Communications in Combinatorics and Optimization
Vol. xx, No. x (xxxx), pp. 1-14
https://doi.org/10.22049/cco.2025.30397.2451

CCO
Commun. Comb. Optim.

Research Article

Weak signed total Italian domination in graphs

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Received: 1 August 2024; Accepted: 15 March 2025
Published Online: 23 March 2025

Abstract: A *weak signed total Italian dominating function* (WSTIDF) of a graph G with vertex set $V(G)$ is defined as a function $f : V(G) \to \{-1, 1, 2\}$ having the property that $\sum_{x \in N(v)} f(x) \geq 1$ for each $v \in V(G)$, where $N(v)$ is the neighborhood of v. The weight of a WSTIDF is the sum of its function values over all vertices. The *weak signed total Italian domination number* of G, denoted by $\gamma_{wstI}(G)$, is the minimum weight of a WSTIDF in G. We initiate the study of the weak signed total Italian domination number, and we present different sharp bounds on $\gamma_{wstI}(G)$. In addition, we determine the weak signed total Italian domination number of some classes of graphs.

Keywords: weak signed total Italian domination, signed total Italian domination, signed total Roman domination, total domination.

AMS Subject classification: 05C69

1. Introduction

For notation and graph theory terminology, we in general follow Haynes, Hedetniemi and Slater [9]. Specifically, let G be a graph with vertex set $V(G) = V$ and edge set $E(G) = E$. The integers $n = n(G) = |V(G)|$ and $m = m(G) = |E(G)|$ are the *order* and the *size* of the graph G, respectively. The *open neighborhood* of vertex v is $N_G(v) = N(v) = \{u \in V(G) | uv \in E(G)\}$, and the *closed neighborhood* of v is $N_G[v] = N[v] = N(v) \cup \{v\}$. The *degree* of a vertex v is $d_G(v) = d(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta(G) = \delta$ and $\Delta(G) = \Delta$, respectively. A graph G is *regular* or *r-regular* if $\delta(G) = \Delta(G) = r$. For a subset $X \subseteq V(G)$, we use $G[X]$ to denote the subgraph of G induced by X. Let K_n be the complete graph of order n, C_n the cycle of order n, P_n the path of order n, and $K_{p,q}$ the complete bipartite graph with partite sets X and Y, where $|X| = p$ and $|Y| = q$. Let $S(r, s)$ be the *double star* with exactly two adjacent vertices u and v that are not leaves such that u is adjacent to $r \geq 1$ leaves and v is adjacent to $s \geq 1$ leaves.

Cockayne, Dreyer, S.M. Hedetniemi and S.T. Hedetniemi [6] introduced the concept of *Roman domination* in graphs, and since then a lot of related variations and generalizations in graphs have been studied (see, [2–4]). In this paper we continue the study of Roman and Italian dominating functions in graphs.

A set D of vertices of G is called by Cockayne, Dawes and Hedetniemi [5] a *total dominating set* if each vertex in $V(G)$ is adjacent to some vertex of D. The *total domination number* $\gamma_t(G)$ equals the minimum cardinality of a total dominating set in G. We note that this parameter is only defined for graphs without isolated vertices. Total domination is very well studied in the literature. For more details on total domination, the reader is referred to the two domination books by Haynes, Hedetniemi and Slater [8, 9], the survey article on total domination by Henning [10] and the book on total domination by Henning and Yeo [13].

A *signed total Roman dominating function* (STRDF) on a graph G is defined in [15] as a function $f : V(G) \to \{-1, 1, 2\}$ having the property that $f(N(v)) = \sum_{x\in N(v)} f(x) \geq 1$ for each $v \in V(G)$ and if $f(u) = -1$, then the vertex u must have a neighbor w with $f(w) = 2$. The weight of a signed total Roman dominating function is the value $\sum_{u\in V(G)} f(u)$. The *signed total Roman domination number* $\gamma_{stR}(G)$ is the minimum weight of a signed total Roman dominating function on G.

A *signed total Italian dominating function* (STIDF) of a graph G is defined in [17] as a function $f : V(G) \to \{-1, 1, 2\}$ having the property that (i) $f(N(v)) \geq 1$ for each $v \in V(G)$ and (ii) every vertex u for which $f(u) = -1$ is adjacent to a vertex v for which $f(v) = 2$ or adjacent to two vertices w and z with $f(w) = f(z) = 1$. The weight of an STIDF f is the value $\sum_{v\in V(G)} f(v)$. The *signed total Italian domination number* of G, denoted by $\gamma_{stI}(G)$, is the minimum weight of an STIDF in G.

A *weak signed total Italian dominating function* (WSTIDF) of a graph G is defined as a function $f : V(G) \to \{-1, 1, 2\}$ having the property that $f(N(v)) \geq 1$ for each $v \in V(G)$. The weight of a WSTIDF f is $\omega(f) = \sum_{v\in V(G)} f(v)$. The *weak signed total Italian domination number* of G, denoted by $\gamma_{wstI}(G)$, is the minimum weight of a WSTIDF in G. A $\gamma_{wstI}(G)$-function is a WSTIDF of weight $\gamma_{wstI}(G)$. For a WSTIDF f on G, let $V_i = \{v \in V(G) : f(v) = i\}$ for $i = -1, 1, 2$. A WSTIDF f can be represented by the ordered partition $f = (V_{-1}, V_1, V_2)$.

The signed total Roman, signed total Italian and weak signed total Italian domination numbers are well-defined for graphs G without isolated vertices, since the function $f : V(G) \to \{-1, 1, 2\}$ with $f(x) = 1$ for each $x \in V(G)$ is an STRDF, an STIDF as well as a WSTIDF. Thus we assume throughout this paper that $\delta(G) \geq 1$. The definitions lead to $\gamma_{wstI}(G) \leq \gamma_{stI}(G) \leq \gamma_{stR}(G) \leq n(G)$. Therefore each lower bound of $\gamma_{wstI}(G)$ is also a lower bound of $\gamma_{stI}(G)$ and $\gamma_{stR}(G)$.

