Let S be a distance-2 irredundant set, let $S' \subset S$ and let $v \in S'$. It is easy to see that if v is a self distance-2 private neighbor in S then it is also a self distance-2 private neighbor in S', since it as the same distance to vertices in $S' - \{v\}$ as it has to the same vertices in $S - \{v\}$. Similarly, if vertex v has an external distance-2 private neighbor w in $V - S$, then vertex w is also an external distance-2 private neighbor in $V - S'$.
Thus, a set S is a maximal distance-2 irredundant set if and only if for all $w \in V - S$, $S \cup \{w\}$ is not a distance-2 irredundant set. Accordingly, we have the following result.

Proposition 7. *If S is a minimal distance-2 dominating set, then S is a maximal distance-2 irredundant set.*

An immediate consequence of Proposition 7 is the following Distance-2 Domination Chain involving the well-known 2-packing numbers of a graph.

Corollary 5. *For any graph G,*

$$ir_2(G) \leq \gamma_{\leq 2}(G) \leq \rho_2(G) \leq \rho_2^+(G) \leq \Gamma_{\leq 2}(G) \leq IR_2(G).$$

It is interesting to consider the conditions under which a distance-2 irredundant set is a maximal distance-2 irredundant set, in the same way that we considered the maximality conditions of a 2-packing earlier. In this case adding any vertex $w \in V - S$ to a distance-2 irredundant set S creates a set $S \cup \{w\}$ that is no longer distance-2 irredundant. This in turn means that there is at least one vertex $x \in S \cup \{w\}$ that is neither a self distance-2 private neighbor nor has an external distance-2 private neighbor. This remains to be considered.
Distance domination was introduced by Slater in 1976 [15]. For an extended treatment of distance domination in graphs, the reader is referred to two chapters by Henning [12, 13].

3.3. Enclaveless Sets

Definition 9. A set $S \subset V$ is called *enclaveless* if for every vertex $v \in S$, $N(v) \cap (V - S) \neq \emptyset$, that is, every vertex v in S has at least one neighbor in $V - S$. The minimum and maximum cardinalities of a maximal enclaveless set are denoted $\psi(G)$ and $\Psi(G)$, respectively.

Enclaveless sets were introduced by Slater in 1977 [16]. The property of being an enclaveless set is clearly hereditary, since if every vertex in a set S has a neighbor in $V - S$, then every vertex in a subset $S' \subset S$ will still have the same neighbor in $V - S'$. Thus, every subset of an enclaveless set is also an enclaveless set. It follows therefore from Proposition 1 that a set S is a *maximal* enclaveless set if and only if for every vertex $w \in V - S$, the set $S \cup \{w\}$ is not an enclaveless set, that is, either
(i) $N(w) \cap S = N(w)$, that is, every neighbor of w is in S, or equivalently, $N[w]$ is an enclave in $S \cup \{w\}$, or
(ii) there is a vertex $v \in S$ whose only neighbor in $V - S$ is w.
This leads to the following definition.

Definition 10. A set $S \subset V$ is called an *enclave dominating set* if every vertex w in $V - S$ creates an enclave in the set $S \cup \{w\}$, and this enclave contains w, which means that either (i) w is the center of an enclave in $S \cup \{w\}$, that is, $N[w] \subseteq S \cup \{w\}$, or equivalently, every neighbor of w is in S, or (ii) w is adjacent to a vertex $v \in S$ which is the center of an enclave in $S \cup \{w\}$, that is, $w \in N[v] \subseteq S \cup \{w\}$, or equivalently, w is adjacent to a vertex $v \in S$ whose only neighbor in $V - S$ is w. Let $\gamma_\psi(G)$ and $\Gamma_\psi(G)$ equal the minimum and maximum cardinality of a minimal enclave dominating set in G.

It is easy to see that enclave domination is superhereditary, in that every superset of an enclave dominating set is also an enclave dominating set. Thus, an enclave dominating set S is a minimal enclave dominating set if and only if for every vertex $v \in S$, $S - \{v\}$ is not an enclave dominating set. Consequently, we have the following result.

Proposition 8. *If S is a maximal enclaveless set, then S is a minimal enclave dominating set.*

Corollary 6. *For any graph G, $\gamma_\psi(G) \leq \psi(G) \leq \Psi(G) \leq \Gamma_\psi(G)$.*

As we noted above, since the property of being an enclave dominating set is super-hereditary, from Proposition 1-(2) it follows that an enclave dominating set S is a minimal enclave dominating set if and only if for every vertex $v \in S$, the set $S - \{v\}$ is not an enclave dominating set.

4. Conclusions and Open Problems

In this short paper we have shown that for almost any seed hereditary property $\mathcal{P}$, one can, with care, derive a corresponding domination chain, sometimes with surprises. For example, in this paper we have illustrated chains containing new results about such the well-known types of sets as open irredundant sets, enclaveless sets, k-dependent sets, 2-packings, and distance-2 dominating sets.
Several new or unstudied types of sets are also introduced that bear further study, including: (i) total irredundance and total redundance, (ii) total k-dependence, (iii) open redundance, (iv) distance-2 irredundance, and (v) enclave domination.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

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