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Research Article

Pebbling in Sierpiński-type graphs

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Abstract: Graph pebbling is a network optimization technique for the movement of resources in transit. A pebbling move in a connected graph G can be defined as a distribution of pebbles on the vertices of a graph, which involves removing two pebbles from a vertex, placing one pebble on one of its adjacent vertices, and discarding the other pebble. For a graph G, the pebbling number $f(G)$ is the minimum number of pebbles required such that one pebble is moved to any arbitrary vertex of the graph G. Fractals are described as intricate patterns that are identical at different dimensions or identical in all dimensions. In this paper, the strategy of pebbling is applied to Sierpiński graphs which are well known fractals and several critical points are scrutinized and verified for Generalized Sierpiński graph $S(G,t), t \geq 2$, Sierpiński graph $S(K_n, t)$, $t \geq 1$, $n \geq 2$ and Sierpiński triangle graph S_m, $m \geq 2$.

Keywords: fractal, pebbling, generalized Sierpiński graph, Sierpiński triangle graph.

AMS Subject classification: 05C12, 05C57, 05C70

1. Introduction

Graph pebbling is a combinatorial game played on graphs. The technique of pebbling can be used to minimize memory traffic in computers, solve the register allocation problem and transmit data from one directory to another. The pebbling steps examine the cost of pebble loss and have been the focus of substantial research in the context of proving lower bounds for computation on graphs. Chung was the first to establish the idea of graph pebbling and found the pebbling number of hypercubes [3]. For a graph G, consider a distribution of p pebbles over the vertices of G. A pebbling move from a vertex u_1 to an adjacent vertex v_1 is defined as the removable of two pebbles from u_1 and placing one pebble on v_1. The pebbling number $f(G)$ is the minimum

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number of pebbles which ensures that every vertex of G can be pebbled through a sequence of pebbling move. The lower bound obtained in the pebbling number of a graph satisfies the condition $f(G) \geq \max\{n(G), 2^d\}$, where $n(G)$ is the number of vertices of G and d is the diameter of G. A path $v_0, v_1, v_2 \ldots v_n$ is a transmitting subgraph if it has one pebble transmitted from v_0 to v_n, distributed as two pebbles in v_0 and at least one pebble on each of the other vertices in the path, perhaps excluding v_n. This pebble distribution allows a pebble to be transferred from v_0 to v_n.
In recent years, there have been multiple variants to the pebbling moves, which have stimulated the curiosity of many scholars who want to extend their relevance to diverse graph theoretic concepts. For a general background on pebbling, the reader is invited to read [1, 2, 4, 7–9, 12–14, 16] and references therein. In this paper, the concept of pebbling is applied to fractals, in particular to the class of Sierpiński graphs. These graphs are generalizations of the graphs of Tower of Hanoi problem. Interesting pebbling bounds are obtained for the Generalized Sierpiński graph $S(G, t), t \geq 2$, Sierpiński graph $S(K_n, t)$, $t \geq 1$, $n \geq 2$ and Sierpiński triangle graph S_m, $m \geq 2$.

2. Sierpiński graphs

Motivated by the topological studies of Limpscomb's space [11] it was Klavzar and Milutinovic [10] who introduced the graphs $S(K_n, t)$ in 1997. The construction was then generalized by Gravier et.al.in [5] for any graph G. The Generalized Sierpiński graph $S(G, t)$, is a graph G of dimension n with vertex set as $\{1, 2, \ldots, n\}^t$. The letters of a word u of $\{1, 2, \ldots, n\}^t$are denoted by $u = u_1, u_2, \ldots, u_t$. The edge set is defined by $\{u, v\}$ is an edge if and only if there exists $i \in \{1, 2, \ldots, t\}$ such that

- $u_j = v_j$ if $j < i$
- $u_i \neq v_i$ and $\{u_i, v_i\} \in E(G)$
- $u_j = v_i$ and $v_j = u_i$ if $j > i$.

