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Research Article

On some topological indices of Mycielskian graph and its complement

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Abstract: In this paper, we obtain a formula for the Harary index and hyper-Wiener index of Mycielskian of G, $\mu(G)$, and complement of $\mu(G)$. More precisely, we determine a formula for the hyper-Wiener index of $\mu(G)$ in terms of Zagreb indices of G if the girth of G is greater than 6 and we deduce the result in [M. Azari in Discrete Math. Algorithms and Appl. 09 (2017) 1750022]. In addition, we find a formula for the vertex Padmakar-Ivan index of $\mu(G)$ if the girth of G is greater than 7 and the complement of $\mu(G)$.

Keywords: Mycielskian of a graph, Wiener index, vertex Padmakar-Ivan index, Harary index, hyper-Wiener index, Zagreb indices.

AMS Subject classification: 05C09, 05C12, 05C25

1. Introduction

All graphs considered in this paper are finite, simple, undirected, and connected. Let $G = (V(G), E(G))$ be a graph, where $V(G)$ is the vertex set and $E(G)$ is the edge set of G. For each $v \in V(G)$, *degree* of v in G, denoted by $d_G(v)$, is the number of edges incident with v. For a positive integer ℓ, a graph in which every vertex has degree ℓ is called an *ℓ-regular* graph. For a positive integer n, K_n, P_n, and C_n denote the *complete* graph on n vertices, path of order n, and cycle of length n, respectively. The *distance* between u and v in G, denoted by $d_G(u, v)$, is the length of a shortest path between u and v in G, the *open neighbourhood* of u in G is $N_G(u) = \{x \in V(G) : ux \in E(G)\}$, the *closed neighbourhood* of u in G is $N_G[u] = \{u\} \cup N_G(u)$. The *diameter* of a graph G, $diam(G)$, is defined as $max\{d_G(u, v) : u, v \in V(G)\}$. The *girth* of G, denoted by $g(G)$, is the length of a shortest cycle in G. For a non-empty subset T of $V(G)$, $\langle T\rangle$ denotes the subgraph induced by T. The *complement* of the graph G, denoted by G^c, is a graph with $V(G^c) = V(G)$ and $E(G^c) = \{xy : xy \notin E(G)\}$.