In this paper we continue the study of signed (total) Roman (Italian) domination in graphs (see, for example, [1, 7, 11, 12, 14–16]). Our purpose in this work is to initiate the study of the weak signed total Italian domination number. We present basic properties and sharp bounds for the weak signed total Italian domination number of a graph. In particular, we show that many lower bounds on $\gamma_{stI}(G)$ and on $\gamma_{stR}(G)$ are also valid for $\gamma_{wstI}(G)$. In addition, we prove $\gamma_{wstI}(G) \geq (8n - 9m)/3$ for connected graphs of order n and size m, and we characterize the graphs achieving equality. Furthermore, we observe that the difference $\gamma_{stI}(G) - \gamma_{wstI}(G)$ can be arbitrarily large, and we determine the weak signed total Italian domination number of some classes of graphs.

We make use of the following known results.

Proposition 1. *[17]* If $n \geq 2$, then $\gamma_{stI}(K_n) = 2$ when n is even and $\gamma_{stI}(K_n) = 3$ when n is odd.

Proposition 2. *[17]* If $p, q \geq 2$ are integers, then $\gamma_{stI}(K_{p,q}) = 2$.

Proposition 3. *[17]* Let $S(r, s)$ be the double star. If $r, s \geq 3$, then $\gamma_{stI}(S(r, s)) = 2$. In addition, $\gamma_{stI}(S(1, s)) = 2$ for $s = 1$ or $s \geq 3$.

Proposition 4. *[17]* If C_n is a cycle of length $n \geq 3$, then $\gamma_{stI}(C_n) = n/2$ when $n \equiv 0 \pmod 4$, $\gamma_{stI}(C_n) = (n + 3)/2$ when $n \equiv 1, 3 \pmod 4$ and $\gamma_{stI}(C_n) = (n + 6)/2$ when $n \equiv 2 \pmod 4$.

Proposition 5. *[15]* Let P_n be a path of order $n \geq 3$. Then $\gamma_{stR}(P_n) = n/2$ when $n \equiv 0 \pmod 4$ and $\gamma_{stR}(P_n) = \lceil (n + 3)/2 \rceil$ otherwise.

2. Preliminary results and first bounds

In this section we present basic properties and some first bounds on the weak signed total Italian domination number.

Observation 1. *If $f = (V_{-1}, V_1, V_2)$ is a WSTIDF of a graph G of order n with $\delta(G) \geq 1$, then the following holds.*

(a) $|V_{-1}| + |V_1| + |V_2| = n$.

(b) $\omega(f) = |V_1| + 2|V_2| - |V_{-1}|$.

(c) $V_1 \cup V_2$ *is a total dominating set of* G.

Proof. Since (a) and (b) are immediate, we only prove (c). By the definition, each vertex of V_{-1} is adjacent to a vertex of $V_1 \cup V_2$. Suppose that $G[V_1 \cup V_2]$ has an isolated vertex v. As $\delta(G) \geq 1$, the vertex v is adjacent to a vertex in V_{-1} and all its neighbors are in V_{-1}. This leads to the contradiction $f(N(v)) \leq -1$. Therefore $G[V_1 \cup V_2]$ does not contain an isolated vertex and hence $V_1 \cup V_2$ is a total dominating set of G. □

The proof of the next lower bound is identically with the proof of Proposition 3 in [17] and is therefore omitted.

Proposition 6. If G is graph of order n with $\delta(G) \geq 1$, then

$$\gamma_{wstI}(G) \geq \max\{\Delta + 1 - n, \delta(G) + 3 - n\}.$$

Proposition 7. If G is graph of order n with $\delta(G) \geq 1$, then $\gamma_{wstI}(G) \geq 2\gamma_t(G) - n$.

Proof. Let $f = (V_{-1}, V_1, V_2)$ be a $\gamma_{wstI}(G)$-function. Then it follows from Observation 1 that

$$\gamma_{wstI}(G) = |V_1| + 2|V_2| - |V_{-1}| = 2|V_1| + 3|V_2| - n \geq 2|V_1 \cup V_2| - n \geq 2\gamma_t(G) - n, \quad (2.1)$$

and the desired inequality is proved. □

Example 1. *Let $p \geq 3$ be an integer, and let $v_1, v_2, \ldots, v_p$ be the vertex set of the complete graph K_p. Now let H be the graph consisting of K_p and the $p(p-2)$ new vertices $w_i^1, w_i^2, \ldots, w_i^{p-2}$ for $1 \leq i \leq p$ such that v_i is adjacent to the vertices $w_i^1, w_i^2, \ldots, w_i^{p-2}$ for $1 \leq i \leq p$. Now define $f : V(H) \to \{-1, 1, 2\}$ by $f(v_i) = 1$ for $1 \leq i \leq p$ and $f(x) = -1$ otherwise. Then f is a WSTIDF on H of weight $p - p(p-2) = 3p - p^2$ and thus $\gamma_{wstI}(H) \leq 3p - p^2$. In addition, we observe that $\gamma_t(H) = p$. Combining this with Proposition 7, we obtain*

$$3p - p^2 = 2\gamma_t(H) - n(H) \leq \gamma_{wstI}(H) \leq 3p - p^2$$

and thus $\gamma_{wstI}(H) = 3p - p^2$ and $\gamma_{wstI}(H) = 2\gamma_t(H) - n(H)$.

Example 1 shoes that Proposition 7 is sharp. Example 1 will also demonstrate that the difference $\gamma_{stI}(G) - \gamma_{wstI}(G)$ can be arbitrarily large.