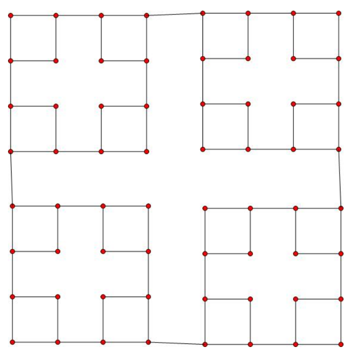

Figure 1. **Generalized Sierpiński graph** $S(C_4, 3)$

The graphs $S(G,t)$ can be recursively constructed. It is to note that, $S(G,1)$ is isomorphic to the graph G and we construct $S(G,t+1)$ by copying n times $S(G,t)$ and adding one edge between a copy x and a copy y of $S(G,t)$ whenever (x,y) is an edge of G. See Figure 1. Each copy of $S(G,t)$ can be referred as $S_1(G,t), S_2(G,t), \ldots, S_n(G,t)$. Refer each vertex $(u_1,u_2,\ldots,u_t)$ of $S(G,t)$ as $u_1u_2\ldots u_t$. The vertices $1\ldots1, 2\ldots2, n\ldots n$ of $S(G,t)$ are called extreme vertices and the remaining vertices are called inner vertices. For any u and v with $u \neq v$, $(uvv\ldots v, vv\ldots vu) \in E(S(G,t))$ with $uvv\ldots v \in V(S_u(G,t))$ and $vv\ldots vu \in V(S_v(G,t))$. These vertices are referred as binding vertices. For $S(K_5,2)$ shown in Figure 2, vertices with labels 12 and 21 are a pair of binding vertices.

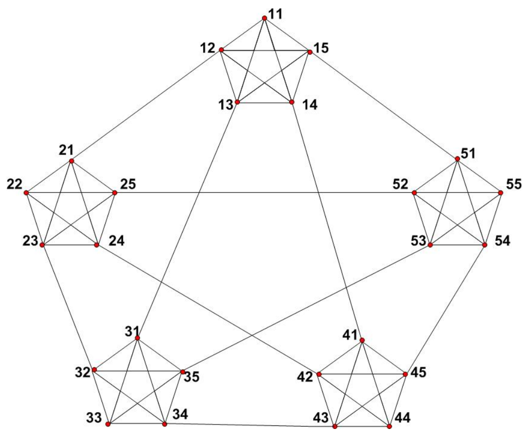

Figure 2. **Sierpiński graph** $S(K_5,2)$

3. Pebbling number of Sierpiński graph

Theorem 1. *For any $x,y \in V(S(G,t))$ and any integer $t \geq 2$, $f(S(G,t)) \geq 2^{(2^t-1)d_G(x,y)} + n^{t-2}f(G)$.*

Proof. Let $f(S(G,1)) = f(G)$. Choose the target vertex as $11\ldots1 \in S_1(G,t)$. For the reason $S(G,t)$, $t \geq 2$ is recursively constructed the result holds true for any vertex $u \in S_u(G,t)$ chosen as target vertex. If at least $n^{t-2}f(G)$ pebbles are distributed on $S_1(G,t)$ then the target can be pebbled from any vertex $1jj\ldots j \in S_1(G,t)$. Now excluding this trivial case consider the possibility when the target has to be pebbled from some vertex $ijj\ldots j \in S_i(G,t)$. To begin with, assume that $f(S_1(G,t)) < n^{t-2}f(G)$.
If $d_G(x,y)$ is the distance between the vertices x and y in G then the distance between the corresponding vertices x^t and y^t at level t is $d_G(x^t,y^t) = (2^t-1)d_G(x,y)$. This

