Communications in Combinatorics and Optimization
Vol. xx, No. x (xxxx), pp. 1-16
https://doi.org/10.22049/cco.2025.29137.1857

CCO
Commun. Comb. Optim.

Research Article

Reciprocal distance Laplacian spectral radius of graphs

Hilal Ahmad Ganie[1,*], Bilal Ahmad Rather[2], Yilun Shang[3]

[1]Department of School Education JK Govt. Kashmir, India
*hilahmad1119kt@gmail.com

[2]Department of Mathematics, Smarkand International University of Technology, Samarkand 140100, Uzbekistan
bilalahmadrr@gmail.com

[3]Department of Computer and Information Sciences, Northumbria University, Newcastle NE1 8ST, UK
yilun.shang@northumbria.ac.uk

Received: 31 October 2023; Accepted: 14 March 2025
Published Online: 26 March 2025

Abstract: For a simple connected graph G with $V(G) = \{v_1, v_2, \ldots, v_n\}$, let d_{ij} be the distance between any pair of distinct vertices v_j and v_j. The reciprocal distance Laplacian matrix $RD^L(G)$ of G is defined by $RD^L(G) = RTr(G) - RD(G)$, where $RTr(G)$ is the diagonal matrix having i-the entry $RTr(v_i) = \sum_{j\in V(G)} \frac{1}{d_{ij}}$ and $RD(G)$ is the reciprocal distance matrix (also called Harary matrix) having (i,j)-th entry $\frac{1}{d_{ij}}$ if $i \neq j$ and zero, otherwise. The set of all $RD^L(G)$-eigenvalues $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_{n-1} > \delta_n$ is known as the RD^L-spectrum (also called reciprocal distance Laplacian spectrum) of G and δ_1 is called the RD^L-spectral radius (also called reciprocal distance Laplacian spectral radius) of G. We explore various interesting properties of RD^L-eigenvalues along with the bounds for RD^L-spectral radius. We characterize the corresponding extremal graphs attaining these bounds.

Keywords: distance matrix, distance Laplacian matrix, reciprocal distance Laplacian matrix, largest eigenvalue.

AMS Subject classification: 05C50, 05C12, 15A18

1. Introduction

All our graphs in this article are connected, simple and undirected graphs. A graph is denoted by $G = G(V(G), E(G)$, where $V(G) = \{v_1, v_2, \ldots, v_n\}$ is the vertex set $E(G)$

* *Corresponding Author*

is the edge set. The cardinality of $V(G)$ is the *order* n and the cardinality of $E(G)$ is the *size* m of G. The complement of G is denoted by $\overline{G}$ and by K_n, we denote the complete graph, by $K_{1,n-1}$, we denote the star graph. For other undefined notations and definitions, see [5, 7].

The *distance* $d(v_i, v_j)$ (shortly d_{ij}) between the two distinct vertices v_i and v_j in a connected graph G, is the length of the smallest path connecting them. The *distance matrix* $D(G)$ indexed by order n of G and is defined as $D(G) = (d_{ij})$. A nice survey of the distance matrix can be seen in [3, 9]. The *transmission* (or transmission degree) $Tr(v_i)$ of a vertex v_i is defined to be the sum of the distances from v_i to all other vertices of G, that is, $Tr(v_i) = \sum\limits_{v_j \in V(G)} d_{ij}$. Let $Tr(G) = diag(Tr(v_1), Tr(v_2) \dots, Tr(v_n))$ be the diagonal matrix of vertex transmission degrees of G. Aouchiche and Hansen [2] defined the distance Laplacian matrix $D^L(G) = Tr(G) - D(G)$. It immediately follows that $D^L(G)$ is a real symmetric and positive semi-definite matrix. Besides, each row sum of $D^L(G)$ is zero, so 0 must be the smallest distance Laplacian eigenvalue of $D^L(G)$.
The reciprocal distance matrix $RD(G)$ (or the Harary matrix) is an $n \times n$ matrix whose (i, j)-th entry is $\frac{1}{d_{ij}}$, if $v_i \neq v_j$ and 0 otherwise. The *reciprocal transmission degree* $RTr(v_i)$ (or RTr_i) of v_i is defined by $RTr(v_i) = \sum\limits_{v_j \in V(G)} \frac{1}{d_{ij}}$ or equivalently the sum of the entries of i-th of the matrix $RD(G)$. The *Harary index* $H(G)$ of G is the sum of reciprocal distances between all unordered pairs of vertices. Clearly,

$$2H(G) = \sum_{v \in V(G)} RTr(v) = \sum_{v_i, v_j \in V(G)} \frac{1}{d_{ij}} = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{d_{ij}}.$$

The relation between the Harary matrix, the Harary index and the Harary energy can be seen in [6] and the references therein.
Let $RTr(G) = diag\big(RTr(v_1), RTr(v_2), \dots, RTr(v_n)\big)$ be the diagonal matrix of reciprocal transmission degrees of G. Bapat and Panda [4] defined the *reciprocal distance Laplacian* matrix as $RD^L(G) = RTr(G) - RD(G)$. Since each row sum of $RD^L(G)$ is zero, it follows that 0 is its eigenvalue and $(1, 1, \dots, 1)$ is its corresponding eigenvector. The reciprocal distance Laplacian matrix is a real symmetric positive semi-definite matrix, so its eigenvalues can be indexed from the largest to the smallest in the following manner

$$\delta_1 \geq \delta_2 \geq \cdots \geq \delta_{n-1} > \delta_n = 0.$$

The set of all eigenvalues (including algebraic multiplicities) of $RD^L(G)$ is known as the reciprocal distance Laplacian spectrum (or RD^L spectrum) of G, the largest RD^L eigenvalue δ_1 is known as the *reciprocal distance Laplacian spectral radius* of G. We note that for real symmetric positive semi-definite matrix M, the largest singular of M (known as the spectral norm) is precisely the spectral radius of M. Thus, δ_1 is also known as the RD^L spectral norm of G. For some recent work on the spectral properties of $RD^L(G)$, see [1, 4, 8, 10, 12–14].