Example 2. *If f is an STIDF on the graph H of Example 1, then we show that $f(v_j) + \sum_{i=1}^{p-2} f(w_j^i) \geq 4 - p$ for $1 \leq j \leq p$. If $f(w_j^i) = -1$ for an index $1 \leq i \leq p-2$, then $f(v_j) = 2$ and therefore $f(v_j) + \sum_{i=1}^{p-2} f(w_j^i) \geq 2 - (p-2) = 4 - p$. If $f(w_j^i) \geq 1$ for each $i \in \{1, 2, \ldots, p-2\}$, then $f(v_j) + \sum_{i=1}^{p-2} f(w_j^i) \geq 1 + (p-2) = p - 1 \geq 4 - p$. This leads to $\sum_{x \in V(H)} f(x) \geq p(4-p) = 4p - p^2$ and thus $\gamma_{stI}(H) \geq 4p - p^2$. Using the fact that $\gamma_{wstI}(H) = 3p - p^2$, we observe that $\gamma_{stI}(H) - \gamma_{wstI}(H) \geq 4p - p^2 - (3p - p^2) = p$.*

We present a further example which will show that the difference $\gamma_{stI}(G) - \gamma_{wstI}(G)$ can be arbitrarily large.

Example 3. *Let $p \geq 2$ be an integer, and let SP_{2p+1} be the spider with the central vertex w, the neighbors $u_1, u_2, \ldots, u_p$ of w and the leaves v_i adjacent to u_i for $1 \leq i \leq p$. If f is a $\gamma_{stI}(SP_{2p+1})$-function, then we observe that $f(u_i) + f(v_i) \geq 1$ for $1 \leq i \leq p$ and $f(u_i) + f(v_i) = 1$ if and only if $f(w) = 2$. Therefore $\gamma_{stI}(SP_{2p+1}) \geq p + 2$, and in fact we observe $\gamma_{stI}(SP_{2p+1}) = p + 2$.*
On the other hand, the function g defined by $g(v_i) = -1$, $g(u_i) = 1$ for $1 \leq i \leq p$ and $g(w) = 2$ is a WSTIDF on SP_{2p+1} of weight 2 and thus $\gamma_{wstI}(SP_{2p+1}) \leq 2$. In fact we have $\gamma_{wstI}(SP_{2p+1}) = 2$.
Consequently, we deduce that $\gamma_{stI}(SP_{2p+1}) - \gamma_{wstI}(SP_{2p+1}) \geq p + 2 - 2 = p$.

The proof of the next proposition is identically with the proof of Proposition 8 in [15] and is therefore omitted.

Proposition 8. Let $f = (V_{-1}, V_1, V_2)$ be a WSTIDF of a graph G of order n, $\Delta = \Delta(G)$ and $\delta = \delta(G) \geq 1$. Then the following holds.

(a) $(2\Delta - 1)|V_2| + (\Delta - 1)|V_1| \geq (\delta + 1)|V_{-1}|$.

(b) $(2\Delta + \delta)|V_2| + (\Delta + \delta)|V_1| \geq (\delta + 1)n$.

(c) $(\Delta + \delta)\omega(f) \geq (\delta - \Delta + 2)n + (\delta - \Delta)|V_2|$.

(d) $\omega(f) \geq (\delta - 2\Delta + 2)n/(2\Delta + \delta) + |V_2|$.

As an immediate consequence of Proposition 8 (c), we obtain a lower bound on the weak signed total Italian domination number of regular graphs.

Corollary 1. If G is an r-regular graph of order n with $r \geq 1$, then $\gamma_{wstI}(G) \geq \lceil n/r \rceil$.

Using Corollary 1 and the inequalities $\gamma_{wstI}(G) \leq \gamma_{stI}(G) \leq \gamma_{stR}(G)$, we obtain the next known bounds immediately.

Corollary 2. *[15, 17]* If G is an r-regular graph of order n with $r \geq 1$, then $\gamma_{stR}(G) \geq \gamma_{stI}(G) \geq \lceil n/r \rceil$.

In the case that G is not regular, Proposition 8 (c) and (d) lead to the following lower bound.

Corollary 3. Let G be a graph of order n, maximum degree Δ and minimum degree $\delta \geq 1$. If $\delta < \Delta$, then

$$\gamma_{wstI}(G) \geq \left\lceil \frac{-2\Delta + 2\delta + 3}{2\Delta + \delta} n \right\rceil .$$

Proof. Multiplying both sides of the inequality in Proposition 8 (d) by $\Delta - \delta$ and adding the resulting inequality to the inequality in Proposition 8 (c), we obtain the desired lower bound. □

Corollary 3 leads to the next known results.

Corollary 4. *[15, 17]* Let G be a graph of order n, maximum degree Δ and minimum degree $\delta \geq 1$. If $\delta < \Delta$, then

$$\gamma_{stR}(G) \geq \gamma_{stI}(G) \geq \left\lceil \frac{-2\Delta + 2\delta + 3}{2\Delta + \delta} n \right\rceil .$$

The examples in [15, 17] which show the sharpness of Corollary 4 yield to the sharpness of Corollary 3.

Theorem 2. If G is a graph with $\delta(G) \geq 2$, then $\gamma_{wstI}(G) = \gamma_{stI}(G)$.

Proof. Clearly, $\gamma_{wstI}(G) \leq \gamma_{stI}(G)$. Let now f be a $\gamma_{wstI}(G)$-function. Then $\sum_{x \in N(v)} f(x) \geq 1$ for each vertex $v \in V(G)$. If $f(u) = -1$, then it follows from $d(u) \geq 2$ and $\sum_{x \in N(u)} f(x) \geq 1$ that u is adjacent to a vertex v with $f(v) = 2$ or u is adjacent to two vertices w and z with $f(w) = f(z) = 1$. Hence f is also an STIDF on G and thus $\gamma_{stI}(G) \leq \gamma_{wstI}(G)$. This leads to $\gamma_{wstI}(G) = \gamma_{stI}(G)$. □

3. Special classes of graphs

In this section, we determine the weak signed total italian domination number for special classes of graphs. Since $\gamma_{wstI}(K_2) = 2$, Theorem 2 and Proposition 1 lead to the first result in this section immediately.

Proposition 9. If $n \geq 2$, then $\gamma_{wstI}(K_n) = 2$ when n is even and $\gamma_{wstI}(K_n) = 3$ when n is odd.

For even n, Proposition 9 shows that Proposition 6 is sharp.