infers that, the atmost distance to reach the corresponding vertex in the x^{th} copy of $S(G,t)$ from the corresponding vertex in y^{th} copy of $S(G,t)$ is $(2^t-1)d_G(x,y)$. At this moment, excluding the pebbles considered on $S_1(G,t)$ there are $2^{(2^t-1)d_G(x,y)}$ remaining pebbles. Distribute $2^{(2^t-1)d_G(x,y)}$ pebbles over the vertices of $S(G,t)$ in such a way that each $S_i(G,t)$ has at least $n^{t-2}f(G)+1$ pebbles. Consequently after a pebbling move at least two pebbles can be placed on every binding vertex of $S_i(G,t), i \in \{2,3,\dots,n\}$. This allows to transmit pebbles from some $S_i(G,t)$ to $S_k(G,t)$ and sequentially a pebble is moved to the target by pebbling moves.
Suppose $f(S_i(G,t)) < n^{t-2}f(G)+1, i \in \{2,3,\dots,n\}$. For instant, consider the worst case possibility where there are no pebbles on any copy of $S_j(G,t), j \neq 1$. If $d_G(x,y)$ is the distance between the vertices x and y in G then $d_G(x^t,y^t) = (2^t-1)d_G(x,y)$. Let us place $2^{(2^t-1)d_G(x,y)}$ pebbles on some vertex $x^t \in S_i(G,t)$. By a sequence of pebbling moves one pebble is moved to the vertex $y^t \in S_1(G,t)$. Further to place a pebble on the target consider the pebbles on $S_1(G,t)$. It is concluded that with at least $2^{(2^t-1)d_G(x,y)} + n^{t-2}f(G)$ pebbles it is always possible to pebble the target. □

In Generalized Sierpiński graphs $S(G,t)$ when G is a complete graph we obtain Sierpiński graphs $S(K_n,t)$. Sierpiński graphs have some interesting structural properties such as planarity, hamiltonicity and connectivity. There are also findings related to the existence of perfect codes, crossing number, metric properties, domination, number of spanning trees and matchings [6]. In this paper, we obtain a sharp bound for pebbling number of Sierpiński graphs based on its topological properties.

Theorem 2. *For $t=2$ and $n \geq 3$, $f(S(K_n,2)) = nf(K_n)$.*

Proof. The Sierpiński graphs $S(K_n,2)$ contains n copies of $S(K_n,1)$ which are referred as $S_1(K_n,2), S_2(K_n,2), \dots, S_n(K_n,2)$. Without loss of generality, we assume $x = 11 \in V(S_1(K_n,2))$ to be the target vertex. Let $p(u)$ denote the number of pebbles on the vertex u, for $u \in S_i(K_n,2)$, $1 \leq i \leq n$. Let p_i be the pebbling number of $S_i(K_n,2)$ for $1 \leq i \leq n$. Since each $S_i(K_n,2)$ for $1 \leq i \leq n$ is a complete graph on n vertices it follows that $p_i = f(S_i(K_n,2)) = f(K_n), 1 \leq i \leq n$.
Initially if $n=3$, let $p_1 = 2, p_2 = 1, p_3 = 1$ be the distribution of pebbles on $S(K_3,2)$. This means that pebbles from $S_2(K_3,2)$ and $S_3(K_3,2)$ cannot be transmitted to the target vertex in $S_1(K_3,2)$. It implies that if $f(S(K_n,2)) < nf(K_n)$ then pebbling the target is not possible. Therefore, $f(S(K_n,2)) \geq nf(K_n)$.
By distributing $nf(K_n)$ pebbles on the vertices of $S(K_n,2)$ there are various possibilities which arise due to the pebbling move. We discuss elaborately these possibilities. If $p_1 \geq f(K_n)$ or $p(x) = 1$, then it is trivial to see that the target can be pebbled. We now assume that $p_1 < f(K_n)$ and $p(x) = 0$. Assume that the least possibility of distribution $p_1 = 1$ and that one pebble is placed on any of the binding vertex of $S_1(K_n,2)$ in order to facilitate the pebbling move. Since $p_1 < f(K_n)$, pebbling move is not possible in $S_1(K_n,2)$. In this case, pebbles have to be extracted from the sets $S_2(K_n,2), S_3(K_n,2) \dots S_n(K_n,2)$ to pebble the target vertex x.