Any column vector $X = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ can be regarded as function defined on $V(G)$ which associates every v_i to x_i, that is $X(v_i) = x_i$ for all $i = 1, 2, \ldots, n$. Also, it is easy to see that

$$X^T RD^L(G) X = \sum_{i,j,i \neq j} \frac{1}{d(v_i, v_j)} (x_i - x_j)^2,$$

and δ is an eigenvalue of $RD^L(G)$ with its associated eigenvector X if and only if $X \neq 0$ and for every $v_i \in V(G)$, we have

$$\delta X(v_i) = \sum_{v_j \in V(G)} \frac{1}{d(v_i, v_j)} (X(v_i) - X(v_j)), \tag{1.1}$$

or equivalently

$$\delta X(v_i) - RTr(v_i) = - \sum_{v_j \in V(G)} \frac{1}{d(v_i, v_j)} X(v_j), \tag{1.2}$$

Equations (1.1) and (1.2) are known as (δ, X)-eigenequations of G.
In the rest of the paper, we explore some interesting properties of the reciprocal distance eigenvalues of G. We obtain some bounds for RD^L-spectral radius and characterize the extremal graphs attaining these bounds.

2. Bounds for the reciprocal distance Laplacian spectral radius

We begin the section with the following result, which is helpful in finding some RD^L-eigenvalues of G, when G has some special structure.

Theorem 1. *[8] Let G be a connected graph with the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and let $S = \{v_1, v_2, \ldots, v_p\}$ be any subset of G such that $N(v_i) = N(v_j)$ for all $i, j \in \{1, 2, \ldots, p\}$ and $\delta = RTr(v_i) = RTr(v_j)$ for all $i, j \in \{1, 2, \ldots, p\}$. Then the following holds.*

(i) *If S is independent, then $\delta + \frac{1}{2}$ is the RD^L-eigenvalue of G with multiplicity at least $p - 1$.*

(ii) *If S is a clique, then $\delta + 1$ is an eigenvalue of the $RD^L(G)$ matrix with multiplicity at least $p - 1$.*

For the connected graphs of diameter 2, the following interesting result is mentioned in [4], which gives that the reciprocal distance Laplacian eigenvalues of a graph of diameter 2 can be found from the corresponding Laplacian eigenvalues of the graph. For the sake of completeness, we will provide a proof of this result here.

Theorem 2. *Let G be a connected graph of order $n \geq 3$ with diameter 2 having reciprocal distance Laplacian and Laplacian eigenvalues δ_i and μ_i, $i = 1, 2, \ldots, n$, respectively. Then $\delta_i = \frac{n+\mu_i}{2}$, for all $i = 1, 2, \ldots, n-1$. Moreover, the eigenvalues μ_i and δ_i have the same multiplicity, for $i = 1, 2, \ldots, n$.*

Proof. For any vertex $v_i \in V(G)$, the reciprocal transmission degree is given by $RTr(v_i) = d_i + \frac{1}{2}(n - d_i - 1) = \frac{n+d_i-1}{2}$, as G is of diameter 2. Therefore, diagonal matrix of reciprocal transmission degrees is $RTr(G) = \frac{1}{2}\Big((n-1)I_n + Deg(G)\Big)$, where $Deg(G)$ is the diagonal matrix of vertex degrees. Also, since G is of diameter 2, so any two vertices are either adjacent or share a common neighbour. Thus, it follows that $RD(G) = A + \frac{1}{2}\overline{A}$, where $\overline{A}$ is the adjacency matrix of the complement $\overline{G}$ of G. Therefore, the reciprocal distance Laplacian matrix of G can be put as $RD^L(G) = RTr(G) - RD(G) = \frac{1}{2}\Big(nI_n - J + L(G)\Big)$, where $L(G) = Deg(G) - A(G)$ is the Laplacian matrix and J is the all one matrix. The result now follows directly from this last equation. □

A graph G is said to be determined by its spectrum (adjacency spectrum, Laplacian spectrum) if there is no other graph up to isomorphism with spectrum (adjacency spectrum, Laplacian spectrum) same as G. It is an interesting and hard problem in spectral graph theory to characterize graphs which are determined by their spectrum with respect to a given graph matrix. Various papers can be found in the literature regarding this problem, see [16] and the references therein. Likewise, a graph G is determined by its reciprocal distance Laplacian spectrum if there is no graph up to isomorphism with reciprocal distance Laplacian spectrum same as G. The following observation is immediate from Theorem 2.

Theorem 3. *A connected graph G of order n and diameter 2 is determined by its reciprocal distance Laplacian spectrum if and only if it is determined by its Laplacian spectrum. In particular, the complete graph, the complete bipartite graph and the complete split graph are determined its reciprocal distance Laplacian spectrum.*

Proof. The first half of the theorem is clear from Theorem 2. While as the second half follows by using the fact that the complete graph, the complete bipartite graph and the complete split graphs are determined by their Laplacian spectrum. □

The following important result about the reciprocal distance Laplacian spectral radius was obtained in [4].

Theorem 4. *Let G be a connected graph on n vertices. Then, the complement graph $\overline{G}$ is disconnected if and only if the reciprocal distance Laplacian spectral radius of G is n.*

We note that it is clear from Theorem 2 and the proof of Theorem 4 that the algebraic multiplicity of the eigenvalue n is one less than the number of components of $\overline{G}$. The following observation is immediate from Theorem 4.

Corollary 1. *If G is a bipartite graph and n is among its reciprocal distance Laplacian eigenvalues, then G is a complete bipartite graph. Moreover, the star $K_{1,n-1}$ is the only tree for which n is a reciprocal distance Laplacian eigenvalue.*

Proof. It is clear from the definition of bipartite graphs that the only bipartite graph with a disconnected complement is the complete bipartite graph. Using this observation together with Theorem 4 the first part of the result now follows. Further, since trees are bipartite graphs and star $K_{1,n-1}$ is the only tree which is a complete bipartite graph, the second part follows from the first part. □

The following result gives a lower bound for the reciprocal distance Laplacian spectral radius in terms of the reciprocal transmission degrees.

Theorem 5. *Let G be a connected graph of order $n \geq 3$ having reciprocal transmission degrees $RTr_1 \geq RTr_2 \geq \cdots \geq RTr_n$. Then*

$$\delta_1 \geq \max_{v_i v_j \in E(G)} \left\{ \frac{RTr_i + RTr_j}{2} \right\} + 1. \tag{2.1}$$

Equality occurs in (2.1) if and only if $RTr_i = RTr_j$ and $d(v_i, v_k) = d(v_j, v_k)$, for all $v_k \in V(G) - \{v_i, v_j\}$. In particular, if $G \cong K_2 \vee H$, where H is a graph of order $n-2$, then equality occurs (2.1).

