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Research Article

A study on strong and geodetic domination integrity sets in graphs

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Abstract: Consider a graph $\Omega = (\mathcal{V}, \mathcal{E})$ that is simple, and let ϑ_1 and ϑ_2 be elements of $\mathcal{V}(\Omega)$ suth that $\vartheta_1\vartheta_2 \in \mathcal{E}(\Omega)$. Then, ϑ_1 is said to strongly dominate ϑ_2 if $deg\,(\vartheta_1) \geq deg\,(\vartheta_2)$. A set K of $\mathcal{V}(\Omega)$ is identified as a strong dominating set (sd-set) if every vertex ϑ_2 outside of K is strongly dominated by at least one node ϑ_1 within K. The concept of strong domination integrity for Ω is defined as $\widetilde{SDI}(\Omega) = min_{K\subseteq\mathcal{V}}\{|K|+m(\Omega-K) : K$ is a sd-set of $\Omega\}$. Similarly, the set $K \subseteq \mathcal{V}(\Omega)$ is identified as a geodetic dominating set (gd-set) if K is both geodetic and dominating set. The geodetic domination integrity of Ω is defined as $\widetilde{GDI}(\Omega) = min\{|K| + m(\Omega - K) : K$ is a gd-set of $\Omega\}$. This paper delves into the study of strong and geodetic domination integrity sets, as well as the impact of node removal on these sets. Additionally, it introduces the concepts of $\widetilde{SDI}$-Excellent and $\widetilde{GDI}$-Excellent graphs, provides examples, and derives theorems from these graphs.

Keywords: strong domination integrity sets, geodetic domination integrity sets, $\widetilde{SDI}$-Ext graphs, $\widetilde{GDI}$-Ext graphs.

AMS Subject classification:

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1. Introduction

Graph theory stands as a crucial mathematical concept with broad use in various fields, such as Operations Research, Physics, Chemistry, Biology, Electrical Engineering, Sociology, Architecture, and many others. A key feature of a communication network is its ability to function well even when some nodes or connections are not working. The vulnerability of a network, starting with its connectivity, offers numerical assessments of how well the network can withstand difficult situations. These assessments aim to explain how the network behaves when a part of its nodes or connections is removed. In this scenario, domination measures how connected a chosen group of nodes remains with the rest of the network. A lower domination value suggests a higher potential for efficient communication among most nodes, with only a few exceptions. Examining the effects of removing a node or a dominating group from a network is particularly interesting, as it shows that the network experiences greater harm when its critical components are impacted.

Ore and Berge [26] laid the groundwork for understanding domination in graphs, while Cockayne and Hedetniemi [12] expanded on this by introducing the notions of domination number and independent domination number. Sampathkumar and Pushpa Latha [28] further developed the concept of strong (weak) domination. Hattingh *et al.* provided an in-depth analysis of the strong domination number in [19, 20, 27]. The notion of the connected domination number in graph theory was examined by Sampathkumar and Walikar [29]. Somasundaram *et al.* [31, 32] examined the concepts of domination within fuzzy graphs by employing effective arcs. Within the realm of network design, the concepts of geodesic and domination have gained recognition and have been applied in a variety of contexts. It's evident that in most graphs, a geodesic set does not dominate, and similarly, a dominating set does not necessarily have a geodesic. Harary *et al.* [7, 8, 17] defined the geodetic number for graphs, while Santhakumaran and John [30] and Atici [1] provided formal definitions for the geodetic number of graphs with edges. Mariano and Canoy [25] conducted further research on edge geodetic coverings for graphs. Talebiy and Rashmanlouz presented the notions of dominating set, perfect dominating set, minimal perfect dominating set, and independent dominating set within the context of vague graphs [35, 36]. The concept of graph integrity, which has been extensively explored, was first introduced by Barefoot *et al.* [4]. Sundareswaran and Swaminathan [33, 34], who defined domination integrity in graphs. The notion of connected domination integrity was examined by Harisaran *et al.* [18]. In 2022, Balaraman *et al.* [16] delved into the study of strong domination integrity and introduced the concepts of geodetic domination integrity, expanding on this research [3, 15].

2. Basic Definitions

Let $\Omega = (\mathcal{V}, \mathcal{E})$ be a graph with node set $\mathcal{V}(\Omega)$ and the edge set $\mathcal{E}(\Omega)$. The number of nodes within $\mathcal{V}(\Omega)$ is referred to as the graph's order and is denoted by $O(\Omega)$,

whereas the number of edges within $\mathcal{E}(\Omega)$ is referred to as the graph's size. A graph is considered connected if there exists a path between any two distinct nodes within Ω. A maximal connected subgraph of Ω is identified as a component of Ω. The distance, denoted as $d(\vartheta_1, \vartheta_2)$, between two nodes ϑ_1 and ϑ_2 in $\mathcal{V}(\Omega)$, is defined as the length of the shortest path (geodesic) between ϑ_1 and ϑ_2 within Ω. The length of the longest geodesic path, referred to as its diameter $diam(\Omega)$. The open neighborhood of a node ϑ within Ω is defined as the set of all nodes adjacent to ϑ. Conversely, the closed neighborhood of ϑ, denoted as $N[\vartheta]$, is defined as the set $N(\vartheta) \cup \{\vartheta\}$. A subset $K \subseteq \mathcal{V}(\Omega)$ is identified as a dominating set (d-set) if for every node $\vartheta_2 \in \mathcal{V} - K$, there exists a node $\vartheta_1 \in K$ such that ϑ_2 is dominated by ϑ_1. A d-set $K \subseteq \mathcal{V}(\Omega)$ is minimal if $K - \vartheta$ is not a d-set for any ϑ in K. The size of the smallest minimal d-set is known as domination number $\gamma(\Omega)$.
Let $\vartheta_1, \vartheta_2 \in \mathcal{V}(\Omega)$. Suppose $\vartheta_1\vartheta_2 \in \mathcal{E}(\Omega)$, and $deg(\vartheta_1) \geq deg(\vartheta_2)$. In this case, ϑ_1 is said to strongly dominate ϑ_2. A subset $K \subseteq \mathcal{V}(\Omega)$ is referred to as a strong dominating set (sd-set) if every node $\vartheta_2 \in \mathcal{V} - K$ is strongly dominated by some $\vartheta_1 \in K$. A sd-set is minimal if $K - \{\vartheta\}$ is not a sd-set for each ϑ in K. The strong domination number $\gamma_s(\Omega)$ is the smallest cardinality of a sd-set. A set $K \subseteq \mathcal{V}(\Omega)$ is considered a geodetic set if $I[K] = \mathcal{V}(\Omega)$. The geodetic number $g(\Omega)$ is the smallest possible cardinality of a geodetic set of Ω. A geodetic set K is minimal if no proper subset of K is geodetic set of Ω. A set $K \subseteq \mathcal{V}(\Omega)$ is geodetic dominating set (gd-set) if K is both a geodetic and a dominating set of Ω. The geodetic domination number $\gamma_g(\Omega)$, is the smallest possible cardinality of a gd-set.