Suppose there are $(n-1)f(K_n)$ pebbles which are distributed over the vertices of the Sierpiński graphs $S_2(K_n,2), S_3(K_n,2), \ldots, S_n(K_n,2)$, such that $p_2+p_3+\cdots+p_n \geq (n-1)f(K_n)$. Now assume $p_i > f(K_n)$ for every p_i, $2 \leq i \leq n$. Then subsequently two pebbles are moved to the vertex $i1 \in S_i(K_n,2)$, for $2 \leq i \leq n$ and as a consequence one pebble is transmitted from every binding vertices of $S_i(K_n,2)$ to the binding vertices of $S_1(K_n,2)$ so that the target can be pebbled.

Next assume that there are some vertices in $S_i(K_n,2), 2 \leq i \leq n$, for which p_i is not greater than $f(K_n)$. But $p_2+p_3+\cdots+p_n \geq (n-1)f(K_n)$. Hence it can be concluded that there exists at least one $p_i, 2 \leq i \leq n$ such that $p_i = f(K_n)$. Hence with this n pebbles in $S_i(K_n,2)$ one pebble is moved to the vertex $i1 \in S_i(K_n,2)$. Further pebbling move from $S_i(K_n,2)$ to $S_1(K_n,2)$ is not possible. Excluding the pebbles considered on $S_i(K_n,2)$ and $S_1(K_n,2)$, there are at least $(n-1)f(K_n) - p_i$ pebbles distributed over the vertices of $S(K_n,2)$. Therefore it is possible to move at least four pebbles to some vertex say lj of $S_l(K_n,2)$, $l \in 2,3\ldots i-1, i+1\ldots n, 2 \leq j \leq n, l \neq j$. It implies that $p(lj) \geq 4$. Since each $S_l(K_n,2)$ is a complete graph, it is easy to move two pebbles to $l1 \in S_l(K_n,2)$ such that $p(l1) = 2$. Now through a transmitting subgraph $\{lj, l1, 1l, 11\}$ one pebble is transmitted from the binding vertex of $S_l(K_n,2)$ to the corresponding binding vertex of $S_1(K_n,2)$ which has already a pebble on it. On the other hand, there is another possibility of pebbling move which occurs between the binding vertices of $S_l(K_n,2)$ to some $S_j(K_n,2)$ and to $S_1(K_n,2)$ which can be exhibited through the transmitting subgraph $\{lj, jl, j1, 1j, 11\}$, for $l \neq j, 2 \leq j \leq n$ and so the target x is pebbled.

Now assume that the distribution has some $p_i < f(K_n)$, $2 \leq i \leq n$. From our initial assumption there should exists some subgraphs of $S(K_n,2)$ with $p_i > f(K_n)$. Consider there are at most two subgraphs say $S_{i_1}(K_n,2)$ and $S_{i_2}(K_n,2)$ with $p_{i_1} + p_{i_2} \geq 2(f(K_n)+1)$ for $i_1, i_2 \in 2,3\ldots i-1, i+1\ldots n$. By using $p_{i_1} \geq f(K_n) + 1$ pebbles on $S_{i_1}(K_n,2)$, two pebbles can be moved to the binding vertex $i_1 1 \in S_{i_1}(K_n,2), i_1 \in 2,3\ldots i-1, i+1\ldots n$. Similarly for $p_{i_2} \geq f(K_n)+1$ after a pebbling move it infers that $p(i_2 1) = 2$ for $i_2 \in 2,3\ldots i-1, i+1\ldots n$ and $i_1 \neq i_2$. Hence from the transmitting subgraph $\{i_1 1, 1 i_1\}$ and $\{i_2 1, 1 i_2\}$ we find that $S_1(K_n,2)$ will receive a pebble from the binding vertices of $S_{i_1}(K_n,2)$ and $S_{i_2}(K_n,2)$ so that the vertex with one pebble on it will have two pebbles after a sequence of pebbling move. Hence, the target is pebbled.