Proposition 10. If $n \geq 3$, then $\gamma_{wstI}(K_{1,n-1}) = 2$.

Proof. Let $G = K_{1,n-1}$, and let f be a $\gamma_{wstI}(G)$-function. If w is the central vertex of the star G, then clearly $f(w) \geq 1$. This implies $\gamma_{wstI}(G) = f(w) + f(N(w)) \geq 1 + 1 = 2$.
Now let $v_1, v_2, \ldots, n_{n-1}$ be the neighbors of w. If n is even, then define g by $g(w) = g(v_1) = g(v_2) = \ldots = g(v_{n/2}) = 1$ and $g(v_{n/2+1}) = g(v_{n/2+2}) = \ldots = g(v_{n-1}) = -1$. Then $g(N(w)) = \frac{n}{2} - (\frac{n}{2} - 1) = 1$ and hence g is a WSTIDF on G of weight 2 and thus $\gamma_{wstI}(G) \leq 2$. If n is odd, then define g by $g(v_1) = 2$, $g(w) = g(v_2) = g(v_3) = \ldots = g(v_{(n-1)/2}) = 1$ and $g(v_{(n+1)/2}) = g(v_{(n+3)/2}) = \ldots = g(v_{n-1}) = -1$. Then $g(N(w)) = 2 + \frac{n-1}{2} - 1 - \frac{n-1}{2} = 1$. Therefore g is a WSTIDF on G of weight 2 and so $\gamma_{wstI}(G) \leq 2$. In both cases we obtain $\gamma_{wstI}(G) = 2$, and the proof is complete. □

Theorem 2 and Proposition 2 yield to the next result immediately.

Proposition 11. If $p, q \geq 2$ are integers, then $\gamma_{wstI}(K_{p,q}) = 2$.

Proposition 12. If $S(r, s)$ is the double star, then $\gamma_{wstI}(S(r, s)) = 2$.

Proof. Let u and v be two adjacent vertices of $S(r,s)$ such that u is adjacent to r leaves and v is adjacent to s leaves. If f is a $\gamma_{wstI}(S(r,s))$-function, then the definition implies $\gamma_{wstI}(S(r,s)) = \omega(f) = f(N(u)) + f(N(v)) \geq 2$.
Assume, without loss of generality, that $r \leq s$. If $r \geq 3$, then if follows from Proposition 3 that $\gamma_{wstI}(S(r,s)) \leq \gamma_{stI}(S(r,s)) = 2$ and thus $\gamma_{wstI}(S(r,s)) = 2$ in this case.
Assume next that $r = 2$ and let u be adjacent to the leaves x and y, and let v be adjacent to the leaves $w_1, w_2, \ldots, w_s$. If $s = 2q$ is even, then define g by $g(u) = g(v) = g(x) = g(w_1) = g(w_2) = \ldots = g(w_q) = 1$ and $g(y) = g(w_{q+1}) = g(w_{q+2}) = \ldots - g(w_{2q}) = -1$. Then g is a WSTIDF on $S(2,s)$ of weight $q+3-(q+1) = 2$ and so $\gamma_{wstI}(S(2,s)) \leq 2$. If $s = 2q+1$ is odd, then define g by $g(w_1) = 2$, $g(u) = g(v) = g(x) = g(w_2) = g(w_3) = \ldots = g(w_q) = 1$ and $g(y) = g(w_{q+1}) = g(w_{q+2}) = \ldots - g(w_{2q+1}) = -1$. Then g is a WSTIDF on $S(2,s)$ of weight $2+3+(q-1)-(q+1)-1 = 2$ and so $\gamma_{wstI}(S(2,s)) \leq 2$. This leads to $\gamma_{wstI}(S(r,s)) = 2$ in both cases.
Finally, assume that $r = 1$. If $s = 1$ or $s \geq 3$, then Proposition 3 yiels the desired result. If $s = 2$, then let u be adjacent to the leaf x and v be adjacent to the leaves w and z. Define g by $g(v) = 2$, $g(u) = g(w) = 1$ and $g(x) = g(z) = -1$. Obviously, g is a WSTIDF on $S(1,2)$ of weight 2 and consequently $\gamma_{wstI}(S(1,2)) = 2$. □

We obtain the weak signed total Italian domination number of cycles from Theorem 2 and Proposition 4.

Proposition 13. If C_n is a cycle of length $n \geq 3$, then $\gamma_{wstI}(C_n) = n/2$ when $n \equiv 0 \pmod 4$, $\gamma_{wstI}(C_n) = (n+3)/2$ when $n \equiv 1,3 \pmod 4$ and $\gamma_{wstI}(C_n) = (n+6)/2$ when $n \equiv 2 \pmod 4$.

The next lemma is easy to prove but useful.

Lemma 1. Let G be a graph without isolated vertices, and let f be a WSTIDF on G. If $v_1v_2v_3v_4$ is a path of G with $d(v_2) = d(v_3) = 2$, then $f(v_1) + f(v_2) + f(v_3) + f(v_4) \geq 2$.

Proof. Since f is a WSTIDF on G and $d(v_2) = d(v_3) = 2$, we observe that $f(v_1) + f(v_2) + f(v_3) + f(v_4) = f(N(v_2)) + f(N(v_3)) \geq 2$. □

Proposition 14. Let P_n be a path of order $n \geq 4$. Then $\gamma_{wstI}(P_n) = n/2$ when $n \equiv 0 \pmod 4$ and $\gamma_{wstI}(P_n) = (n+2)/2$ when $n \equiv 2 \pmod 4$, $\gamma_{wstI}(P_n) = (n+3)/2$ when $n \equiv 3 \pmod 4$, $\gamma_{wstI}(P_n) = (n+1)/2$ when $9 \leq n \equiv 1 \pmod 4$ and $\gamma_{wstI}(P_5) = 2$.

Proof. Let $P_n = v_1v_2\ldots v_n$ and let f be a $\gamma_{wstI}(P_n)$-function.
Since $f(v_2) \geq 1$ and $f(N(v_2)) \geq 1$, we deduce that $A = f(v_1) + f(v_2) + f(v_3) \geq 2$. Analogously we have $B = f(v_n) + f(v_{n-1}) + f(v_{n-2}) \geq 2$.