Proof. Let $X = (x_1, x_2, \ldots, x_n)^T$ be a non zero vector in $\mathbb{R}^n$, then by Rayleigh-Ritz Theorem, we have

$$\delta_1 \geq \frac{X^T RD^L X}{X^T X}, \tag{2.2}$$

where $\mathbb{R}^n$ is the real vector space of dimension n and X^T is the transpose of X. Let v_i and v_j be two adjacent vertices in G, then $d(v_i, v_j) = 1$. Taking $x_i = 1, x_j = -1$ and $x_k = 0$, for $k \neq i, j$ in (2.2), we get

$$\delta_1 \geq \frac{RTr_i + RTr_j}{2} + 1,$$

with this the inequality (2.1) follows. Suppose that equality occurs in (2.1), then equality occurs in Rayleigh-Ritz Theorem, giving that $X = (1, 0, \ldots, -1, 0, \ldots, 0)^T$ is an eigenvector of the matrix $RD^L(G)$ corresponding to the eigenvalue δ_1. For the vertex v_i, it follows from the equation $RD^L X = \delta_1 X$ that $\delta_1(1) = RTr_i(1) - (-1 + 0 + \cdots + 0)$. This gives that $\delta_1 = RTr_i + 1$. Similarly, for the vertex v_j, we get $\delta_1 = RTr_j + 1$. These two equations together give that $RTr_i = RTr_j$. Let v_k be a vertex different from v_i and v_j. For this vertex, it follows from the equation $RD^L X = \delta_1 X$ that $\delta_1(0) = RTr_k(0) - (\frac{-1}{d(v_i, v_k)} + \frac{1}{d(v_j, v_k)} + 0 + \cdots + 0)$. This gives that $d(v_i, v_k) = d(v_j, v_k)$. Thus, it follows that equality occurs in (2.1) if and only if v_i and v_j are adjacent with $RTr_i = RTr_j$ and $d(v_i, v_k) = d(v_j, v_k)$, for all $v_k \in V(G) - \{v_i, v_j\}$. For the graph $G = K_2 \vee H$, let v_1 and v_2 be the vertices of K_2 and $v_3, \ldots, v_n$ be the vertices of H. It is clear that $RTr_1 = n - 1 = RTr_2$ and $RTr_k \leq n - 1$, for all $k = 3, 4, \ldots, n$. Since complement of the graph $K_2 \vee H$ is disconnected, therefore by Theorem 4, we have $\delta_1 = n$ and also we note that $\max_{v_i v_j \in E(G)} \left\{ \frac{RTr_i + RTr_j}{2} \right\} + 1 = \frac{2n-2}{2} + 1 = n$. This completes the proof. □

The following result [4] gives the relation between the reciprocal distance Laplacian eigenvalues of a connected graph G and its connected spanning subgraph.

Lemma 1. *Let G be a connected graph on n vertices with $m \geq n$ edges and let $G' = G - e$ be the connected graph obtained from G by the deletion of an edge e. Then $\lambda_i(RD^L(G)) \geq \lambda_i(RD^L(G'))$, for all $i = 1, 2, \ldots, n$.*

The following result gives lower bounds for the reciprocal distance Laplacian spectral radius in terms of order n and the Harary index of the graph G.

Theorem 6. *Let G be a connected graph of order $n \geq 3$ having Harary index $H(G)$. Then the following holds.*

(i). *If $G \ncong K_n$ then $\delta_1 \geq 2H - n(n-2) + 1$. Equality occurs if and only if $G \cong K_n - e$.*

(ii). *If G is a bipartite graph with partite sets of cardinality a and b, then $\delta_1 \geq 2H - \frac{n}{2}(n-3) - ab$. Equality occurs if and only if G is a connected bipartite graph with three or four distinct reciprocal distance Laplacian eigenvalue, which are $\delta_1, \delta_2 = \cdots = \delta_a = b + \frac{a}{2}$ and $\delta_{a+1} = \cdots = \delta_{n-1} = a + \frac{a}{2}, 0$. In particular, equality occurs for the graph $K_{a,b}$.*

(iii). *If G has independence number α, then $\delta_1 \geq 2H - n(n + \frac{\omega}{2} - 2) + \frac{\omega^2+1}{2}$. Equality occurs if and only if $G \cong CS_{\omega,\alpha}$, where $\alpha = n - \omega$.*

Proof. (i). For the graph $K_n - e$ which is obtained from the complete graph K_n by deleting an edge e, it is clear that $K_n - e = K_{n-2} \vee \overline{K}_2$. Clearly, the graph $K_n - e$ has a clique of order $n - 2$ such that each vertex within the clique shares the same neighbourhood outside the clique with reciprocal transmission degree of each vertex equal to $n - 1$, given by Lemma 1 that n is an eigenvalue of G with multiplicity $n - 3$. Also the graph $K_n - e$ has an independent set of order 2 such that each vertex within the independent set shares the same neighbourhood with reciprocal transmission degree equal to $n - \frac{3}{2}$, given by Lemma 1 that $n - 1$ is an eigenvalue of G with multiplicity 1. Further, 0 is an eigenvalue of multiplicity 1. Using the fact sum of reciprocal distance Laplacian eigenvalues is $2H(K_n - e) = n^2 - n - 1$, we conclude that n is the remaining eigenvalue. Thus, it follows that the reciprocal distance Laplacian spectrum of $K_n - e$ is $\{n^{[n-2]}, n-1, 0\}$. Since G is a connected graph with $G \ncong K_n$, it follows that G is a spanning subgraph of $K_n - e$. Therefore, by Lemma 1, we have $\delta_i(G) \leq \delta_i(K_n - e)$, for all i. Now, $\delta_1(G) + \delta_2(G) + \cdots + \delta_{n-1}(G) = 2H(G)$ implies that $\delta_1(G) = 2H(G) - \delta_2(G) - \delta_3(G) - \cdots - \delta_{n-1}(G)$. This further gives

$\delta_1(G) \geq 2H(G) - \delta_2(K_n - e) - \cdots - \delta_{n-1}(K_n - e) = 2H(G) - n(n-2) + 1$. It is clear that if $G \cong K_n - e$ then equality occurs. So, suppose that equality occurs then we must have $\delta_2 = \cdots = \delta_{n-2} = n$ and $\delta_{n-1} = n - 1$. Since, $\delta_1 \leq n$, it follows that if equality occurs in (i), then G is a connected graph having reciprocal distance eigenvalue n with multiplicity $n-2$. Using the discussion after the Theorem 4, it follows that the complement $\overline{G}$ has $n-1$ components. This is only possible if $\overline{G} = K_2 \cup (n-2)K_1$. Thus, it follows that if equality occurs in (i), then G must be of the form $K_{n-2} \vee \overline{K}_2 = K_n - e$. This completes the proof in this case.