Consider the case when $p_2+p_3+\cdots+p_n < (n-1)f(K_n)$. In this case, the pebbling move is possible only when each p_i, $i \in \{2,3\ldots n\}$ has at most two pebbles and in particular $p(i1) = 2$, $i \in \{2,3\ldots n\}$, which is a binding vertex of every $S_i(K_n,2)$. Hence each binding vertex of $S_i(K_n,2)$ will contribute a pebble to $S_1(K_n,2)$ and so with the remaining pebbles in $S_1(K_n,2)$ one pebble is moved to the target vertex.

The entire proof of the theorem is carried out by fixing target as $11 \in V(S_1(K_n,2))$, but it is evident to see from the adjacency of the vertices of $S(K_n,2)$ that the proof of the theorem is analogous if the target vertex is chosen as any vertex of $S_i(K_n,2)$, $1 \leq i \leq n$. Thus $f(S(K_n,2)) = nf(K_n)$. $\square$

Theorem 3. *Let $S(K_n, t)$ be the Sierpiński graph. The pebbling number of $S(K_n, t)$ for any $n \geq 3, t > 2$ is $f(S(K_n, t)) = 2^{2^t-1}$.*

Proof. The Sierpiński graph $S(K_n, t)$ and its subgraph $S_i(K_n, t)$ has n^t and n^{t-1} vertices respectively. Let p_i denote the pebbling number of $S_i(K_n, t)$ for $1 \leq i \leq n$. Suppose there are 2^{2^t-1} pebbles distributed over the vertices of $S(K_n, t)$. Without loss of generality, let us assume the target vertex as $v \in S_i(K_n, t), 1 \leq i \leq n$. If $p_i = 2^{2^t-1}$, one pebble is moved to v which is trivial. So, let us discuss the case when the target has to be pebbled from $S_j(K_n, t)$ where $i \neq j$. Assume that there are no pebbles in $S_i(K_n, t)$. Hence consider a distribution of 2^{2^t-1} pebbles over the remaining vertices of $S(K_n, t)$. Initially assume that there are at least $f(K_n)$ pebbles on each $S_k(K_n, t)$, $k \neq i$. In this case there will be additional $2^{2^t-1} - [(n-1)f(K_n)]$ pebbles, let it be placed on the vertices of $S_j(K_n, t)$. With $f(K_n) + 2^{2^t-1} - [(n-1)f(K_n)]$ pebbles on $S_j(K_n, t)$ we can move $\frac{2f(K_n)+2^{2^t-1}-nf(K_n)}{2}$ pebbles to the binding vertex $1jj\ldots j \in S_j(K_n, t)$. Hence after a sequence of pebbling move the target is pebbled. Suppose there are no pebbles on all the vertices of $S(K_n, t)$. Since the diameter of $S(K_n, t)$ is $2^t - 1$ place 2^{2^t-1} pebbles on any vertex initiating the pebbling move in order to pebble the target vertex. Thus it is proved that 2^{2^t-1} pebbles are sufficient to pebble any arbitrary chosen target vertex. □

4. Sierpiński triangle graph

Sierpiński triangle graphs are obtained from Sierpiński graphs $S(K_3, t)$ by identifying every edge of $S(K_3, t)$ that lies in no triangle. Sierpiński triangle graph denoted by S_m serves as an ideal model for both software and hardware architecture. These graphs play a vital role in the study of development strategy of DNA fractal links. Further, Sierpiński triangle graph has a significant contribution in analyzing dynamic systems and are also used as models for Sierpiński fractal antenna [15].

For $m \geq 1$, the Sierpiński triangle graph S_m is defined as the graph whose vertices are the intersection points on the line segments in Sierpiński graph $S(K_3, t)$ and edge set consisting of the line segments connecting two vertices. The Sierpiński triangle graph S_m has $(3^m + 3)/2$ vertices and 3^m edges. The diameter of S_m is 2^{m-1}. See Figure 3.