Assume first that $n \equiv 0 \pmod 4$. Let $n = 4t$ for an integer $t \geq 1$. Using Lemma 1, we obtain

$$\gamma_{wstI}(P_n) = \gamma_{wstI}(P_{4t}) = \sum_{i=0}^{t-1}(f(v_{4i+1}) + f(v_{4i+2}) + f(v_{4i+3}) + f(v_{4i+4})) \geq 2t = \frac{n}{2}.$$

On the other hand define g by $g(v_{4i+1}) = g(v_{4i+4}) = -1$ and $g(v_{4i+2}) = g(v_{4i+3}) = 2$ for $0 \leq i \leq t-1$. Then g is a WSTIDF on P_n of weight $2t$, and therefore $\gamma_{wstI}(P_n) = n/2$ in this case.
Assume second that $n \equiv 2 \pmod 4$. Let $n = 4t+2$ for an integer $t \geq 1$. Using Lemma 1, we obtain

$$\begin{aligned}\gamma_{wstI}(P_n) &= A + B + \sum_{i=1}^{t-1}(f(v_{4i}) + f(v_{4i+1}) + f(v_{4i+2}) + f(v_{4i+3}))\\ &\geq 2 + 2 + 2(t-1) = 2t + 2 = \frac{n+2}{2}.\end{aligned}$$

On the other hand define g by $g(v_{4i}) = g(v_{4i-1}) = 2$ for $1 \leq i \leq t$, $g(v_{4i+1}) = g(v_{4i+2}) = -1$ for $1 \leq i \leq t-1$, $g(v_1) = g(v_{4t+2}) = -1$ and $g(v_2) = g(v_{4t+1}) = 1$. Then g is a WSTIDF on P_n of weight $2t+2$, and therefore $\gamma_{wstI}(P_n) = (n+2)/2$ in this case.
Assume third that $n \equiv 3 \pmod 4$. Let $n = 4t+3$ for an integer $t \geq 1$. Assume next that $f(v_4) = -1$. It follows that $f(v_2) = 2$. If $f(v_3) = 2$, then $A = f(v_1) + f(v_2) + f(v_3) \geq 3$, if $f(v_3) = 1$, then $f(v_1) \geq 1$ and so $A \geq 4$, and if $f(v_3) = -1$ then $f(v_1) = 2$ and thus $A \geq 3$. Therefore Lemma 1 implies

$$\gamma_{wstI}(P_n) = A + \sum_{i=1}^{t}(f(v_{4i}) + f(v_{4i+1}) + f(v_{4i+2}) + f(v_{4i+3})) \geq 3 + 2t = \frac{n+3}{2}.$$

Let now $f(v_4) \geq 1$. Then $A + f(v_4) \geq 3$, and we obtain

$$\begin{aligned}\gamma_{wstI}(P_n) &= A + f(v_4) + \sum_{i=1}^{t-1}(f(v_{4i+1}) + f(v_{4i+2}) + f(v_{4i+3}) + f(v_{4i+4})) + B\\ &\geq 3 + 2(t-1) + 2 = 2t + 3 = \frac{n+2}{2}.\end{aligned}$$

Conversely, Proposition 5 implies $\gamma_{wstI}(P_n) \leq \gamma_{stR}(P_n) = (n+3)/2$ and thus $\gamma_{wstI}(P_n) = (n+3)/2$ when $n \equiv 3 \pmod 4$.
Finally, assume that $9 \leq n \equiv 1 \pmod 4$. Let $n = 4t+1$ for an integer $t \geq 2$. If $f(v_5) \geq 1$, then $f(v_1)+f(v_2)+f(v_3)+f(v_4)+f(v_5) \geq f(N(v_2))+f(N(v_3))+f(v_5) \geq 3$ and thus

$$\begin{aligned}\gamma_{wstI}(P_n) &= \sum_{i=1}^{5} f(v_i) + \sum_{i=1}^{t-1}(f(v_{4i+2}) + f(v_{4i+3}) + f(v_{4i+4}) + f(v_{4i+5}))\\ &\geq 3 + 2(t-1) = 2t + 1 = \frac{n+1}{2}.\end{aligned}$$

Let now $f(v_5) = -1$. This leads to $f(v_3) = f(v_7) = 2$. If $f(v_4) = 2$, then again $f(v_1) + f(v_2) + f(v_3) + f(v_4) + f(v_5) \geq 3$ and we arrive at $\gamma_{wstI}(P_n) \geq (n+1)/2$ as in the last case. If $f(v_4) = -1$, then we see that $f(v_2) = f(v_6) = 2$. This leads to $\sum_{i=1}^{7} f(v_i) \geq 5$. Since $f(v_{4t}) + f(v_{4t+1}) \geq 0$, we deduce that

$$\begin{aligned}\gamma_{wstI}(P_n) &= \sum_{i=1}^{7} f(v_i) + \sum_{i=2}^{t-1}(f(v_{4i}) + f(v_{4i+1}) + f(v_{4i+2}) + f(v_{4i+3})) + f(v_{4t}) + f(v_{4t+1}) \\ &\geq 5 + 2(t-2) = 2t + 1 = \frac{n+1}{2}.\end{aligned}$$

If $f(v_4) = 1$, then we observe $f(v_6) \geq 1$. Since $f(v_2) \geq 1$, we again have $\sum_{i=1}^{7} f(v_i) \geq 5$, and hence we deduce $\gamma_{wstI}(P_n) \geq (n+1)/2$ as in the last case.
On the other hand define g by $g(v_{4i}) = g(v_{4i+3}) = 2$ and $g(v_{4i+1}) = g(v_{4i+2}) = -1$ for $1 \leq i \leq t-1$, $g(v_1) = g(v_{4t+1}) = -1$, $g(v_2) = 1$ and $g(v_3) = g(v_{4t}) = 2$. Then g is a WSTIDF on P_n of weight $2t+1$, and therefore $\gamma_{wstI}(P_n) = (n+1)/2$ in this case.
Clearly, $\gamma_{wstI}(P_5) \geq A + f(v_4) + f(v_5) \geq 2$ and thus $\gamma_{wstI}(P_5) \geq 2$. Conversely, define g by $g(v_1) = g(v_5) = -1$, $g(v_2) = g(v_4) = 1$ and $g(v_3) = 2$. Then g is a WSTIDF on P_5 of weight 2, and therefore $\gamma_{wstI}(P_5) = 2$. □

Proposition 15. If $G = K_{n_1,n_2,\ldots,n_p}$ is a complete p-partite graph with $n_1 \leq n_2 \leq \ldots \leq n_p$, $p \geq 3$ and $n_p \geq 2$, then $\gamma_{wstI}(G) = 2$.