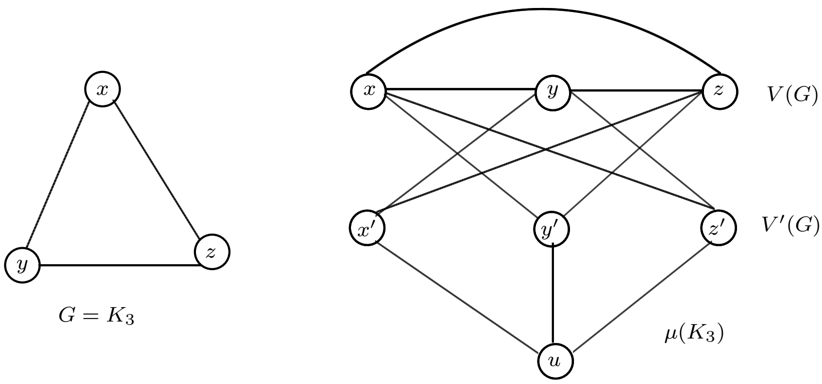

Figure 1. **Mycielskian of K_3**

For $u, v \in V(G)$, define $d_G(u \wedge v) = |N_G(u) \cap N_G(v)|$. For $uv \in E(G)$, define $N_u(uv|G) = \{x \in V(G) : d_G(u, x) < d_G(v, x)\}$ and $n_u(uv|G) = |N_u(uv|G)|$ the number of vertices of G whose distance to the vertex u is smaller than the distance to the vertex v in G. A vertex $x \in V(G)$ is said to be *equidistant* from the edge uv of G if $d_G(u, x) = d_G(v, x)$, otherwise it is said to be non-equidistant from an edge uv. Denote $N_G(uv)$ the set of all equidistant vertices from the edge uv, that is, $N_G(uv) = \{z \in V(G) : d_G(u, z) = d_G(v, z)\}$. A graph G is said to be a *triangle-free graph* if G does not contain a triangle (that is, a cycle of length 3).
In the mid-20th century, there was a question regarding triangle-free graphs with arbitrarily large chromatic numbers. Mycielski [18] developed an interesting graph transformation that answered the above famous question. For a graph G, the Mycielskian of G is the graph $\mu(G)$ with vertex set $V(\mu(G)) = V(G) \cup V'(G) \cup \{u\}$, where $V'(G) = \{x' : x \in V(G)\}$ and is disjoint from $V(G)$, and edge set $E(\mu(G)) = E(G) \cup \{xy' : xy \in E(G)\} \cup \{y'u : y' \in V'\}$. The vertex x' is called the shadow of the vertex x (and x the shadow of x') and the vertex u is the root of $\mu(G)$. For example, the Mycielskian of K_3 is illustrated in Figure 1. We use V in place of $V(G)$ and E in place of $E(G)$ when no ambiguity arises. The Mycielskian graph has fascinated graph theorists a great deal. As their interest, several graph parameters of $\mu(G)$ have been studied in the past, see [2, 4, 8]. Note that, if G has no isolated vertices, then $\mu(G)$ and $\mu^c(G)$ are connected.
A topological index is a molecular descriptor calculated from a molecular graph of a chemical compound that characterizes its topology. Topological indices are very useful tools in graph theory and mathematical chemistry. We consider the following well-known topological indices namely, Wiener index, Harary index, hyper-Wiener index, vertex Padmakar-Ivan index, and Zagreb indices. Some of the chemical applications of topological indices are reported in [6, 7, 21, 22, 24].
The Wiener index is one of the significant and oldest topological indices used in

mathematical chemistry. It was introduced in 1947 by Wiener [24]. For a connected graph G, the *Wiener index* of G is the sum of the distance between any two unordered pairs of vertices of G, that is,

$$W(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u,v) = \frac{1}{2}\sum_{u,v\in V(G)} d_G(u,v).$$

It was used for modeling the shape of organic molecules and for calculating several of their physico-chemical properties, in particular the boiling points of alkane isomers. There have been several papers on the Wiener index from the time of its introduction. Some of the related references are [2, 8]. The *Harary index* of a connected graph G is the sum of the reciprocal of the distances of any two unordered pair of vertices of G, that is,

$$\mathcal{H}(G) = \sum_{\{u,v\}\subseteq V(G)} \frac{1}{d_G(u,v)} = \frac{1}{2}\sum_{\substack{u,v\in V(G)\\ u\neq v}} \frac{1}{d_G(u,v)}.$$

The Harary index of a graph has been introduced independently by Plavšić et al. [21] and by Ivanciuc et al. [11] in 1993. The *hyper-Wiener* index is the generalization of the Wiener index. This index was introduced by Randic [22] in 1993. For a connected graph G, the hyper-Wiener index of the graph G is one-half of the summation of distances and square distances over all its unordered vertex pairs u, v, that is,

$$WW(G) = \frac{1}{2}\sum_{\{u,v\}\subseteq V(G)} \Big[d_G(u,v) + d_G^2(u,v)\Big] = \frac{1}{2}W(G) + \frac{1}{2}\sum_{\{u,v\}\subseteq V(G)} d_G^2(u,v),$$

where $d_G^2(u,v) = (d_G(u,v))^2$. In [15], Khalifeh et al. computed an exact formula for the hyper-Wiener index of various graph operations. The hyper-Wiener index has been studied extensively in the past, see some of the references [3, 14, 23]. The *vertex Padmakar-Ivan index* of a graph G, denoted by $PI_v(G)$, is defined as

$$PI_v(G) = \sum_{uv\in E(G)} [n_u(uv|G) + n_v(uv|G)].$$

In other words, $PI_v(G)$ is the sum over all edges uv of G the number of vertices which are non-equidistant from an edge uv, that is,

$$PI_v(G) = \sum_{uv\in E(G)} \Big(|V(G)| - |N_G(uv)|\Big).$$

The vertex Padmakar-Ivan index was introduced by Khalifeh et al. [15]. In short, we use the PI index instead of the vertex Padmakar-Ivan index. Several authors have studied the PI index of various classes of graphs. For instance, see [5, 12, 13, 16, 17, 20]. The *first Zagreb index* $M_1(G)$ is equal to the sum of squares of the degrees of the vertices. The *second Zagreb index* $M_2(G)$ is equal to the sum of the products of the degrees of pairs of adjacent vertices of the graph G. The Zagreb indices were introduced by Gutman et al. [10]. They are defined as

$$M_1(G) = \sum_{uv\in E(G)} \left[d_G(u) + d_G(v)\right] = \sum_{v\in V(G)} d_G^2(v), M_2(G) = \sum_{uv\in E(G)} d_G(u)d_G(v).$$