An m-dimensional Sierpiński triangle graph consists of three copies of Sierpiński triangle graph of dimension $m - 1$ with $(3^{m-1} + 3)/2$ vertices. These three copies of S_{m-1} in S_m are denoted by $S_{m,T}$, $S_{m,L}$ and $S_{m,R}$ as shown in Figure 4. The binding vertices of $S_{m,T}$, $S_{m,L}$ and $S_{m,R}$ are referred as x_1, x_2, x_3, x_4, x_5 and x_6. These binding vertices are connected by a set of vertices which are referred as extreme vertices. The extreme vertices connecting the binding vertices $\{x_1, x_2, x_3\}$, $\{x_2, x_4, x_5\}$ and $\{x_3, x_5, x_6\}$ are labeled as $x_{T,i}, x_{L,i}$ and $x_{R,i}$ respectively for $1 \leq i \leq 3(2^{m-2} - 1)$. See Figure 5.

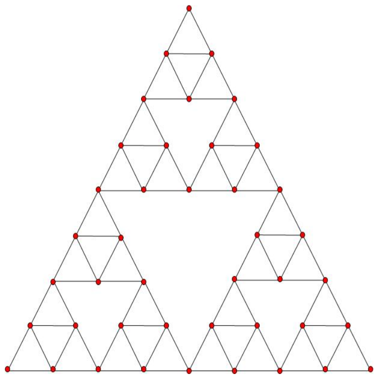

Figure 3. **Sierpiński triangle graph** S_4

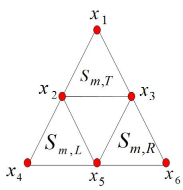

Figure 4. **Sierpiński triangle graph** S_m

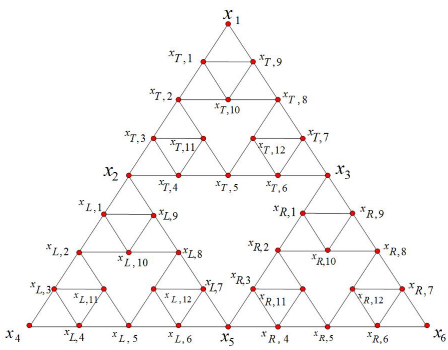

Figure 5. **Labeling of Sierpiński triangle graph** S_4

5. Pebbling number of Sierpiński triangle graph

The pebbling number of S_2 is $f(S_2) = 6$. The proof is direct since for a distribution of 5 pebbles with one pebble on each vertex except the target there is no pebbling move possible. We begin with $m > 2$.

Theorem 4. *For a Sierpiński triangle graph S_m, $m \geq 3$ $f(S_m) = 2^{2^{m-1}}$.*

Proof. Let p_1, p_2 and p_3 be the number of pebbles in $S_{m,T}$, $S_{m,L}$ and $S_{m,R}$ respectively. Let $p(x)$ be the number of pebbles on the vertex x. Choose $x_1 \in S_{m,T}$ as the target vertex. The Sierpiński triangle graph is a fractal, that is, it is a self similar structure that occurs at different levels of iterations. Hence, the proof of the theorem by choosing the target vertex as any vertex of $S_{m,R}$ and $S_{m,L}$ is carried out with the same procedure. Assume that $p(x_1) = 0$, otherwise the solution is trivial. The proof of the theorem is carried out in such a way that the target x_1 can be pebbled from any vertex of $S_{m,T}, S_{m,L}, S_{m,R}$ with a distribution of $2^{2^{m-1}}$ pebbles.