Proof. Let $X_1, X_2, \ldots, X_p$ be the partite sets of G with $|X_i| = n_i$ for $1 \leq i \leq p$, and let f be a $\gamma_{wstI}(G)$-function. If we suppose that $f(X_i) \leq 0$ for each $i \in \{1, 2, \ldots, p\}$, then we obtain the contradiction $f(N(v)) = \sum_{i=2}^{p} f(X_i) \leq 0$ when $v \in X_1$. Hence there exists a partite set, say X_1, with $f(X_1) \geq 1$. If $v \in X_1$, then we deduce that

$$\gamma_{wstI}(G) = f(X_1) + f(N(v)) \geq 2.$$

If $|X_i| = |\{v_1, v_2, \ldots, v_s\}| \geq 2$, then we show next that we can define a function $g : X_i \to \{-1, 1, 2\}$ with $g(X_i) = 0$ or $g(X_i) = 1$. Let first $s = 2k$ be even. If we define g by $g(v_1) = 2$, $g(v_2) = g(v_3) = \ldots = g(v_k) = 1$ and $g(v_{k+1}) = g(v_{k+2}) = \ldots = g(v_{2k}) = -1$, then $g(X_i) = 1$. If we define g by $g(v_1) = g(v_2) = \ldots = g(v_k) = 1$ and $g(v_{k+1}) = g(v_{k+2}) = \ldots = g(v_{2k}) = -1$, then $g(X_i) = 0$. Let second $s = 2k+1$ be odd. If we define g by $g(v_1) = g(v_2) = \ldots = g(v_{k+1}) = 1$ and $g(v_{k+2}) = g(v_{k+3}) = \ldots = g(v_{2k+1}) = -1$, then $g(X_i) = 1$. If we define g by $g(v_1) = 2$, $g(v_2) = g(v_3) = \ldots = g(v_k) = 1$ and $g(v_{k+1}) = g(v_{k+2}) = \ldots = g(v_{2k+1}) = -1$, then $g(X_i) = 0$.
Now assume that $n_1 \geq 2$. Define g such that $g(X_1) = g(X_2) = 1$ and $g(X_i) = 0$ for $3 \leq i \leq p$. We observe that $g(N(w)) = 1$ for $w \in X_1 \cup X_2$ and $g(N(w)) = 2$ for $w \in X_3 \cup X_4 \cup \ldots \cup X_p$. Therefore g is a WSTIDF on G of weight 2 and thus $\gamma_{wstI}(G) \leq 2$ and so $\gamma_{wstI}(G) = 2$ if $n_1 \geq 2$.

Assume next that $n_1 = n_2 = \ldots = n_s = 1$ for an integer $s \geq 1$ and $n_{s+1}, n_{s+2}, \ldots, n_p \geq 2$. Let $X_i = \{x_i\}$ for $1 \leq i \leq s$.
If $s = 2k$ is even, then define g by $g(x_1) = g(x_2) = \ldots g(x_{k+1}) = 1$, $g(x_{k+2}) = g(x_{k+3}) = \ldots = g(x_{2k}) = -1$ and $g(X_i) = 0$ for $s+1 \leq i \leq p$. We note that $g(N(x_i)) = 1$ for $1 \leq i \leq k+1$, $g(N(x_i)) = 3$ for $k+2 \leq i \leq 2k$ and $g(N(x)) = 2$ for $x \in X_{s+1} \cup X_{s+2} \cup \ldots \cup X_p$. Hence g is a WSTIDF on G of weight 2 and thus $\gamma_{wstI}(G) \leq 2$ and so $\gamma_{wstI}(G) = 2$ in this case.
If $s = 2k+1$ is odd, then define g by $g(x_1) = g(x_2) = \ldots g(x_{k+1}) = 1$, $g(x_{k+2}) = g(x_{k+3}) = \ldots = g(x_{2k+1}) = -1$, $g(X_{2k+2}) = 1$ and $g(X_i) = 0$ for $s+2 \leq i \leq p$. We note that $g(N(x_i)) = 1$ for $1 \leq i \leq k+1$, $g(N(x_i)) = 3$ for $k+2 \leq i \leq 2k+1$, $g(N(x)) = 1$ for $x \in X_{2k+2}$ and $g(N(x)) = 2$ for $x \in X_{s+2} \cup X_{s+3} \cup \ldots \cup X_p$. Hence g is a WSTIDF on G of weight 2 and thus $\gamma_{wstI}(G) \leq 2$ and so $\gamma_{wstI}(G) = 2$ in the last case. □

Theorem 2 shows that Proposition 15 is also valid for the signed total Italian domination number.

Corollary 5. If $G = K_{n_1,n_2,\ldots,n_p}$ is a complete p-partite graph with $n_1 \leq n_2 \leq \ldots \leq n_p$, $p \geq 3$ and $n_p \geq 2$, then $\gamma_{stI}(G) = 2$.