(ii). For the graph $K_{a,b}$, there is an independent set of order a with same neighbourhood set such that the reciprocal transmission degree of each vertex is $b + \frac{a-1}{2}$, giving by Lemma 1 that $b + \frac{a}{2}$ is an eigenvalue of G with multiplicity $a-1$. Similarly, there is an independent set of order b in $K_{a,b}$ such that each vertex share same neighbourhood with same reciprocal transmission degree $a + \frac{b-1}{2}$, giving by Lemma 1 that $a + \frac{b}{2}$ is an eigenvalue of G with multiplicity $b-1$. Further, 0 is an eigenvalue with multiplicity 1 and using the fact sum of reciprocal distance Laplacian eigenvalues of $K_{a,b}$ is $2H(K_{a,b}) = 2ab + \frac{a^2+b^2}{2}$, we get that the remaining eigenvalue is $a + b = n$. Thus, the reciprocal distance Laplacian spectrum of $K_{a,b}$ is $\left\{n, b + \frac{a}{2}^{[a-1]}, a + \frac{b}{2}^{[b-1]}, 0\right\}$. Since G is a bipartite graph with partite sets of cardinality a and b, it follows that G is a spanning subgraph of $K_{a,b}$. Now, using Lemma 1 the result follows in this case as well. If $G \cong K_{a,b}$, then it is easy to see that equality occurs in (ii). So, suppose that equality occurs in (ii), then we must have $\delta_2 = \cdots = \delta_a = b + \frac{a}{2}$ and $\delta_{a+1} = \cdots = \delta_{n-1} = a + \frac{a}{2}$, for $a \leq b$. This gives that if equality occurs in (ii), then G is a connected bipartite graph with three or four distinct reciprocal distance Laplacian eigenvalue, which are $\delta_1, \delta_2 = \cdots = \delta_a = b + \frac{a}{2}$ and $\delta_{a+1} = \cdots = \delta_{n-1} = a + \frac{a}{2}, \delta_n = 0$. This completes the proof in this case.

(iii). Let $CS(\omega, n-\omega)$ be the complete split graph with clique number ω and independence number $\alpha = n - \omega$. Clearly, the graph $CS(\omega, n-\omega)$ has a clique on ω vertices with same neighbourhood outside the clique and with the common reciprocal distance transmission degree $n-1$, giving by lemma 1, n is the RD^L-eigenvalue with multiplicity $\omega - 1$. Similarly, the graph $CS(\omega, n-\omega)$ has an independent set on $n-\omega$ vertices sharing the same neighbourhood and with the common reciprocal transmission degree $n + \frac{\omega-1}{2}$, giving by Lemma 1, that $\frac{2n+\omega}{2}$ is RD^L-eigenvalue with multiplicity $n - \omega - 1$. Lastly, it is easy to verify that 0 and n are the remaining two RD^L-eigenvalues of $CS(\omega, n-\omega)$. Thus, it follows that the RD^L-spectrum of $CS(\omega, n-\omega)$ is $\left\{0, n^{[\omega]}, \left(\frac{2n+\omega}{2}\right)^{[n-\omega-1]}\right\}$. Since G is connected graph with independence number α, it follows that G is a spanning subgraph of $CS(\omega, \alpha)$, $\alpha = n - \omega$. Now, using Lemma 1 the result follows in this case as well. If $G \cong CS(\omega, n-\omega)$, then it is clear that equality occurs in (iii). So, suppose that equality occurs in (iii), then we must have $\delta_2 = \cdots = \delta_\omega = n$ and $\delta_{\omega+1} = \cdots = \delta_{n-1} = n + \frac{\omega}{2}$. Since $\delta_1 \leq n$, it follows that if equality occurs in (iii), then G is a connected graph with reciprocal distance Laplacian eigenvalues $\delta_1 = n, \delta_2 = \cdots = \delta_\omega = n$ and $\delta_{\omega+1} = \cdots = \delta_{n-1} = n + \frac{\omega}{2}, \delta_n = 0$. This gives that the reciprocal distance Laplacian spectrum of G is $\left\{0, n^{[\omega]}, \left(\frac{2n+\omega}{2}\right)^{[n-\omega-1]}\right\}$, which is same as the reciprocal distance Laplacian spectrum $CS(\omega, n-\omega)$. Since, by Theorem 3 the graph $CS(\omega, n-\omega)$ is determined by its reciprocal distance Laplacian spectrum, it follows that if equality occurs in (iii) then G must be isomorphic to $CS(\omega, n-\omega)$. This completes the proof in this case. $\square$

Based on Theorem 6, we leave the following interesting problem for future research.

Problem 1. Characterize all the connected graphs which are extremal graphs for (ii) of Theorem 6.

The following interesting Lemma can be found in [15] and is helpful in obtaining our next result.

Lemma 2. *If $x_1 \geq x_2 \geq \cdots \geq x_t$ are real numbers such that $\sum\limits_{i=1}^{t} x_i = 0$, then $x_1 \leq \sqrt{\frac{t-1}{t}\sum\limits_{i=1}^{t} x_i^2}$, with equality if and only if $x_2 = x_3 = \cdots = x_t = \frac{-x_1}{t-1}$.*

The following result can be found in [11].