Case 1: $p_1 \geq \frac{3^{m-1}+3}{2}$
Each copy of S_{m-1} in S_m has $\frac{3^{m-1}+3}{2}$ vertices. Distribute $\frac{3^{m-1}+3}{2}$ pebbles on the vertices of $S_{m,T}$ in such a way that except the target vertex all the other vertices have at least one pebble. Then there exists minimum two vertices of $S_{m,T}$, say x_2 and x_3, with $p(x_2) \geq 2$ and $p(x_3) \geq 2$. Therefore the target can be pebbled through a transmitting subgraph which is exhibited by $\{x_2, x_{T,i}, x_{T,i-1}, x_{T,i-2} \ldots x_{T,1}, x_1\}$, $i = 2^{m-2} - 1$ or through a transmitting subgraph $\{x_3, x_{T,i}, x_{T,i+1}, x_{T,i+2}, \ldots x_{T,3(2^{m-2}-1)}, x_1\}$, $i = 2^{m-1} - 1$. If $p(x_{T,i}) \geq 2$, for either $i = 2^{m-2} - 1$ or $i = 2^{m-1} - 1$, the target can be pebbled using the transmitting subgraph $\{x_{T,i}, x_{T,i-1}, x_{T,i-2} \ldots x_{T,1}, x_1\}$ or $\{x_{T,i}, x_{T,i+1}, x_{T,i+2} \ldots x_{T,3(2^{m-2}-1)}, x_1\}$.
Let $S = \{x/p(x) = 0, x \in V(S_{m,T})\}$. If $1 \leq |S| \leq \frac{3^{m-1}-1}{2}$, then there should exists at least four pebbles distributed on vertices $y \in S^C$, such that $p(y) > 0$, where x and y are adjacent. Now, we consider a pebbling move in such a way that, the vertex y will retain two pebbles on it and out of the remaining two pebbles it will contribute a pebble to the vertex x. In this way, we can see that the target is pebbled through a transmitting subgraph which is exhibited by $\{x_{T,i}, x_{T,i-1}, x_{T,i-2} \ldots x_{T,1}, x_1\}$ or through a transmitting subgraph $\{x_{T,i}, x_{T,i+1}, x_{T,i+2} \ldots x_{T,3(2^{m-2}-1)}, x_1\}$.
If $|S| = \frac{3^{m-1}+1}{2}$, then we need to consider the entire distribution of pebbles stacked at a particular vertex either on x_i or $x_{T,i}$ so that the target is pebbled along its path of length $n \in \{1, 2, \ldots, m\}$. Hence for some i, $p(x_i) = \frac{3^{m-1}+5}{2}$ or $p(x_{T,i}) = \frac{3^{m-1}+5}{2}$, we have one pebble moved to the vertex $x_{T,i}$ which is adjacent to x_1. In order to pebble the target vertex the remaining pebbles considered on p_1 are placed on $x_{T,i}$ which facilitates a pebbling move to the target x_1.

Case 2: $p_1 < \frac{3^{m-1}+3}{2}$
Due to insufficiency of pebbles in $S_{m,T}$, pebbling move cannot be initiated and pebbling x_1 is not possible. In order to pebble the target vertex, pebbles are extracted from $S_{m,L}$ and $S_{m,R}$ which leads to the following subcases.

Subcase 2.1. $p_2 \geq \frac{3^{m-1}+3}{2}$ and $p_3 \geq \frac{3^{m-1}+3}{2}$
The distribution of $\frac{3^{m-1}+3}{2}$ pebbles over the vertices of $S_{m,L}$ and $S_{m,R}$ allows two pebbles moved to $x_5 \in S_{m,L} \cap S_{m,R}$. Hence $p(x_5) = 2$ will initiate the pebbling move

either in $S_{m,L}$ or $S_{m,R}$.
Let $T = \{u/p(u) = 0, u \in V(S_{m,L})\}$. If $|T| = \phi$, then there exists at least one vertex r with $p(r) \geq 2$ and through a sequence of pebbling move two pebbles are transmitted either to the vertex x_2 or x_3. Hence $p(x_2) \geq 2$ or $p(x_3) \geq 2$ allows at least two pebbles transmitted to $S_{m,T}$ which will trigger the pebbling move within the vertices of $S_{m,T}$ and the target is pebbled as in Case 1.
Let $1 \leq |T| \leq \frac{3^{m-1}-1}{2}$. In order to have a pebbling move a distribution of $p_2 + p_3 \geq 3^{m-1} + 3$ either on $S_{m,L}$ or $S_{m,R}$ is considered. On distributing $3^{m-1} + 3$ pebbles there exists at least three pebbles on every vertex of $S_{m,L}$ or $S_{m,R}$ and so the target is pebbled as discussed in Case 1.