4. Further lower bounds

Theorem 3. Let T be a tree of order $n \geq 4$. If T is not a star, then

$$\gamma_{wstI}(T) \geq \Delta(T) + 4 - n.$$

Proof. Let f be a $\gamma_{wstI}(T)$-function, and let v be a vertex of maximum degree $\Delta = \Delta(T)$. If $f(v) \geq 2$, then

$$\gamma_{wstI}(T) = f(v) + f(N(v)) + \sum_{x \in V(T) \setminus N[v]} f(x) \geq 2 + 1 + (\Delta + 1 - n) = \Delta + 4 - n.$$

If $f(v) = -1$, then v has a neighbor u with $f(u) \geq 1$. As T is a tree, u has a neighbor $w \notin N[v]$ with $f(w) = 2$ or u has two neighbors $a, b \notin N[v]$ with $f(a) = f(b) = 1$. This leads to

$$\begin{aligned}\gamma_{wstI}(T) &= f(v) + f(N(v)) + f(w) + \sum_{x \in V(T) \setminus (N[v] \cup \{w\})} f(x) \\ &\geq -1 + 1 + 2 + (\Delta + 2 - n) = \Delta + 4 - n\end{aligned}$$

or

$$\begin{aligned}\gamma_{wstI}(T) &= f(v) + f(N(v)) + f(a) + f(b) + \sum_{x \in V(T) \setminus (N[v] \cup \{a,b\})} f(x) \\ &\geq -1 + 1 + 1 + 1 + (\Delta + 3 - n) = \Delta + 5 - n.\end{aligned}$$

Let now $f(v)=1$. Since T is not a star, there exists a vertex $w\notin N[v]$ adjacent to a neighbor u of v. If $f(w)\geq 1$, then it follows that

$$\begin{aligned}\gamma_{wstI}(T) &= f(v)+f(N(v))+f(w)+\sum_{x\in V(T)\setminus(N[v]\cup\{w\})} f(x)\\ &\geq 1+1+1+(\Delta+2-n)=\Delta+5-n.\end{aligned}$$

However, if $f(w)=-1$, then u has a further neighbor $z\notin(N[v]\cup\{w\})$ with $f(z)\geq 1$, and we obtain again $\gamma_{wstI}(T)\geq\Delta+5-n$. □

If T is a star, then Proposition 10 inplies $\gamma_{wstI}(T)=2=\Delta(T)+3-n(T)$. The next example will demonstrate that Theorem 3 is sharp.

Example 4. *Let T_{3p} be the wounded spider by subdividing p of the edges of the star $K_{1,2p-1}$ for $p\geq 2$. Let w be the center of the star, $v_1,v_2,\ldots,v_p$ be the vertices of degree two, u_i be the neighbor of v_i for $1\leq i\leq p$ and $y_1,y_2,\ldots,y_{p-1}$ be the leaves adjacent to w. Define the function f by $f(w)=2$, $f(v_1)=f(v_2)=\ldots=f(v_p)=1$ and $f(x)=-1$ otherwise. Then f is a WSTIDF on T_{3p} of weight $3-p=\Delta(T_{3p})+4-n(T_{3p})$. Therefore $\gamma_{wstI}(T_{3p})\leq\Delta(T_{3p})+4-n(T_{3p})$ and thus $\gamma_{wstI}(T_{3p})=\Delta(T_{3p})+4-n(T_{3p})$ by Theorem 3.*

For $p\geq 2$, let T_{3p} be the wounded spider of Example 4. Let $\mathcal{T}=\{T_{3p}\mid p\geq 2\}$.

For a subset $S\subseteq V(G)$, we let $d_S(v)$ denote the number of vertices in S that are adjacent to the vertex v. For disjoint subsets U and W of vertices, we let $[U,W]$ denote the set of edges between U and W. Now let $f=(V_{-1},V_1,V_2)$ be a WSTIDF. For notational convenience, we let $V_{12}=V_1\cup V_2$, $|V_{12}|=n_{12}$, $|V_1|=n_1$ and $|V_2|=n_2$. Furthermore, let $|V_{-1}|=n_{-1}$ and so $n_{-1}=n-n_{12}$. Let $G_{12}=G[V_{12}]$ be the subgraph induced by V_{12} and let G_{12} have size m_{12}. For $i=-1,1,2$, if $V_i\neq\emptyset$, let $G_i=G[V_i]$ be the subgraph induced by V_i and let G_i have size m_i. Hence $m_{12}=m_1+m_2+|[V_1,V_2]|$.

Theorem 4. If G is a connected graph of order $n\geq 4$ and size m, then

$$\gamma_{wstI}(G)\geq\frac{8n-9m}{3},$$

with equality if and only if $T\in\mathcal{T}$.

Proof. Let $f=(V_{-1},V_1,V_2)$ be a $\gamma_{wstI}(G)$-function. Since each vertex of V_{-1} has at least one neighbor in $V_1\cup V_2$, we observe that

$$|V_{-1}|\leq|[V_{-1},V_1\cup V_2]|=\sum_{v\in V_1}d_{V_{-1}}(v)+\sum_{v\in V_2}d_{V_{-1}}(v).$$

For each $v \in V_1 \cup V_2$, we have $1 \le f(N(v)) = 2d_{V_2}(v) + d_{V_1}(v) - d_{V_{-1}}(v)$ and so $d_{V_{-1}}(v) \le 2d_{V_2}(v) + d_{V_1}(v) - 1$. Hence we obtain

$$\begin{aligned} n_{-1} &= |V_{-1}| \le \sum_{v\in V_1} d_{V_{-1}}(v) + \sum_{v\in V_2} d_{V_{-1}}(v) \\ &\le \sum_{v\in V_1}(2d_{V_2}(v) + d_{V_1}(v) - 1) + \sum_{v\in V_2}(2d_{V_2}(v) + d_{V_1}(v) - 1) \\ &= 2|[V_1, V_2]| + 2m_1 - n_1 + 4m_2 + |[V_1, V_2]| - n_2 \\ &= 4m_{12} - 4m_1 - 4|[V_1, V_2]| + 2m_1 + 3|[V_1, V_2]| - n_1 - n_2 \\ &= 4m_{12} - 2m_1 - |[V_1, V_2]| - n_1 - n_2 \\ &= 4m_{12} + \frac{1}{2}m_{12} - \frac{1}{2}m_1 - \frac{1}{2}m_2 - \frac{1}{2}|[V_1, V_2]| - 2m_1 - |[V_1, V_2]| - n_1 - n_2 \\ &\le \frac{9}{2}m_{12} - \frac{5}{2}m_1 - \frac{3}{2}|[V_1, V_2]| - n_1 - n_2 \qquad (4.1) \end{aligned}$$

and so

$$m_{12} \ge \frac{2}{9}\left(n_{-1} + n_1 + n_2 + \frac{5}{2}m_1 + \frac{3}{2}|[V_1, V_2]|\right).$$