Theorem 7. *Let $y_1 \geq y_2 \geq \cdots \geq y_n$ be the zeros of polynomial $p(y)$ and let $\overline{y} = \frac{1}{n}\sum\limits_{i=1}^{n} y_i$ and $z = n\sum\limits_{i=1}^{n} y_i^2 - \left(\sum\limits_{i=1}^{n} y_i\right)^2$. Then, followings holds*

$$\overline{y} + \frac{1}{n}\sqrt{\frac{z}{n-1}} \leq y_1 \leq \overline{y} + \frac{1}{n}\sqrt{(n-1)z},$$
$$\overline{y} - \frac{1}{n}\sqrt{\frac{(i-1)}{n-i+1}z} \leq y_i \leq \overline{y} + \frac{1}{n}\sqrt{\frac{(n-i)}{i}z}, \text{ for } i = 2, 3, \ldots, n-1.$$

The following result is the immediate consequence of Theorem 7 and gives upper and lower bounds for the reciprocal distance spectral radius and the second smallest reciprocal distance eigenvalue.

Theorem 8. *Let $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_{n-1} > \delta_n = 0$ be the RD^L-eigenvalues of G. Then the* *following holds for* δ_1 *and* δ_{n-1}

$$\begin{aligned}
\delta_1 \geq & \frac{1}{n-1}\left(2H+\sqrt{\frac{1}{n-2}\Big((n-1)\Big(\|RD(G)\|_F^2+\sum_{i=1}^{n}(RTr_i)^2\Big)-4H^2\Big)}\right),\\
\delta_1 \leq & \frac{1}{n-1}\left(2H+\sqrt{(n-2)\Big((n-1)\Big(\|RD(G)\|_F^2+\sum_{i=1}^{n}(RTr_i)^2\Big)-4H^2\Big)}\right),\\
\delta_{n-1} \geq & \frac{1}{n-1}\left(2H-\sqrt{(n-2)\Big((n-1)\Big(\|RD(G)\|_F^2+\sum_{i=1}^{n}(RTr_i)^2\Big)-4H^2\Big)}\right),\\
\delta_{n-1} \leq & \frac{2H}{n-1}.
\end{aligned}$$

For the first three inequalities, equality occurs for K_n*, while equality holds in the last one if and only if* $G \cong K_n$*.*

Proof. Since $\sum_{i=1}^{n}\delta_i = 2H$ and $\sum_{i=1}^{n}\delta_i^2 = \sum_{i=1}^{n}(Tr_i)^2 + \|RD(G)\|_F^2$, where $\|RD(G)\|_F^2$ is the frobenius norm of the matrix $RD(G)$. Using Theorem 7 together with these observations, the inequalities follow. For K_n, we see that $\frac{2H}{n-1} = n$ and $(n-1)\Big(\|RD(G)\|_F^2+\sum_{i=1}^{n}(RTr_i)^2\Big)-4H^2 = n^2(n-1)^2-(n(n-1))^2 = 0$. Thus equality occurs in first three inequalities. Besides, it is proved that the multiplicity [4] of $\delta_1(K_n)$ is $n-1$ if and only if $G \cong K_n$ and that proves the equality for the last case. □

The following result gives an upper bound for the reciprocal distance Laplacian spectral radius in terms of order, the Harary index and the Frobenious norm $\|RD^L(G)\|_F^2$ of the graph G.

Theorem 9. *Let* G *be a connected graph of order* $n \geq 3$ *having Harary index* $H(G)$*. Then*

$$\delta_1 \leq \frac{2H(G)}{n-1}+\sqrt{\frac{n-2}{n-1}\Big(\|RD^L(G)\|_F^2-\frac{4H(G)^2}{n-1}\Big)}. \tag{2.3}$$

Equality occurs if and only if $G \cong K_n$ *or* G *is a graph with three distinct reciprocal distance Laplacian eigenvalues, which are* δ_1, $\frac{2H(G)-\delta_1}{n-2}$ *and* 0*.*

Proof. Let $\delta_1 \geq \delta_2 \geq \cdots \delta_{n-1} > \delta_n = 0$ be the reciprocal distance Laplacian eigenvalues of G. Then

$$\sum_{i=1}^{n-1}\delta_i = 2H(G) \quad \text{and} \quad \sum_{i=1}^{n-1}\delta_i^2 = \|RD^L(G)\|_F^2.$$

Further, $\sum\limits_{i=1}^{n-1}\left(\delta_i-\frac{2H(G)}{n-1}\right)=\sum\limits_{i=1}^{n-1}\delta_i-2H(G)=0$. Applying Lemma 2 to real numbers $\delta_1-\frac{2H(G)}{n-1}\geq\delta_2-\frac{2H(G)}{n-1}\geq\cdots\geq\delta_{n-1}-\frac{2H(G)}{n-1}$, we get

$$\delta_1-\frac{2H(G)}{n-1}\leq\sqrt{\frac{n-2}{n-1}\sum_{i=1}^{n}\Big(\delta_i-\frac{2H(G)}{n-1}\Big)^2}. \tag{2.4}$$

We have $\sum\limits_{i=1}^{n}\left(\delta_i-\frac{2H(G)}{n-1}\right)^2=\sum\limits_{i=1}^{n}\left(\delta_i^2+\left(\frac{2H(G)}{n-1}\right)^2-\frac{4H(G)}{n-1}\delta_i=\sum\limits_{i=1}^{n}\delta_i^2+\frac{4H(G)^2}{n-1}-\frac{8H(G)^2}{n-1}=\right.$ $\|RD^L(G)\|_F^2-\frac{4H(G)^2}{n-1}$. With this it follows from (2.4) that the inequality (2.3) holds. Assume that the equality holds in (2.3), then equality holds in Lemma 2. Which is so if and only if $\delta_2-\frac{2H(G)}{n-1}=\delta_3-\frac{2H(G)}{n-1}=\cdots=\delta_{n-1}-\frac{2H(G)}{n-1}=\frac{\frac{2H(G)}{n-1}-\delta_1}{n-2}$. This gives that equality holds in (2.3) if and only if $\delta_2=\delta_3=\cdots=\delta_{n-1}=\frac{2H(G)-\delta_1}{n-2}$. Since $\delta_1\geq\delta_2$, it follows that equality holds in (2.3) if and only if G is a connected graph with two distinct reciprocal distance Laplacian eigenvalues of G is a connected graph with two distinct reciprocal distance Laplacian eigenvalues, which are δ_1 with multiplicity one, $\frac{2H(G)-\delta_1}{n-2}$ with multiplicity $n-2$ and simple eigenvalue 0. Since a connected graph has two distinct reciprocal distance Laplacian eigenvalues if and only if it is a complete graph, the result follows. □

The following result gives an upper bound for the reciprocal distance Laplacian spectral radius in terms of reciprocal transmission degree and the reciprocal distance between vertices.