3. Vulnerability parameters

The analysis of vulnerability can be approached from multiple perspectives within the field of Graph Theory. It is essential to take into account specific factors to evaluate the network's vulnerability.

The variables involved in assessing vulnerability include:
1. Group of nodes or links that are broken: $|K|$, $K \subseteq \mathcal{V}(\Omega)$
2. Quantity of components still present: $\omega(\Omega - K)$
3. Largest order of components: $m(\Omega - K)$

Various graph theorists have devised unique metrics to measure vulnerability based on these variables.

- *Connectivity*: $\kappa(\Omega) = min\{|K| : K \subseteq \mathcal{V}(\Omega), \omega(\Omega - K) > 1\}$.
- *Toughness* [5, 10]: $t(\Omega) = min\{\frac{|K|}{\omega(\Omega-K)} : K \subseteq \mathcal{V}(\Omega), \omega(\Omega - K) > 1\}$.
- *Scattering number* [21]: $sc(\Omega) = max\{\omega(\Omega - K) - |K| : K \subseteq \mathcal{V}(\Omega), \omega(\Omega - K) > 1\}$.
- *Integrity* [4]: $I(\Omega) = min\{|K| + m(\Omega - K) : K \subseteq \mathcal{V}(\Omega)\}$.

- *Edge-Integrity* [2]: $I'(\Omega) = min\{|K| + m(\Omega - K) : K \subseteq \mathcal{E}(\Omega)\}$.
- *Tenacity* [13, 14]: $T(\Omega) = min\{\frac{|K|+m(\Omega-K)}{\omega(\Omega-K)} : K \subseteq \mathcal{V}(\Omega), \omega(\Omega - K) > 1\}$.
- *Weak Integrity* [22]: $I_w(\Omega) = min\{|K| + m_e(\Omega - K) : K \subseteq \mathcal{V}(\Omega)\}$, where $m_e(\Omega - K)$ denotes the number of edges of a largest component of $\Omega - K$.
- *Rupture degree* [23, 24]: $r(\Omega) = max\{\omega(\Omega - K) - |K| - m(\Omega - K) : K \subseteq \mathcal{V}(\Omega), \omega(\Omega - K) > 1\}$.
- *Domination Integrity* [33]: $DI(\Omega) = min\{|K| + m(\Omega - K) : K \subseteq \mathcal{V}(\Omega)$ and K is a d-set of $\Omega\}$.
- *Strong domination integrity* [16]: $\widetilde{SDI}(\Omega) = min\{|K| + m(\Omega - K) : K$ is a sd-set of $\Omega\}$
- *Geodetic domination integrity* [3]: $\widetilde{GDI}(\Omega) = min\{|K| + m(\Omega - K) : K$ is a gd-set of $\Omega\}$.

4. The dynamics of strong and geodetic domination integrity: An exploration of change and constancy

Fault tolerance plays a crucial role in the development of a network topology, referring to the network's ability to continue providing services despite the presence of malfunctioning components. To assess the network's performance in the face of a fault, one may investigate the consequences of eliminating either an edge (for example, a link failure) or a node (for instance, a processor malfunction) within the network graph, in relation to the established fault tolerance criteria. This research delves into the analysis of strong domination integrity sets and geodetic domination integrity sets. Additionally, it examines the effect of node removal on the values of the strong and geodetic domination integrity sets.

Definition 1. A subset $K \subseteq \mathcal{V}(\Omega)$ is a Strong Domination Integrity set ($\widetilde{SDI}$-set) of Ω if $\widetilde{SDI}(\Omega) = |K| + m(\Omega - K)$.

Definition 2. The smallest cardinality of a minimal $\widetilde{SDI}$-set is called $\widetilde{SDI}$-set number and is represented by $\underleftrightarrow{SDI}(\Omega)$. The maximum cardinality of a minimal $\widetilde{SDI}$-set is known as upper $\widetilde{SDI}$-set number and is represented by $\overleftrightarrow{SDI}(\Omega)$.

Remark 1.

- $\gamma_s(\Omega) \le \underleftrightarrow{SDI}(\Omega) \le \overleftrightarrow{SDI}(\Omega)$.
- $\underleftrightarrow{SDI}(\Omega) = \overleftrightarrow{SDI}(\Omega) = p$ iff Ω is totally disconnected.
- For a non-complete graph Ω, every $\widetilde{SDI}$-set related to Ω forms a cut-set of Ω, which means it has a cardinality of at least $\kappa(\Omega)$. Therefore, we can conclude that $\underleftrightarrow{SDI}(\Omega) \ge \kappa(\Omega)$.
- $\underleftrightarrow{SDI}(K_p) = 1$ and $\overleftrightarrow{SDI}(K_p) = p$.
- For $K_{p,q}, p \ge q$, $\underleftrightarrow{SDI}(K_{p,q}) = \overleftrightarrow{SDI}(K_{p,q}) = q$.

Definition 3. Consider a simple graph denoted as Ω. The node set $\mathcal{V}(\Omega)$ can be divided into three distinct subsets.
$\widetilde{SDI}^{+}(\Omega) = \{\vartheta \in \mathcal{V}(\Omega) : \widetilde{SDI}(\Omega - \vartheta) > \widetilde{SDI}(\Omega)\}$
$\widetilde{SDI}^{0}(\Omega) = \{\vartheta \in \mathcal{V}(\Omega) : \widetilde{SDI}(\Omega - \vartheta) = \widetilde{SDI}(\Omega)\}$
$\widetilde{SDI}^{-}(\Omega) = \{\vartheta \in \mathcal{V}(\Omega) : \widetilde{SDI}(\Omega - \vartheta) < \widetilde{SDI}(\Omega)\}$.