Subcase 2.2. $p_2 \geq \frac{3^{m-1}+3}{2}$ and $p_3 < \frac{3^{m-1}+3}{2}$
It is observed that, $x_2 \in S_{m,T} \cap S_{m,L}$. If $p(x_2) \geq 2$, then one pebble is transmitted to $S_{m,T}$ and target vertex will be pebbled as discussed in Case 1. In the remaining subcases, assume that $p(x_2) < 2$.

Subcase 2.2.1. $p(x_2) = 0$
Suppose $V(S_{m,L}) = T^C$. With $p_2 \geq \frac{3^{m-1}+3}{2}$ pebbles on $S_{m,L}$, $p(x_2) \geq 2$ as discussed in Case 2.1. Now consider the case when there exists some vertex $u \in S_{m,L}$ for which $p(u) = 0$. First, let us assume that there are pebbles distributed only on the extreme vertices of $S_{m,L}$ and there are no pebbles on the inner vertices of $S_{m,L}$. Now x_2 can receive minimum two pebbles from a series of pebbling move along the extreme vertices through the transmitting subgraphs $\{x_4, x_{L,i}, x_{L,i-1}, x_{L,i-2} \ldots x_{L,1}, x_2\}$, $i = 2^{m-2}-1$ and $\{x_5, x_{L,i}, x_{L,i+1}, x_{L,i+2} \ldots x_{L,3(2^{m-2}-1)}, x_2\}$, $i = 2^{m-1}-1$. Thus $p_1 \geq \frac{3^{m-1}+3}{2}$ and x_1 is pebbled as in Case 1.
Next, let us assume that there are no pebbles on the extreme vertices and $\frac{3^{m-1}+3}{2}$ pebbles are distributed over the inner vertices such that the vertex initiating the pebbling move should have at least two pebbles and all the other vertices $v \in T^C$ has at least one pebble. The path from some inner vertex $x_{L,i}$ to the binding vertex x_2 has at least $m-2$ extreme vertices. As the pebbling move starts from some inner vertex $x_{L,i}$ the transmitting path stops as it reaches some extreme vertex as there are no pebbles on it. Hence to overcome this situation we may assume that $p(x_{L,i}) \geq 3$ for all the inner vertices which are adjacent to the extreme vertices in particular which are included in the transmitting path. Further, in order to facilitate the pebbling move there should exists at least one pebble on these $m-2$ extreme vertices. In this way, $p(x_2) \geq 2$ and the target is pebbled.

Subcase 2.3. $p_2 < \frac{3^{m-1}+3}{2}$ and $p_3 \geq \frac{3^{m-1}+3}{2}$
This case is an analogy to Case 2.2.

Subcase 2.4. $p_2 < \frac{3^{m-1}+3}{2}$ and $p_3 < \frac{3^{m-1}+3}{2}$
Here we assume the worst case when there are no pebbles on the vertices of S_m. In order to pebble the target from some vertex x_i let $p(x_i) = 2^{2^{m-1}}$. Thus the target

$x_1 \in S_{m,T}$ can be pebbled through a sequence of pebbling move.
Finally, pebbling the target vertex for any $x_{T,i} \in S_{m,T}$, $1 \leq i \leq \frac{3^{m-1}-3}{2}$, is same as pebbling the target vertex $x_1 \in S_{m,T}$. Thus, we have obtained the pebbling number for Sierpiński triangle graph as $f(S_m) = 2^{2^{m-1}}$, for $m \geq 3$. □

6. Conclusion

The pebbling number of Sierpiński graphs can be applied by biologist and chemist to develop synthesis strategies specifically arranging the DNA crystals in the pattern of Sierpiński triangle to obtain an efficient growth mechanism through pebbling strategies and further enhances the system information regarding DNA nano structures. In this paper, we have obtained the lower bound for pebbling number of Generalized Sierpiński graph $S(G,t), t \geq 2$ and pebbling number of Sierpiński graph $S(K_n,t)$ for any $t \geq 1$ and $n \geq 2$, Sierpiński triangle graph S_m, $m \geq 2$. The problem of determining the pebbling number of other variants of Sierpiński graphs are open.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability. Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

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