Theorem 10. *Let G be a connected graph of order n. Then*

$$\delta_1<RTr_i+\sqrt{2\sum_{1\leq i<j\leq n}\frac{1}{d_{ik}^2}-\frac{1}{n}\sum_{k=1}^{n}(RTr_i)^2}.$$

Proof. Let $X=(x_1,x_2,\ldots,x_n)^T$ be a unit eigenvector corresponding to eigenvalue δ_1 of the reciprocal distance Laplacian matrix $RD^L(G)$ of G. Then by the eigenequation $RD^L(G)X=\delta_1X$, for each $v_i\in V(G)$, we have

$$\delta_1x_i=RTr_ix_i-\sum_{i=1}^{n}\frac{x_k}{d_{ik}},$$

or equivalently

$$\Big(\delta_1-RTr_i\Big)^2x_i^2=\Big(\sum_{i=1}^{n}\frac{x_k}{d_{ik}}\Big)^2. \tag{2.5}$$

By using Cauchy Schwartz inequality and noting that X is a unit vector, we have

$$\Big(\sum_{k=1}^{n} x_k\Big(\frac{1}{d_{ik}}-\frac{1}{n}\sum_{k=1}^{n}\frac{1}{d_{ik}}\Big)\Big)^2 \le \sum_{k=1}^{n} x_k^2 \sum_{k=1}^{n}\Big(\frac{1}{d_{ik}^2}+\frac{1}{n^2}\Big(\sum_{k=1}^{n}\frac{1}{d_{ik}}\Big)^2-\frac{2}{nd_{ik}}\sum_{k=1}^{n}\frac{1}{d_{ik}}\Big) \tag{2.6}$$

$$=\sum_{k=1}^{n}\frac{1}{d_{ik}^2}-\frac{1}{n}\Big(\sum_{k=1}^{n}\frac{1}{d_{ik}}\Big)^2$$

Besides, it is well known that $\sum\limits_{i=1}^{n} x_i=0$, so we have

$$\begin{aligned}\Big(\sum_{k=1}^{n}\frac{x_k}{d_{ik}}\Big)^2 &=\Big(\sum_{k=1}^{n} x_k\Big(\frac{1}{d_{ik}}-\frac{1}{n}\sum_{k=1}^{n}\frac{1}{d_{ik}}\Big)\Big)^2\\ &\le\sum_{k=1}^{n}\frac{1}{d_{ik}^2}-\frac{1}{n}\Big(\sum_{k=1}^{n}\frac{1}{d_{ik}}\Big)^2.\end{aligned}$$

Summing from 1 to n, we obtain

$$\sum_{i=1}^{n}\Big(\sum_{k=1}^{n}\frac{x_k}{d_{ik}}\Big)^2 \le 2\sum_{1\le i<j\le n}\frac{1}{d_{ik}^2}-\frac{1}{n}\sum_{k=1}^{n}(RTr_i)^2.$$

Therefore,

$$(\delta_1-RTr_i)^2\le\sum_{i=1}^{n}\Big(\delta_1-RTr_i\Big)^2 x_i^2\le 2\sum_{1\le i<j\le n}\frac{1}{d_{ik}^2}-\frac{1}{n}\sum_{k=1}^{n}(RTr_i)^2, \tag{2.7}$$

and from this, it follows that

$$\delta_1\le RTr_i+\sqrt{2\sum_{1\le i<j\le n}\frac{1}{d_{ik}^2}-\frac{1}{n}\sum_{k=1}^{n}(RTr_i)^2}. \tag{2.8}$$

Now, Suppose that equality occurs in (2.8), then all the above inequalities are equalities. From (2.7) , we get $RTr_1=RTr_2=\cdots=RT_n=T$ and by (2.6), we have $\frac{1}{x_1}\Big(\frac{1}{d_{i1}}-\frac{T}{n}\Big)=\frac{1}{x_2}\Big(\frac{1}{d_{i2}}-\frac{T}{n}\Big)=\cdots=\frac{1}{x_n}\Big(\frac{1}{d_{in}}-\frac{T}{n}\Big)=l_i$ (say), for every i. If $l_i=0$, then $\frac{1}{d_{i1}}=\frac{1}{d_{i2}}=\cdots=\frac{1}{d_{in}}=\frac{1}{n}T$, a contradiction. Else, $l_i\neq 0$ for any i. Also, diagonal entries of the reciprocal distance matrix are zero. So, there are no entries of type $\frac{1}{d_{ii}}$ in $RD^L(G)$ and hence $l_ix_i=-\frac{T}{n}$, for each i. Moreover, for $i\neq j$, $d_{ij}=l_ix_j+\frac{T}{n}=\frac{T}{n}\Big(1-\frac{x_j}{x_i}\Big)$ and likewise $d_{ji}=\frac{T}{n}\Big(1-\frac{x_i}{x_j}\Big)$. As $d_{ij}=d_{ji}$, we obtain $d_{ij}=0$, again a contradiction. This completes the proof. □

The following result gives a lower bound for the reciprocal distance Laplacian spectral radius in terms of the Harary index, the order and the Frobenious norm parameter $\|RD(G)\|_F^2$.

Theorem 11. *Let G be a graph of order $n \geq 3$. Then*

$$\delta_1 \geq \frac{2H}{n-1} + \sqrt{\frac{1}{(n-1)(n-2)}\Big(\|RD(G)\|_F^2 + \sum_{i=1}^{n}(RTr_i)^2\Big) - \frac{4H^2}{n-1}\Big)},$$

$$\delta_{n-1} \leq \frac{2H}{n-1} - \sqrt{\frac{1}{(n-1)(n-2)}\Big(\|RD(G)\|_F^2 + \sum_{i=1}^{n}(RTr_i)^2 - \frac{4H^2}{n-1}\Big)},$$

with equality if and only if $G \cong K_n$.