Example 1. A simple graph $\Omega = (\mathcal{V}, \mathcal{E})$ is given in Figure 1. Assume $k \ge 3$. $\widetilde{SDI}(\Omega) = 3$. $\widetilde{SDI}^{0}(\Omega) = \{\vartheta_i : 1 \le i \le k\} \cup \{\zeta_2\}$, $\widetilde{SDI}^{+}(\Omega) = \{\zeta_1\}$ and $\widetilde{SDI}^{-}(\Omega) = \{\zeta_3\}$.

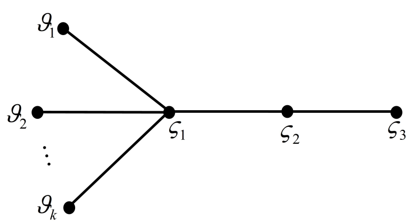

Figure 1. **Example of changing and unchanging of** $\widetilde{SDI}(\Omega)$

Theorem 1. *If ϑ is an isolate node of Ω, then $\vartheta \in \widetilde{SDI}^{-}(\Omega)$.*

Proof. Let ϑ be an isolate node, then $\vartheta \in I$, where I is a *sd*-set of Ω. Let I be any $\widetilde{SDI}$-set of Ω. Let $K = I - \{\vartheta\}$. Then K is a *sd*-set of $\Omega - \vartheta$. $\widetilde{SDI}(\Omega - \vartheta) \le |K| + m(\Omega - \vartheta) - K) = |I| - 1 + m(\Omega - I) = \widetilde{SDI}(\Omega) - 1$. Therefore, $\widetilde{SDI}(\Omega - \vartheta) < \widetilde{SDI}(\Omega)$. Hence $\vartheta \in \widetilde{SDI}^{-}(\Omega)$. □

Remark 2. The converse is not true. Consider $\Omega = C_4$ is shown in the Figure 2. $\widetilde{SDI}(\Omega) = 3$, $\widetilde{SDI}(\Omega - \vartheta_i) = 2$, $1 \le i \le 4$.

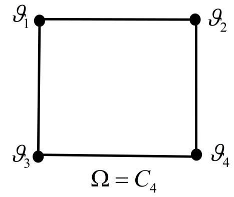

Figure 2. **The cycle graph C_4**

Theorem 2. *Let $\vartheta \in \mathcal{V}(\Omega)$. If there is no $\widetilde{SDI}$-set I within Ω that includes ϑ and for which the set $I - \{\vartheta\}$ is a sd-set of $\Omega - \vartheta$, it follows that $\vartheta \notin \widetilde{SDI}^{-}(\Omega)$.*

Proof. Assuming the hypothesis is valid, let K represent a $\widetilde{SDI}$-set of $\Omega - \vartheta$. It is evident that $K \cup \{\vartheta\}$ constitutes a *sd*-set of Ω that includes ϑ. Given that K serves as a *sd*-set for $\Omega - \vartheta$, it follows that $K \cup \{\vartheta\}$ cannot be classified as a $\widetilde{SDI}$-set of Ω. Consequently,

$$\begin{aligned}\widetilde{SDI}(\Omega) &< |K \cup \{\vartheta\}| + m(\Omega - (K \cup \{\vartheta\}))\\ &= |K| + 1 + m((\Omega - \vartheta) - K)\\ &= \widetilde{SDI}(\Omega - \vartheta) - 1.\end{aligned}$$

Therefore, $\widetilde{SDI}(\Omega - \vartheta) \geq \widetilde{SDI}(\Omega)$. We have $\vartheta \notin \widetilde{SDI}^{-}(\Omega)$. □

Theorem 3. *If ϑ is not a member of any $\widetilde{SDI}$-set within Ω, it follows that ϑ is an element of $\widetilde{SDI}^{0}(\Omega)$.*

Proof. Assume that ϑ is not a member of any $\widetilde{SDI}$-set associated with Ω. Let K represent a $\widetilde{SDI}$-set of $\Omega - \vartheta$. Then $K - \{\vartheta\}$ is a dominating set of Ω. According to the given hypothesis, the union $K \cup \{\vartheta\}$ is not a $\widetilde{SDI}$-set of Ω. Thus,

$$\begin{aligned}\widetilde{SDI}(\Omega) &< |K \cup \{\vartheta\}| + m(\Omega - (K \cup \{\vartheta\}))\\ &< |K| + 1 + m(\Omega - \vartheta) - K)\\ \widetilde{SDI}(\Omega) - 1 &< \widetilde{SDI}(\Omega - \vartheta)\\ \widetilde{SDI}(\Omega) &\leq \widetilde{SDI}(\Omega - \vartheta) \longrightarrow (1)\end{aligned}$$

Let I be a $\widetilde{SDI}$-set of Ω. Then $\vartheta \notin I$. Therefore, $I \subseteq \mathcal{V}(\Omega - \vartheta)$ and I is a *sd*-set of $\Omega - \vartheta$.

Case (i): Assume that $\Omega - I$ contains a unique component of maximum order K, and let ϑ be an element of K. In this case, the union of the set I with the singleton set $\{\vartheta\}$ forms a $\widetilde{SDI}$-set within Ω, which leads to a contradiction.

Case (ii) : Assume that $\Omega - I$ contains a minimum of two components of maximum order. In this case, $m((\Omega - \vartheta) - I) = m(\Omega - I)$.

$$\begin{aligned}\widetilde{SDI}(\Omega) &= |I| + m(\Omega - I)\\ &= |I| + m((\Omega - \vartheta) - (I - \vartheta))\\ &= |I| + m((\Omega - \vartheta) - I)\\ &\geq \widetilde{SDI}(\Omega - \vartheta).\end{aligned}$$

Therefore, $\widetilde{SDI}(\Omega) \geq \widetilde{SDI}(\Omega - \vartheta) \longrightarrow (2)$.
From (1) and (2), $\widetilde{SDI}(\Omega) = \widetilde{SDI}(\Omega - \vartheta)$. Hence $\vartheta \in \widetilde{SDI}^0(\Omega)$. $\square$

Theorem 4. *Assume $\vartheta \in \widetilde{SDI}^+(\Omega)$. Let I represent a sd-set with the condition that $\vartheta \notin I$. Under these circumstances, it follows that I cannot be classified as a $\widetilde{SDI}$-set of Ω.*

Proof. Assume I is a $\widetilde{SDI}$-set. Then $|I| + m(\Omega - I) = \widetilde{SDI}(\Omega)$. Therefore, $|I| + m((\Omega - \vartheta) - I) \leq |I| + m(\Omega - I) = \widetilde{SDI}(\Omega)$. Thus, $\widetilde{SDI}(\Omega - \vartheta) \leq \widetilde{SDI}(\Omega)$, a contradiction. $\square$

Corollary 1. *Let $\vartheta \in \widetilde{SDI}^+(\Omega)$. Then ϑ is a member of every $\widetilde{SDI}$-set of Ω.*

Proof. Let us consider a $\widetilde{SDI}$-set I with the condition that ϑ is not an element of I. Under these circumstances, it follows that I also constitutes a *sd*-set of Ω, while still excluding ϑ. According to Theorem 4, this implies that I cannot be classified as a $\widetilde{SDI}$-set of Ω, leading to a contradiction. Therefore, we conclude that ϑ must be an element of every $\widetilde{SDI}$-set of Ω. $\square$

Remark 3. Assume that $I \subset \mathcal{V}(\Omega)$ and $\vartheta \in I$. We have $m((\Omega - \vartheta) - I) = m(\Omega - I)$.