The PI index and Zagreb index are well-studied topological indices, both from a theoretical point of view and applications, see [9, 19].
This paper is organized as follows. In Section 2, we recall some properties of $\mu(G)$ and its complement. Also, we recall the results in [1, 2, 8]. In Section 3, we compute a formula for the Harary index of the Mycielskian of G. Using this result, we deduce a result in [1]. Moreover, we obtain a formula for the Harary index of the complement of $\mu(G)$. In Section 4, we find a formula for the hyper-Wiener index of Mycielskian of G. Also, we provide a formula for the hyper-Wiener index of Mycielskian of a graph with $g(G) \geq 7$ in terms of Zagreb indices of G. In Section 5, we determine a formula for the PI index of the Mycielskian of G if $g(G) \geq 8$, and we find a formula for the PI index of the complement of $\mu(G)$.

2. Preliminaries

Let us first recall the following observations.

Observation 1. If $a \in V(\mu(G))$, then

$$d_{\mu(G)}(a) = \begin{cases} n, & \text{for } a = u; \\ d_G(x) + 1, & \text{for } a = x' \in V'; \\ 2d_G(x), & \text{for } a = x \in V. \end{cases}$$

and for any two distinct vertices a and b in $\mu(G)$,

$$d_{\mu(G)}(a,b) = \begin{cases} 1, & \text{for } a = x' \in V',\ b = u; \\ 2, & \text{for } a = x \in V,\ b = u; \\ 2, & \text{for } a = x' \in V', b = y' \in V'; \\ d_G(x,y), & \text{for } a = x \in V, b = y \in V, d_G(x,y) \leq 3\ ; \\ 4, & \text{for } a, b \in V, d_G(a,b) \geq 4; \\ 2, & \text{for } a = x \in V, b = x' \in V'; \\ d_G(x,y), & \text{for } a = x \in V\ ,\ b = y' \in V', y' \neq x', d_G(x,y) \leq 2; \\ 3, & \text{for } a = x \in V,\ b = y' \in V'\ ,\ y' \neq x', d_G(x,y) \geq 3. \end{cases}$$

Hence, the diameter of Mycielskian of G is at most 4.

Observation 2. If $a \in V(\mu^c(G))$, then

$$d_{\mu^c(G)}(a) = \begin{cases} n, & \text{for } a = u; \\ 2n - (d_G(x) + 1), & \text{for } a = x' \in V'; \\ 2n - 2d_G(x), & \text{for } a = x \in V. \end{cases}$$

and for any two distinct vertices a, b in $\mu^c(G)$,

$$d_{\mu^c(G)}(a,b) = \begin{cases} 2, & \text{for } a = x' \in V',\ b = u; \\ 1, & \text{for } a = x \in V,\ b = u; \\ 1, & \text{for } a = x' \in V',\ b = y' \in V'; \\ 1, & \text{for } a = x \in V,\ b = y \in V,\ d_G(x,y) > 1; \\ 2, & \text{for } a = x \in V,\ b = y \in V,\ d_G(x,y) = 1; \\ 1, & \text{for } a = x \in V,\ b = x'; \\ 1, & \text{for } a = x \in V,\ b = y' \in V'\ ,\ y' \neq x', d_G(x,y) > 1; \\ 2, & \text{for } a = x \in V,\ b = y' \in V'\ ,\ y' \neq x', d_G(x,y) = 1. \end{cases}$$

Hence, the diameter of the complement of Mycielskian of G, $\mu^c(G)$ is 2.
Now, we recall the following Lemma in [8].

Lemma 1 ([8]). *Let G be a graph with n vertices and m edges. Then*
(i) The number of paths of order two in G is equal to $\frac{1}{2}M_1(G) - m$.
(ii) If G is a triangle-free graph, then the number of paths of order three in G is equal to $M_2(G) - M_1(G) + m$.