Hence we deduce that

$$\begin{aligned} m &\ge m_{12} + |[V_{-1}, V_{12}]| \ge m_{12} + n_{-1} \\ &\ge \frac{2}{9}\left(\frac{11}{2}n_{-1} + n_1 + n_2 + \frac{5}{2}m_1 + \frac{3}{2}|[V_1, V_2]|\right) \\ &= \frac{2}{9}\left(\frac{11}{2}n - \frac{9}{2}n_{12} + \frac{5}{2}m_1 + \frac{3}{2}|[V_1, V_2]|\right). \qquad (4.2) \end{aligned}$$

This yields

$$n_{12} \ge \frac{11}{9}n - m + \frac{1}{9}\left(5m_1 + 3|[V_1, V_2]|\right)$$

and thus

$$\begin{aligned} \gamma_{wstI}(G) &= 2n_2 + n_1 - n_{-1} = 3n_2 + 2n_1 - n = 3n_{12} - n - n_1 \\ &\ge \frac{11}{3}n - 3m - n + \frac{1}{3}(5m_1 + 3|[V_1, V] - 3n_1) \\ &= \frac{8n - 9m}{3} + \frac{1}{3}\left(5m_1 + 3|[V_1, V_2]| - 3n_1\right). \end{aligned}$$

Let $\phi(n_1) = 5m_1 + 3|[V_1, V_2]| - 3n_1$. It suffices to show that $\phi(n_1) \ge 0$, since then $\gamma_{wstI}(G) \ge (8n - 9m)/3$, which is the desired bound. If $n_1 = 0$, then $\phi(n_1) = 0$, and we are done. Assume now that $n_1 \ge 1$. Let $H_1, H_2, \ldots, H_t$ be the components of the induced subgraph $G[V_1]$ of order $h_1, h_2, \ldots, h_t$ and size $p_1, p_2, \ldots, p_t$. Since G is connected, each component H_i contains a vertex adjacent to a vertex of V_2 or to a vertex of V_{-1} for $1 \le i \le t$. Assume that $H_1, H_2, \ldots, H_s$ are the components which does not contain a vertex adjacent to a vertex of V_2 and that $H_{s+1}, H_{s+2}, \ldots, H_t$ are the components which contain a vertex adjacent to a vertex in V_2. Let $n_1^1 = h_1 + h_2 + \ldots + h_s$, $n_1^2 = n_1 - n_1^1$, $m_1^1 = p_1 + p_2 + \ldots + p_s$ and $m_1^2 = m_1 - m_1^1$. We observe that $h_i \geq 3$ for $1 \leq i \leq s$ and thus $n_1^1 \geq 3s$. This leads to

$$m_1^1 = p_1 + p_2 + \ldots + p_s \geq (h_1 - 1) + (h_2 - 1) + \ldots + (h_s - 1) = n_1^1 - s \geq \frac{2}{3} n_1^1. \quad (4.3)$$

In addition, we observe that

$$m_1^2 + |[V_1, V_2]| \geq (h_{s+1} - 1) + (h_{s+2} - 1) + \ldots + (h_t - 1) + (t - s) = n_1^2. \quad (4.4)$$

Combining the inequalities (4.3) and (4.4), we obtain

$$\begin{aligned} \phi(n_1) &= 5m_1 + 3|[V_1, V_2]| - 3n_1 \\ &= 5m_1^1 + 5m_1^2 + 3|[V_1, V_2]| - 3n_1^1 - 3n_1^2 \\ &\geq \left(5m_1^1 - \frac{10}{3} n_1^1\right) + \left(3m_1^2 + 3|[V_1, V_2]| - 3n_1^2\right) \geq 0, \end{aligned} \quad (4.5)$$

and the desired bound is proved,
Assume now that $\gamma_{wstI}(G) = \frac{8n-9m}{3}$. Then all inequalities above must be equalities. In particular, $m_2 = 0$ (according to (4.1)), $m_{-1} = 0$ (according to (4.2)), $n_1^1 = 0$, $m_1^2 = 0$ (according to (4.5)), $m_1^1 = 0$ and so $m_1 = 0$ and $n_1^2 = n_1$. In addition, (4.2) yields $|[V_{-1}, V_{12}]| = n_{-1}$ and hence each vertex of V_{-1} is a leaf of G. Now the condition $\phi(n_1) = 5m_1 + 3|[V_1, V_2]| - 3n_1 = 0$, leads to $|[V_1, V_2]| = n_1$. Since G is connected, G_{12} is connected and thus $m_{12} = |[V_1, V_2]| \geq n_1 + n_2 - 1 \geq n_1 + 1$ when $n_2 \geq 2$. Consequently, $n_2 = 1$. The condition $f(N(v)) \geq 1$ for each vetex v leads to $|[V_{-1}, V_1]| \leq n_1$ and $|[V_{-1}, V_2]| \leq n_1 - 1$. Using the identity $n_{-1} = 3|[V_1, V_2]| - n_1 - n_2 = 3n_1 - n_1 - 1 = 2n_1 - 1$ (see (4.1)), it follows that $|[V_{-1}, V_1]| = n_1$ and $|[V_{-1}, V_2]| = n_1 - 1$. All together, we note that $G \in \mathcal{T}$ with $p = n_1$.
Conversely, it is easy to see that $\gamma_{wstI}(T_{3p}) = \frac{8n(T_{3p}) - 9m(T_{3p})}{3} = \frac{24p - 37p + 9}{3} = 3 - p$. □

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

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