Theorem 5. $|\widetilde{SDI}^+(\Omega)| \leq \underleftrightarrow{SDI}(\Omega)$, *where Ω is any graph.*

Proof. Let I be any $\widetilde{SDI}$-set of Ω. Let $\vartheta \in \mathcal{V} - I$. Then

$$\begin{aligned}\widetilde{SDI}(\Omega - \vartheta) &\leq |I| + m((\Omega - \vartheta) - I) \quad (\text{since } I \text{ is a } sd-\text{set of } \Omega - \vartheta)\\ &\leq |I| + m(\Omega - I) = \widetilde{SDI}(\Omega).\end{aligned}$$

Therefore, $\vartheta \notin \widetilde{SDI}^+(\Omega)$.
Thus, $|\widetilde{SDI}(\Omega)| \leq |I|$, for any $\widetilde{SDI}$-set I of Ω.
Hence $|\widetilde{SDI}^+(\Omega)| \leq \underleftrightarrow{SDI}(\Omega)$. $\square$

Definition 4. Let $K \subseteq \mathcal{V}(\Omega)$. Then K is a Geodetic Domination Integrity set ($\widetilde{GDI}$-set) if $\widetilde{GDI}(\Omega) = |K| + m(\Omega - K)$.

Definition 5. $\widetilde{GDI}$-set number is defined as the minimum cardinality of a minimal $\widetilde{GDI}$-set and is denoted by $\underleftrightarrow{GDI}(\Omega)$. The upper $\widetilde{GDI}$-set number is the maximum cardinality of a minimal $\widetilde{GDI}$-set and is denoted by $\overleftrightarrow{GDI}(\Omega)$.

Remark 4.

- $\gamma(\Omega) \leq \gamma_g(\Omega) \leq \underleftrightarrow{GDI}(\Omega) \leq \overleftrightarrow{GDI}(\Omega)$.
- $\kappa(\Omega) \leq \underleftrightarrow{GDI}(\Omega) \leq \overleftrightarrow{GDI}(\Omega)$.
- For $K_{p,q}$, $\ p \geq q \geq 2$, $\underleftrightarrow{GDI}(K_{p,q}) = \overleftrightarrow{GDI}(K_{p,q}) = q$.

Definition 6. The node set $\mathcal{V}(\Omega)$ of Ω can be divided into the three sets.
$\widetilde{GDI}^{+}(\Omega) = \{\vartheta \in \mathcal{V}(\Omega) : \widetilde{GDI}(\Omega - \vartheta) > \widetilde{GDI}(\Omega)\}$
$\widetilde{GDI}^{0}(\Omega) = \{\vartheta \in \mathcal{V}(\Omega) : \widetilde{GDI}(\Omega - \vartheta) = \widetilde{GDI}(\Omega)\}$
$\widetilde{GDI}^{-}(\Omega) = \{\vartheta \in \mathcal{V}(\Omega) : \widetilde{GDI}(\Omega - \vartheta) < \widetilde{GDI}(\Omega)\}$.

Example 2. Consider a graph $\Omega = (\mathcal{V}, \mathcal{E})$ is shown in Figure 3. $\widetilde{GDI}(\Omega) = 6$. $\widetilde{GDI}^{0}(\Omega) = \{\vartheta_i, \zeta_i : 1 \leq i \leq 3\}$, $\widetilde{GDI}^{+}(\Omega) = \{r\}$ and $\widetilde{GDI}^{-}(\Omega) = \{s\}$.

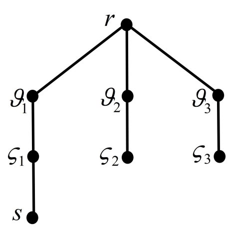

Figure 3. **Illustration of changes in** $\widetilde{GDI}(\Omega)$

Theorem 6. *Any $\widetilde{GDI} - set$ of a connected graph Ω with $O(\Omega) = p$ contains the extreme nodes of Ω.*

Proof. Let us consider ϑ as an extreme node and K as a $\widetilde{GDI}$-set of Ω. If it is the case that ϑ does not belong to K, then K is not a geodetic set of Ω. As a result, K cannot be recognized as a $\widetilde{GDI}$-set of Ω. This situation presents a contradiction. Hence, it can be inferred that every extreme node of Ω must be included in every $\widetilde{GDI}$-set of Ω. $\square$

Corollary 2. *Every terminal node of a connected graph Ω is a member of every $\widetilde{GDI}$-set of Ω.*

Theorem 7. *Let $\Omega = K_p$ is complete graph with $O(\Omega) = p$ then $\underleftrightarrow{GDI}(\Omega) = \overleftrightarrow{GDI}(\Omega) = p$.*

Proof. In a complete graph K_p, each node is classified as an extreme node, which means that the set of nodes $\mathcal{V}(K_p)$ constitutes the unique $\widetilde{GDI}$-set of Ω. Consequently, we have $\underleftrightarrow{GDI}(\Omega) = \overleftrightarrow{GDI}(\Omega) = p$. □

Theorem 8. *In a connected graph Ω with k extreme nodes, any $\widetilde{GDI}$-set must contain at least k elements.*

Proof. Consider a connected graph Ω that contains k extreme nodes. According to Theorem 6, every extreme node in Ω is included in every $\widetilde{GDI}$-set of the graph. Consequently, a $\widetilde{GDI}$-set must contain at least k elements. This condition is satisfied with equality in the case of a complete graph K_p. □

Theorem 9. *For a connected graph Ω, $\underleftrightarrow{GDI}(\Omega) = \overleftrightarrow{GDI}(\Omega) = p$ if and only if $\Omega = K_p$*