Next, we recall the following results in [1]. Let G be a graph with $V(G) = \{x_1, x_2, \ldots, x_n\}$ and m edges. For each positive integer k, define

$$A_k(G) = \Big\{\{i,j\} \subseteq \{1,2,\ldots,n\} : d_G(x_i,x_j) = k\Big\}$$

and

$$A(G) = \Big\{\{i,j\} \subseteq \{1,2,\ldots,n\} : d_G(x_i,x_j) \geq 4\Big\}.$$

Clearly,

$$|A_1(G)| = m. \tag{2.1}$$

If $g(G) \geq 5$, part (i) of Lemma 1 yields

$$|A_2(G)| = \frac{1}{2}M_1(G) - m, \tag{2.2}$$

and if $g(G) \geq 7$, part (ii) of Lemma 1 yields

$$|A_3(G)| = M_2(G) - M_1(G) + m. \tag{2.3}$$

Therefore,

$$|A(G)| = \binom{n}{2} - |A_1(G)| - |A_2(G)| - |A_3(G)|.$$

So if $g(G) \geq 7$, by Equations (2.1)–(2.3), we obtain

$$|A(G)| = \binom{n}{2} - m + \frac{1}{2}M_1(G) - M_2(G). \tag{2.4}$$

For $1 \leq k \leq 3$, we define

$$M_1^{(k)}(G) = \sum_{\{i,j\} \in A_k(G)} \Big[d_G(x_i) + d_G(x_j)\Big],$$

$$M_2^{(k)}(G) = \sum_{\{i,j\} \in A_k(G)} d_G(x_i) d_G(x_j).$$

Obviously, $M_1^{(1)}(G) = M_1(G)$ and $M_2^{(1)}(G) = M_2(G)$. If G is an ℓ-regular graph with $g(G) \geq 7$, then

$$|A_k(G)| = \frac{n\ell}{2}(\ell - 1)^{k-1}, 1 \leq k \leq 3. \tag{2.5}$$

Hence for $1 \leq k \leq 3$,

$$M_1^{(k)}(G) = n\ell^2(\ell - 1)^{k-1} \text{ and } M_2^{(k)}(G) = \frac{n\ell^3}{2}(\ell - 1)^{k-1}. \tag{2.6}$$

For a positive integer k, $p(k, G)$ is the number of pairs of vertices which are at distance k in G.

Observation 3. If G is any graph, then $|A_k(G)| = p(k, G)$, for $1 \leq k \leq 3$ and $|A(G)| = \sum_{k \geq 4} p(k, G)$.

Next, we recall the following result proved in [2].

Theorem 4 ([2]). *Let G be a connected graph with n vertices and m edges. Then the Wiener index of the Mycielskian of G is given by*

$$W(\mu(G)) = 6n^2 - n - 7m - 4p(2, G) - p(3, G). \tag{2.7}$$

3. Harary index of $\mu(G)$ and $\mu^c(G)$

In [1], the authors obtained a formula for the Harary index of $\mu(G)$ when $g(G) \geq 7$. In this section, we find a formula for the Harary index of Mycielskian of G without assuming $g(G) \geq 7$ and complement of $\mu(G)$. The proof of the following result is similar to the one given in [2]. For the sake of completion, we give the proof.

Theorem 5. *Let G be a graph with n vertices and m edges. Then the Harary index of $\mu(G)$ is given by*

$$\mathcal{H}(\mu(G)) = \frac{1}{24}\Big[17n^2 + 31n + 50m + 14p(2,G) + 2p(3,G)\Big].$$

Proof. By the definition of Harary index of $\mu(G)$,

$$\mathcal{H}(\mu(G)) = \frac{1}{2}\sum_{\substack{x,y\in V(\mu(G))\\ x\neq y}} \frac{1}{d_{\mu(G)}(x,y)} = \sum_{\{x,y\}\subseteq V(\mu(G))} \frac{1}{d_{\mu(G)}(x,y)}. \tag{3.1}$$

For each $\{x, y\} \subseteq V(\mu(G))$, we have five cases, namely, 1) $x' \in V'(G)$, $y = u$; 2) $x \in V(G)$, $y = u$; 3) $x' \in V'(G)$, $y' \in V'(G)$; 4) $x \in V(G)$, $y \in V(G)$; 5) $x \in V(G)$, $y' \in V(G)$. Hence, the summation in Equation (3.1) can be divided into five sums as follows.