Proof. For every fixed $k > 0$, we have

$$\Big(\sum_{i=1}^{n-1}\delta_i - (n-1)\delta_k\Big)^2 = \Big(\sum_{i=1}^{n-1}(\delta_i - \delta_k\Big)^2 = \sum_{i=1}^{n-1}(\delta_i - \delta_k)^2 + 2\sum_{1\leq i<j\leq n-1}(\delta_i - \delta_k).$$

Now, for $k = 1$ or $k = n-1$, it is easy to see that

$$\sum_{1\leq i<j\leq n-1}(\delta_i - \delta_k)(\delta_j - \delta_k) \geq 0. \tag{2.9}$$

Thus, with this information, we obtain

$$\Big(\sum_{i=1}^{n-1}\delta_i - (n-1)\delta_k\Big)^2 \geq \sum_{i=1}^{n-1}(\delta_i - \delta_k)^2. \tag{2.10}$$

Also, from (2.10), we have

$$\Big(\sum_{i=1}^{n-1}\delta_i\Big)^2 - 2(n-1)\delta_k\sum_{i=1}^{n-1}\delta_i + (n-1)^2\delta_k^2 \geq \sum_{i=1}^{n}\delta_i^2 - 2\delta_k\sum_{i=1}^{n}\delta_i + (n-1)\delta_k^2,$$

which is further equivalent to

$$\delta_k^2(n-1)(n-2) + \Big(\sum_{i=1}^{n-1}\delta_i\Big)^2 - 2\delta_k(n-2)\sum_{i=1}^{n}\delta_i \geq \sum_{i=1}^{n-1}\delta_i^2,$$

or we have

$$\delta_k^2 + \frac{1}{(n-1)(n-2)}\Big(\sum_{i=1}^{n-1}\delta_i\Big)^2 - \frac{2}{(n-1)}\delta_k\sum_{i=1}^{n}\delta_i \geq \frac{1}{(n-1)(n-2)}\sum_{i=1}^{n-1}\delta_i^2.$$

This further implies that

$$\Big(\delta_k - \frac{1}{(n-1)}\sum_{i=1}^{n-1}\delta_i\Big)^2 \geq \frac{1}{(n-1)(n-2)}\Big(\sum_{i=1}^{n-1}\delta_i^2 - \frac{1}{(n-1)}\Big(\sum_{i=1}^{n-1}\delta_i\Big)^2\Big).$$

Now, using the fact that $\sum\limits_{i=1}^{n}\delta_i = 2H$ and $\sum\limits_{i=1}^{n}\delta_i^2 = \sum\limits_{i=1}^{n}(RT_i)^2 + \|RD(G)\|_F^2$, we have

$$\Big(\delta_k - \frac{2H}{(n-1)}\Big)^2 \geq \frac{1}{(n-1)(n-2)}\Big(\sum_{i=1}^{n}(RT_i)^2 + \|RD(G)\|_F^2 - \frac{(2H)^2}{(n-1)}\Big).$$

Also, by Theorem 8 with $\delta_1 - \frac{2H}{n-1} \geq 0$ and $\delta_{n-1} - \frac{2H}{n-1} \leq 0$, we obtain

$$\begin{aligned}\delta_1 \geq & \frac{2H}{n-1} + \sqrt{\frac{1}{(n-1)(n-2)}\Big(\|RD(G)\|_F^2 + \sum_{i=1}^{n}(RTr_i)^2\Big) - \frac{4H^2}{n-1}\Big)},\\ \delta_{n-1} \leq & \frac{2H}{n-1} - \sqrt{\frac{1}{(n-1)(n-2)}\Big(\|RD(G)\|_F^2 + \sum_{i=1}^{n}(RTr_i)^2 - \frac{4H^2}{n-1}\Big)}.\end{aligned}$$

If equality occurs, then Inequality (2.9) must be equality, so we have

$$\delta_1 = \delta_2 = \cdots = \delta_{n-1} = \frac{2H}{n-1}.$$

Since, the multiplicity [4] of δ_1 is equal to $n-1$ if and only if $G \cong K_n$, so equality occurs if and only if $G \cong K_n$. For other way round, we see that $\|RD(G)\|_F^2 + \sum\limits_{i=1}^{n}(RTr_i)^2 - \frac{4H^2}{n-1} = 0$ for K_n and equality holds. □

The next result gives upper and lower bounds for the reciprocal distance Laplacian spectral radius.

Theorem 12. *Let G be a connected graph of order $n \geq 3$. Then the following holds*

$$\frac{1}{2(n-1)}\Big(4H - \sqrt{D}\Big) \leq \delta_1 \leq \frac{1}{2(n-1)}\Big(4H + \sqrt{D}\Big),$$

where $D = (4H)^2 - 4(n-1)(4H^2 - (n-2)B)$ and equality occurs if and only if $G \cong K_n$.

Proof. If $B = \sum\limits_{i=1}^{n} \delta_i^2 = \|RD(G)\|_F^2 + \sum\limits_{i=1}^{n} (RTr_i)^2$, then we have

$$\begin{aligned}\delta_1^2 = B - \sum_{i=2}^{n-1} \delta_i^2 \leq & B - \frac{1}{n-2}\Big(\sum_{i=2}^{n-1} \delta_i\Big)^2 \\ = & B - \frac{1}{(n-2)}(2H-\delta_1)^2 = B - \frac{1}{n-2}(4H^2 - 4H\delta_1 + \delta_1^2).\end{aligned} \tag{2.11}$$

This, further implies that

$$\delta_1^2(n-1) + 4H^2 - 4H\delta_1 - (n-2)B \leq 0. \tag{2.12}$$

For the above inequality, it follows that

$$\begin{aligned}\delta_1 \leq & \frac{1}{2(n-1)}\Big(4H + \sqrt{(4H)^2 - 4(n-1)(4H^2-(n-2)B)}\Big), \\ \delta_1 \geq & \frac{1}{2(n-1)}\Big(4H - \sqrt{(4H)^2 - 4(n-1)(4H^2-(n-2)B)}\Big).\end{aligned}$$

Equality holds if equality holds in (2.11), that $\delta_2 = \delta_3 = \cdots = \delta_{n-1}$, which is true only for K_n. Conversely, for $G \cong K_n$, we see that $(4H)^2 - 4(n-1)(4H^2-(n-2)B) = 4n^2(n-1)^2 - 4n(n-1)(n^2(n-1)^2 - (n-2)n^2(n-1)) = 0$ and equality holds. □

Next, we find the relation between the reciprocal distance Laplacian spectral radius with the reciprocal distance spectral radius and for that we need the following result.