Proof. Let Ω represent any connected graph with order p, where it holds that $\underleftrightarrow{GDI}(\Omega) = \overleftrightarrow{GDI}(\Omega) = p$. Consider a $\widetilde{GDI}$-set I consisting of p elements. If Ω is not equal to K_p, it follows that there exist two vertices, ϑ_1 and ϑ_2, such that the distance $d(\vartheta_1, \vartheta_2) \geq 2$. Given the connectivity of Ω, there exists a geodesic path M connecting ϑ_1 and ϑ_2. Consequently, there exists a vertex η along this geodesic path such that $\eta \neq \vartheta_1, \vartheta_2$ and is adjacent to either ϑ_1 or ϑ_2. For the sake of argument, let us assume that η is adjacent to ϑ_1. Define $K = \mathcal{V}(\Omega) - \{\eta\}$. It follows that $\eta \in I[\vartheta_1, \vartheta_2] \subset I[K]$ and $\eta \in N[\vartheta_1] \subset N[K]$. Thus, K serves as a gd-set and also a $\widetilde{GDI}$-set. Since $|K| = p - 1 < |I|$, this leads to a contradiction. Conversely, if we let $\Omega = K_p$, then according to Theorem 7, we can conclude that $\underleftrightarrow{GDI}(\Omega) = \overleftrightarrow{GDI}(\Omega) = p$. □

Theorem 10. *For any connected graph Ω, the geodetic number $g(\Omega) = p$ if and only if $\Omega = K_p$[6]*

Theorem 11. *For any connected graph Ω with order p, $\underleftrightarrow{GDI}(\Omega) = \overleftrightarrow{GDI}(\Omega) = p$ if and only if $g(\Omega) = p$*

Proof. Consider a connected graph Ω with order p. If it holds that $\underleftrightarrow{GDI}(\Omega) = \overleftrightarrow{GDI}(\Omega) = p$, then according to Theorem 9, it follows that Ω must be isomorphic to the complete graph K_p. In the case of a complete graph, the geodetic number is given by $g(\Omega) = p$. Conversely, if we have $g(\Omega) = p$, then by Theorem 10, it can be concluded that Ω is indeed K_p, which leads to the result that $\underleftrightarrow{GDI}(\Omega) = \overleftrightarrow{GDI}(\Omega) = p$. □

Theorem 12. *Let Ω be a connected graph with a diameter of at most 3. Then no cut node of Ω is included in any minimum gd-set of the graph Ω [9].*

Theorem 13. *Let $\Omega = T$ be a tree with $p \geq 3$ nodes. Then $\underleftrightarrow{GDI}(\Omega) = p - 1$ if and only if Ω is a star.*

Proof. Let $\Omega = T$ be a tree with $p \geq 3$ and $\underleftrightarrow{GDI}(\Omega) = p - 1$. If Ω is not a star, it follows that $\text{diam}(\Omega) \geq 3$. As Ω is a tree, it must possess at least two terminal nodes, denoted as ϑ_1 and ϑ_2.
Case (i): Assume $\text{diam}(\Omega) = 3$. In this scenario, it is evident that Ω contains precisely two cut nodes. According to Theorem 12, no cut node of Ω can be part of any geodetic dominating set of Ω. Consequently, this implies that $\underleftrightarrow{GDI}(\Omega) \leq p - 2$, leading to a contradiction.
Case (ii): In the case where the diameter of $\Omega = T$ exceeds 3, it follows that there exist two nodes ϑ_1 and ϑ_2 within the vertex set $\mathcal{V}(\Omega)$ such that the distance $d(\vartheta_1, \vartheta_2)$ is greater than 3. The geodesic connecting ϑ_1 and ϑ_2 can be represented as $\vartheta_1, \eta_1, \eta_2, \ldots, \vartheta_2$. Consequently, this geodesic path contains at least three internal nodes. Given that η_1 and η_2 belong to the interval $I[\vartheta_1, \eta_3]$, with η_1 being a neighbor of ϑ_1 and η_2 being adjacent to η_3, it follows that the set $\mathcal{V}(\Omega) - \{\eta_1, \eta_2\}$ constitutes a *gd*-set for Ω. If this set is classified as a $\widetilde{GDI}$-set, it leads to the conclusion that $\underleftrightarrow{GDI}(\Omega) \leq p - 2$, which presents a contradiction.
Thus, in both cases, it can be concluded that Ω must be a star. Conversely, if Ω is indeed a star, specifically $K_{1,p-1}$, then it holds that $\underleftrightarrow{GDI}(\Omega) = p - 1$. □

Theorem 14. *Consider a connected graph Ω that includes a cut node denoted as ϑ. Let K represent a $\widetilde{GDI}$-set associated with Ω. It follows that each component of the graph $\Omega - \vartheta$ must contain at least one element from the set K.*

Proof. Let ϑ represent a cut node within a connected graph Ω, and let K denote a $\widetilde{GDI}$-set associated with Ω. It follows that K includes its extreme nodes from Ω. Conversely, assume there exists a component B of $\Omega - \vartheta$ that does not contain any nodes from K. Given that K encompasses its extreme nodes, it follows that B does not contain any extreme nodes of Ω. Let us consider a node $\eta \in \mathcal{V}(B)$. Since K is a $\widetilde{GDI}$-set, there must exist a pair of nodes $x, y \in K$ such that $\eta \in I[x, y]$ and $\eta \in N[K]$. Denote the $x - y$ geodesic in Ω as $P : x = \eta_0, \eta_1, \eta_2, \ldots, \eta_n = y$. Because ϑ is a cut node of Ω, both the $x - \eta$ subpath and the $\eta - y$ subpath of P must include ϑ, which implies that P cannot be a valid path. This leads to a contradiction. Therefore, it can be concluded that every component of $\Omega - \vartheta$ must contain at least one element from K. □

5. $\widetilde{SDI}$-excellent and $\widetilde{GDI}$-excellent graphs

In this section we define and studied $\widetilde{SDI}$-excellent ($\widetilde{SDI}$-Ext) and $\widetilde{GDI}$-excellent ($\widetilde{GDI}$-Ext) graphs with examples and theorems derived from $\widetilde{SDI}$-excellent ($\widetilde{SDI}$-Ext) and $\widetilde{GDI}$-excellent ($\widetilde{GDI}$-Ext) graphs.

Definition 7. Let $\Omega = (\mathcal{V}, \mathcal{E})$ be a graph. A node $\vartheta \in \mathcal{V}(\Omega)$ is a $\widetilde{SDI}$-good node if it a member of some $\widetilde{SDI}$-set of Ω and ϑ is $\widetilde{SDI}$-bad if ϑ is not a member of any $\widetilde{SDI}$-set of Ω.

Example 3. Consider the Figure 4. In Ω_1 all the nodes are $\widetilde{SDI}$-good nodes, but in Ω_2 the nodes $\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \zeta_4, \zeta_5$ are $\widetilde{SDI}$-good nodes and $\zeta_1, \zeta_2, \zeta_3, \zeta_6, \zeta_7, \zeta_8$ are $\widetilde{SDI}$-bad nodes.

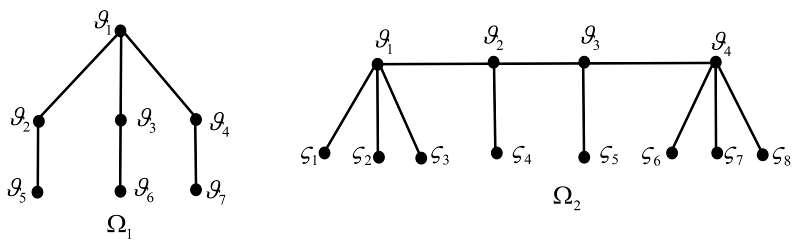

Figure 4. $\widetilde{SDI}$**-good nodes,** $\widetilde{SDI}$**-bad nodes**

Definition 8. A graph Ω is said to be a $\widetilde{SDI}$-Ext graph if every node of Ω is $\widetilde{SDI}$-good nodes.

Example 4. Consider the following Figure 5. Here Ω_1 is γ_s-excellent and $\widetilde{SDI}$-Ext graph. Ω_2 is not γ_s-excellent and not $\widetilde{SDI}$-Ext. Ω_3 is not γ_s-excellent but $\widetilde{SDI}$-Ext graph.

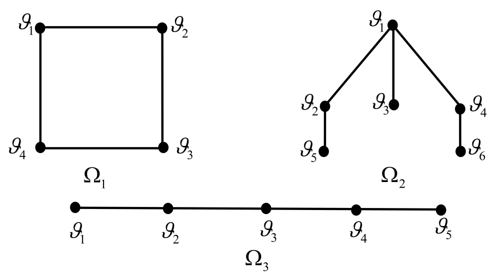

Figure 5. **Illustration of** $\widetilde{SDI}$**-Ext graphs**

Theorem 15. *Every graph Ω can be considered an induced subgraph of a $\widetilde{SDI}$-Ext graph.*

Proof. Consider the node set of Ω denoted as $\mathcal{V}(\Omega) = \{\vartheta_1, \vartheta_2, \vartheta_3, ..., \vartheta_p\}$. We introduce an additional set of nodes $\{\zeta_1, \zeta_2, \zeta_3, ..., \zeta_p\}$ to $\mathcal{V}(\Omega)$. Each node ζ_i is connected to ϑ_j and ζ_j for all $j \neq i$, where $1 \leq j \leq p$. Let the resulting graph be represented as K. Consequently, the $\widetilde{SDI}$-sets are defined as $\{\vartheta_i, \zeta_i\}$ for each i, where $1 \leq j \leq p$. It follows that K qualifies as an $\widetilde{SDI}$-Ext graph. Thus, Ω is established as an induced subgraph of K. □

Proposition 1. *Every graph with order p can be considered an induced subgraph of a $\widetilde{SDI}$-Ext graph that has an order of $2p$.*

Theorem 16. *The cycle $\Omega = C_p$ is $\widetilde{SDI}$-Ext graph.*

Proof. Every γ_s-set qualifies as a $\widetilde{SDI}$-set of $\Omega = C_p$. Given that C_p is γ_s-excellent for all values of p, it follows that C_p is also classified as a $\widetilde{SDI}$-excellent graph. □

Theorem 17. *In the context of any $\widetilde{SDI}$-Ext graph Ω, it can be stated that each pendant node belongs to at least one $\widetilde{SDI}$-set of Ω, while simultaneously, no pendant node is included in every $\widetilde{SDI}$-set of Ω.*

Proof. Given that Ω is classified as a $\widetilde{SDI}$-Ext graph, it follows that every node, including each pendant node of Ω, is included in some $\widetilde{SDI}$-set associated with Ω. Let us denote a pendant node of Ω as ϑ, and assume that $\vartheta \in K$, where K represents a $\widetilde{SDI}$-set of Ω. Define ζ as the support of ϑ. If it holds that $\zeta \in K$, then the equation $|K - \{\vartheta\}| + m(\Omega - (K - \{\vartheta\})) = |K| - 1 + m(\Omega - K) < \widetilde{SDI}(\Omega)$ would yield a contradiction. Consequently, we conclude that $\zeta \notin K$. We can then define a new set $K_1 = K \cup \{\zeta\} - \{\vartheta\}$. It is evident that K_1 constitutes a *sd*-set for Ω, and it follows that $m(\Omega - K_1) = m(\Omega - K)$. Thus, K_1 qualifies as an $\widetilde{SDI}$-set of Ω that does not include ϑ. □

Theorem 18. *Consider a tree K with an order of at least 3, and let ϑ represent a pendant node of K that is situated within a $\widetilde{SDI}$-set of K. Define ζ as the support associated with ϑ. If the intersection of K and the neighborhood of ζ satisfies the condition $|K \cap N(\zeta)| \geq 2$, it follows that there exists a $\widetilde{SDI}$-set of K that is not independent.*

Proof. Assume that the cardinality of the vertex set $|\mathcal{V}(K)|$ is at least 3. Let ϑ represent a pendant node in the graph K, located within a $\widetilde{SDI}$-set I of K. If the set I is not independent, we can conclude our analysis at this point. On the other hand, if I is independent, our assumptions imply that the intersection $|I \cap N(\zeta)|$ is at least 2, where ζ denotes the support of the node ϑ (notably, since I is independent, it follows that $\zeta \notin I$). We can choose an element t from the intersection $I \cap N(\zeta)$, ensuring that $t \neq \vartheta$. We then define a new set $I_1 = (I - \{\vartheta\}) \cup \{\zeta\}$, which qualifies as a $\widetilde{SDI}$-set of K. This is valid because removing ϑ and adding ζ ensures that ϑ remains a singleton in the complement. It is clear that I_1 is not independent. □

Theorem 19. *If K is an $\widetilde{SDI}$-Ext tree. Then there exists an $\widetilde{SDI}$-set I such that I is not independent.*