Lemma 3. *[17] Let be a connected graph with reciprocal degree sequence $\{RT_i\}_{i=1}^n$ and let $\lambda(D(G))$ be the reciprocal distance spectral radius of G. Then*

$$\lambda_1(RD(G)) \geq \sqrt{\frac{(RT_1)^2 + (RT_2)^2 + \cdots + (RT_n)^2}{n}},$$

with equality if and only if G is a reciprocal distance regular graph.

Theorem 13. *Let G be a connected graph and $\lambda(RD(G))$ be its reciprocal distance spectral radius. Then*

$$\lambda(RD(G)) \geq \sqrt{\frac{1}{n(n-1)}\Big(\delta_1^2(n-1) + 4H^2 - 4H\delta_1 - (n-2)\|RD(G)\|_F^2\Big)},$$

with equality if and only if G is reciprocal transmission distance regular.

Proof. From Inequality (2.12) and by Lemma 3, we have

$$
\begin{aligned}
\delta_1^2(n-1)+4H^2-4H\delta_1 &\leq (n-2)B \qquad (2.13)\\
&= (n-2)\Big(\|RD(G)\|_F^2+\sum_{i=1}^{n}(RTr_i)^2\Big)\\
&\leq (n-2)\Big(\|RD(G)\|_F^2+n\lambda^2(RD(G))\Big). \qquad (2.14)
\end{aligned}
$$

This, further implies that

$$\lambda(RD(G)) \geq \sqrt{\frac{1}{n(n-1)}\Big(\delta_1^2(n-1)+4H^2-4H\delta_1-(n-2)\|RD(G)\|_F^2\Big)}.$$

Equality holds if all the above inequalities hold. Inequality (2.13) is equality if and only if (2.11) is equality, that is $B-\sum\limits_{i=2}^{n-1}\delta_i^2 = B-\frac{1}{n-2}\Big(\sum\limits_{i=2}^{n-1}\delta_i\Big)^2$, which holds if and if $\delta_2=\delta_3=\ldots\delta_{n-1}$, and that is true if and only if $G\cong K_n$. Also, by Lemma 3, Inequality (2.14) is equality if and only if G is reciprocal transmission degree regular. Thus equality holds if and only if G is a reciprocal transmission degree regular graph. □

Acknowledgements: We express our sincere gratitude to the anonymous referee for his insightful feedback and recommendations that enhance the paper's presentation. We certify that the article's findings were presented at the BITS Pilani, Dubai Campus, United Arab Emirates, during the "International Conference on Graphs, Combinatorics and Optimization, February 6-8, 2022."

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

References

[1] M. Andelic, S. Khan, and S. Pirzada, *On graphs with a few distinct reciprocal distance Laplacian eigenvalues*, AIMS Math. **8** (2023), no. 12, 29008–29016. https://doi.org/10.3934/math.20231485.

[2] M. Aouchiche and P. Hansen, *Two Laplacians for the distance matrix of a graph*, Linear Algebra Appl. **439** (2013), no. 1, 21–33. https://doi.org/10.1016/j.laa.2013.02.030.

[3] ———, *Distance spectra of graphs: A survey*, Linear Algebra Appl. **458** (2014), 301–386. https://doi.org/10.1016/j.laa.2014.06.010.

[4] R. Bapat and S.K. Panda, *The spectral radius of the reciprocal distance Laplacian matrix of a graph*, Bull. Iranian Math. Soc. **44** (2018), no. 5, 1211–1216. https://doi.org/10.1007/s41980-018-0084-z.

[5] G. Chartrand and P. Zhang, *Introductory Graph Theory*, Tata McGraw-Hill edition, New Delhi, 2006.

[6] Z. Cui and B. Liu, *On Harary matrix, Harary index and Harary energy*, MATCH Commun. Math. Comput. Chem. **68** (2012), no. 3, 815–823.

[7] C. Cvetković, R. Peter, and S. Slobodan, *An Introduction to the Theory of Graph Spectra*, Cambridge University Press, 2009.

[8] H.A. Ganie, B.A. Rather, and M. Aouchiche, *Reciprocal distance Laplacian spectral properties double stars and their complements*, Carpathian Math. Publ. **15** (2023), no. 2, 576–593.
https://doi.org/10.15330/cmp.15.2.576-593.

[9] L. Huiqiu, S. Jinlong, X. Jie, and Z. Yuke, *A survey on distance spectra of graphs*, Adv. Math. **50** (2021), no. 1, 29–76.

[10] S. Khan, S. Pirzada, and Y. Shang, *On the sum and spread of reciprocal distance Laplacian eigenvalues of graphs in terms of harary index*, Symmetry **14** (2022), no. 9, 1937.
https://doi.org/10.3390/sym14091937.

[11] A. Lupaş, *Inequalities for the roots of a class of polynomials*, Publ. Elektroteh. Fak., Univ. Beogradu, Ser. (1977), no. 577/598, 79–85.

[12] L. Medina and M. Trigo, *Upper bounds and lower bounds for the spectral radius of reciprocal distance, reciprocal distance Laplacian and reciprocal distance signless laplacian matrices*, Linear Algebra Appl. **609** (2021), 386–412.
https://doi.org/10.1016/j.laa.2020.09.024.

[13] ______, *Bounds on the reciprocal distance energy and reciprocal distance Laplacian energies of a graph*, Linear Multilinear Algebra **70** (2022), no. 16, 3097–3118. https://doi.org/10.1080/03081087.2020.1825607.

[14] S. Pirzada and S. Khan, *On the distribution of eigenvalues of the reciprocal distance Laplacian matrix of graphs*, Filomat **37** (2023), no. 23, 7973–7980.

[15] O. Rojo and H. Rojo, *A decreasing sequence of upper bounds on the largest Laplacian eigenvalue of a graph*, Linear Algebra Appl. **381** (2004), 97–116.
https://doi.org/10.1016/j.laa.2003.10.026.

[16] R. Sharafdini and A.Z. Abdian, *Tarantula graphs are determined by their Laplacian spectrum*, Electron. J. Graph Theory Appl. **9** (2021), no. 2, 419–432.
https://doi.org/10.5614/ejgta.2021.9.2.14.

[17] B. Zhou and N. Trinajstić, *Maximum eigenvalues of the reciprocal distance matrix and the reverse Wiener matrix*, Int. J. Quantum Chem. **108** (2008), no. 5, 858–864.
https://doi.org/10.1002/qua.21558.