Proof. Without loss of generality, we can assert that $|\mathcal{V}(K)| \geq 3$. Let ϑ represent a pendant node of the graph K, while ζ denotes its support. Given that $|\mathcal{V}(K)| \geq 3$, it follows that there exists a node t that is adjacent to ζ. Since K is classified as $\widetilde{SDI}$-Ext, there exists a $\widetilde{SDI}$-set I within K that includes the node t. It is important to note that exactly one of the nodes ϑ or ζ is a member of the set I. If ζ is included in I, it is evident that I cannot be independent. Conversely, if ϑ is part of I, then by applying Theorem 18, we can conclude that there exists a $\widetilde{SDI}$-set of K that is not independent. □

Theorem 20. *Consider a tree denoted as K and an $\widetilde{SDI}$-set represented by I within this tree. If we identify ϑ as a pendant node that has a support denoted by ζ, it follows that the pair consisting of ϑ and ζ cannot simultaneously be elements of the set I.*

Proof. Suppose ϑ and ζ are elements in I, then $|I - \{\vartheta\}| + m(\Omega - (I - \{\vartheta\})) < |I| + m(\Omega - I)$, a contradiction. □

Theorem 21. *Let Ω represent a graph that is classified as $\widetilde{SDI}$-non excellent, containing a single $\widetilde{SDI}$-bad node, denoted as ζ. If there exists a node ϑ within Ω such that ϑ is adjacent to ζ and the degree of ϑ exceeds that of ζ, then it follows that there exists a graph K for which Ω serves as an induced subgraph of K, and the relationship $\widetilde{SDI}(K) = \widetilde{SDI}(\Omega) + 1$ holds true.*

Proof. Introduce a new node w into the graph Ω and connect it to the node ζ. Let the resulting graph be denoted as K. It is evident that Ω serves as an induced subgraph of K, and the relationship $\widetilde{SDI}(K) = \widetilde{SDI}(\Omega) + 1$ holds. Consider any $\widetilde{SDI}$-set I within Ω. The union $S \cup \{\zeta\}$ constitutes a $\widetilde{SDI}$-set for the graph K. Consequently, every node in Ω is classified as $\widetilde{SDI}$-good within the context of K. Let I_1 represent a $\widetilde{SDI}$-set of Ω that includes the node ϑ. The set $I_1 \cup \{w\}$ then qualifies as a $\widetilde{SDI}$-set for K, indicating that the node w is also a good node in K. Thus, it can be concluded that K is classified as a $\widetilde{SDI}$-Ext graph. □

Definition 9. Consider a connected graph denoted as $\Omega = (\mathcal{V}, \mathcal{E})$. A node $\vartheta \in \mathcal{V}(\Omega)$ is classified as a $\widetilde{GDI}$-good node if it is a member of at least one $\widetilde{GDI}$-set associated with Ω. Conversely, ϑ is termed a $\widetilde{GDI}$-bad node if it is not included in any $\widetilde{GDI}$-set of Ω.

Example 5. Figure 6 shows the graph $K_{2,3}$. In $K_{2,3}$ the nodes $\{\vartheta_1, \vartheta_2\}$ are $\widetilde{GDI}$-good nodes, and $\{\zeta_1, \zeta_2, \zeta_3\}$ are $\widetilde{GDI}$-bad nodes.

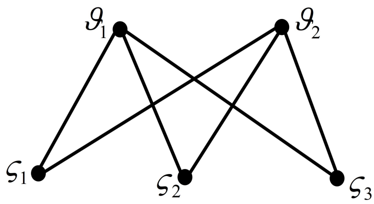

Figure 6. $\widetilde{GDI}$**-good nodes,** $\widetilde{GDI}$**-bad nodes**

Definition 10. A graph Ω is $\widetilde{GDI}$-Ext graph if every node of Ω is $\widetilde{GDI}$-good nodes.

Example 6. Consider the graphs in the Figure 7. Here Ω_1 is $\widetilde{SDI}$-Ext and $\widetilde{GDI}$-Ext graph. Ω_2 is not $\widetilde{SDI}$-Ext and $\widetilde{GDI}$-Ext. Ω_3 is not $\widetilde{SDI}$-Ext and not $\widetilde{GDI}$-Ext graph.

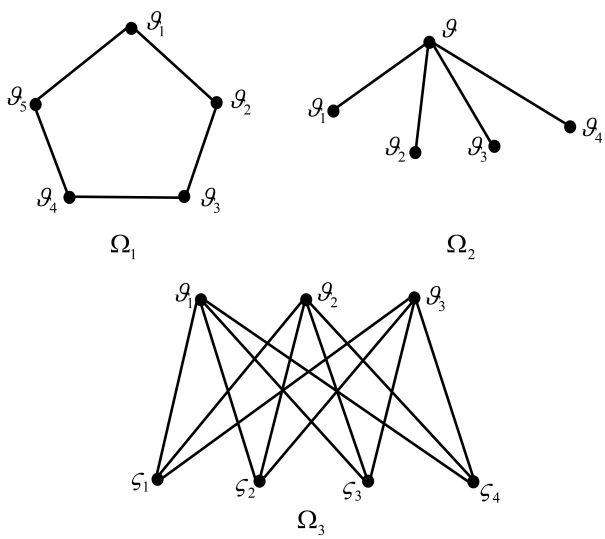

Figure 7. **Illustration of** $\widetilde{SDI}$**-Ext and** $\widetilde{GDI}$**-Ext graphs**

Theorem 22. *The cycle C_p is $\widetilde{GDI}$-Ext graph.*

Proof. In a cycle graph C_p, every γ_g-set is a $\widetilde{GDI}$-set. Hence C_p is $\widetilde{GDI}$-Ext. □

Theorem 23. *K_p is $\widetilde{GDI}$-Ext.*

Proof. Since $\gamma_g(K_p) = p$ and the entire set is the only $\widetilde{GDI}$-set. Hence K_p is $\widetilde{GDI}$-Ext graph. $\square$

Theorem 24. *The complete bipartite graph $K_{p,p}$ is $\widetilde{GDI}$-Ext.*

Proof. Given that $\widetilde{GDI}(k_{p,p}) = p + 1$, it follows that each partite set of $\mathcal{V}(K_{p,p})$ qualifies as a $\widetilde{GDI}$-set for the graph $K_{p,p}$. $\square$

Theorem 25. *If P_p is the path with $O(P_p) = p$. Then $\widetilde{GDI}(\overline{P_p}) = p - 1$, $p > 5$ where $\overline{P_p}$ is the complement of P_p.*

Proof. Let $p > 5$. We denote the endpoints of the path P_p as ϑ and ζ. In the complement graph $\overline{P_p}$, the nodes ϑ and ζ are connected to $p - 2$ other nodes, while the other nodes are connected to $p - 3$ nodes. It is clear that the diameter of $\overline{P_p}$ is 2. Choose a node t that is adjacent to ζ but not to ϑ. As a result, there are $p - 3$ nodes located on the geodesic between ϑ and t, which are dominated by both ϑ and t in $\overline{P_p}$. This indicates the presence of a node $x \in \mathcal{V}$ such that $x \notin I[\vartheta, t]$. Therefore, the pair $\{\vartheta, t\}$ does not form a geodetic dominating set for $\overline{P_p}$. Since it is impossible to create a *gd*-set with only two nodes, we conclude that $K = \{\vartheta, t, x\}$ is a minimal *gd*-set of $\overline{P_p}$. Thus, we establish that $\gamma_g(\overline{P_p}) = 3$. Furthermore, we find that $m(\overline{P_p} - K) = p - 4$. Consequently, the value of $\widetilde{GDI}(\overline{P_p})$ is computed as $|K| + m(\overline{P_p} - K) = 3 + (p - 4) = p - 1$. This concludes the proof of the theorem. $\square$

Theorem 26. *If C_p is the Cycle graph of order p. Then $\widetilde{GDI}(\overline{C_p}) = p - 1$, $p > 5$ where $\overline{C_p}$ is the complement of C_p.*

Proof. For any integer p greater than 5, it is demonstrated that C_p constitutes a 2-regular graph, while its complement $\overline{C_p}$ is characterized as a $p-3$ regular graph with a diameter of 2. It is clear that in $\overline{C_p}$, each node is connected to $p-3$ other nodes, and any two nodes, denoted as ϑ and ζ, that are not directly connected share $p-4$ common neighbors. As a result, there are $p - 4$ nodes located along the geodesic connecting ϑ and ζ, which are dominated by both ϑ and ζ. The pair $\{\vartheta, \zeta\}$ does not form a geodetic dominating set. Next, a node t is chosen such that it is adjacent to ζ but not to ϑ. The remaining nodes are then aligned along the geodesic from ϑ to t. Consequently, the set $K = \{\vartheta, \zeta, t\}$ is identified as the minimal geodetic dominating set of $\overline{C_p}$, yielding $\gamma_g(\overline{C_p}) = 3$. It follows that $m(\overline{C_p} - K) = p - 4$. The geodetic domination integrity of $\overline{C_p}$ is computed as $\widetilde{GDI}(\overline{C_p}) = |K| + m(\overline{C_p} - K) = 3 + (p-4) = p-1$. This concludes the proof of the theorem. $\square$

Remark 5. The decision problem $\widetilde{GDI}(\Omega)$ is formulated as follows:
Input: Given a connected graph $\Omega = (\mathcal{V}, \mathcal{E})$ with an integer k
Question: Is it true that $\widetilde{GDI}(\Omega) \leq k$?

For a connected graph $\Omega = (\mathcal{V}, \mathcal{E})$ and any specified subset $K \subseteq \mathcal{V}(\Omega)$, it is possible to verify in polynomial time whether K constitutes a *gd*-set. Additionally, there exists a polynomial time algorithm capable of computing $m(\Omega - K)$ for any subset K of $\mathcal{V}(\Omega)$. Consequently, the decision problem $\widetilde{GDI}(\Omega)$ is classified within the complexity class NP.

Theorem 27. *Let Ω be a connected graph with $O(\Omega) = p$. Then $\gamma_g(M(\Omega)) = p$, where $M(\Omega)$ is the middle graph of Ω.[9]*

Theorem 28. *$\widetilde{GDI}(\Omega)$ is NP-complete.*

Proof. Consider a connected graph Ω characterized by an order of p and a size of q. Denote the vertex set of Ω as $\mathcal{V}(\Omega) = \{\vartheta_1, \vartheta_2, \ldots, \vartheta_p\}$. The middle graph, denoted as $\Omega' = M(\Omega)$, is defined to have an order of $p + q$ and a size of $2q + |E(L(\Omega))|$. This graph is constructed by subdividing each edge of Ω exactly once and linking all adjacent edges within Ω in the resulting graph $M(\Omega)$. Notably, the line graph $L(\Omega)$ is inherently included as an induced subgraph of $M(\Omega)$. According to Theorem 27, the extreme vertices of Ω' constitute a minimum *gd*-set, leading to the conclusion that $\gamma_g(\Omega') = p$. Let H represent the graph formed by the exclusion of all extreme nodes from Ω'. Consequently, we find that $\widetilde{GDI}(\Omega) = \gamma_g(\Omega') + I(H) = p + I(H)$. The determination of $I(H)$ is known to be NP-complete[11], which implies that $\widetilde{GDI}(\Omega)$ is also NP-complete. □

6. Conclusion

A network used for transferring information between different nodes (such as computers, servers, or devices). It could be a physical or a virtual network. A crucial characteristic of any robust network is its ability to continue functioning effectively even if some parts of the network fail (nodes or links become inactive). For instance, in a communication system, there might be temporary failures or disconnections in some devices (nodes) or transmission paths (links), but the network should still be able to maintain its functionality. The idea is to ensure that the failure of one or more parts does not cause a catastrophic breakdown in the whole system. This is critical in applications like the internet, communication infrastructure, or transportation networks, where reliability and uptime are essential. Vulnerability metrics are used to quantify how vulnerable or robust a network is to failures. Vulnerability metrics help in assessing how well a network can withstand disruptions or faults, and how it performs when parts of the system are no longer operational.
A strong domination set is a more restrictive form where each node in the set must be strongly connected to every other node in the network. The geodetic domination integrity set takes into account how the removal of nodes affects the geodetic distance between the remaining nodes. Essentially, these sets help analyze the system's

integrity when parts of the network fail. In this study, we examined strong domination integrity sets, along with the changes and stability in their strong domination integrity values. Additionally, we investigated geodetic domination integrity sets and the variations or stability in their geodetic domination integrity values. Moreover, we analyzed the effects of node removal on both strong and geodetic domination integrity. We proposed new graph types, $\widetilde{SDI}$-Ext and $\widetilde{GDI}$-Ext, and developed theorems to enhance the modeling and assessment of the network's robustness in the event of partial failures.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

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