$$\begin{aligned}\mathcal{H}(\mu(G)) &= \sum_{x'\in V',\ y=u} \frac{1}{d_{\mu(G)}(x',u)} + \sum_{x\in V,\ y=u} \frac{1}{d_{\mu(G)}(x,u)} + \frac{1}{2}\sum_{x',y'\in V'} \frac{1}{d_{\mu(G)}(x',y')} \\ &\qquad + \tfrac{1}{2}\sum_{x,y\in V} \frac{1}{d_{\mu(G)}(x,y)} + \sum_{x\in V, y'\in V'} \frac{1}{d_{\mu(G)}(x,y')} \\ &= \textstyle\sum_1 + \sum_2 + \sum_3 + \sum_4 + \sum_5 \ \text{(say)}.\end{aligned} \tag{3.2}$$

It is easy to observe that $\sum_1 = n$, $\sum_2 = \frac{1}{2}(n)$, $\sum_3 = \frac{1}{2}\binom{n}{2} = \frac{n^2-n}{4}$. Since the maximum distance of any pair of vertices in $V(G)$ is 4 in $\mu(G)$, $\sum_4 = p(1,G) + \frac{1}{2}p(2,G) + \frac{1}{3}p(3,G) + \frac{1}{4}\big[\binom{n}{2} - p(1,G) - p(2,G) - p(3,G)\big]$. Note that if $xy \in E(G)$, then xy', $yx' \in V(\mu(G))$. Also, for every $x \in V(G)$, $d_{\mu(G)}(x,x') = 2$ and for every $x, y \in V(G)$ such that $d_G(x,y) = 2$, we have $d_{\mu(G)}(x,y') = 2, d_{\mu(G)}(y,x') = 2$. Thus $\sum_5 = \frac{1}{2}n + 2\big[p(1,G) + \frac{1}{2}p(2,G)\big] + \frac{1}{3}\big[n^2 - n - 2p(1,G) - 2p(2,G)\big]$.
Then by equation (3.2), the result follows. □

By Equations (2.2) and (2.3) and Observation 3, we deduce the result in [1].

Corollary 1 (Theorem 3.3, [1]). *Let G be a graph on n vertices, m edges with $g(G) \geq 7$. Then*

$$\mathcal{H}(\mu(G)) = \frac{1}{24}\big[5M_1(G) + 2M_2(G) + 17n^2 + 31n + 38m\big].$$

Proof. By Theorem 5,

$$\mathcal{H}(\mu(G)) = \frac{1}{24}\Big[17n^2 + 31n + 50m + 14p(2,G) + 2p(3,G)\Big].$$

By Observation 3, $p(2,G) = |A_2(G)|$ and $p(3,G) = |A_3(G)|$. By Equations (2.2) and (2.3), we have $p(2,G) = \frac{1}{2}M_1(G) - m$ and $p(3,G) = M_2(G) - M_1(G) + m$. Hence the result. □

Next, we derive a formula for the Harary index of the complement of the Mycielskian of G. Let us first observe the following.

Observation 6.

$$p(i,\mu^c(G)) = \begin{cases} 2n^2 - 3m, & \text{if } i = 1; \\ 3m + n, & \text{if } i = 2. \end{cases} \tag{3.3}$$

Proposition 1. *If G is graph on n vertices and m edges, then the Wiener index of $\mu^c(G)$ is given by*

$$W(\mu^c(G)) = 2n^2 + 2n + 3m. \tag{3.4}$$

Proof. By the definition of Wiener index of $\mu^c(G)$,

$$W(\mu^c(G)) = \sum_{\{x,y\}\subseteq V(\mu^c(G))} d_{\mu^c(G)}(x,y). \tag{3.5}$$

Since the diameter of $\mu^c(G)$ is 2, then

$$W(\mu^c(G)) = \sum_{\{x,y\}\in A_1(\mu^c(G))} d_{\mu^c(G)}(x,y) + \sum_{\{x,y\}\in A_2(\mu^c(G))} d_{\mu^c(G)}(x,y).$$

By Observation 3,

$$\begin{aligned} W(\mu^c(G)) &= p(1,\mu^c(G))(1) + p(2,\mu^c(G))(2) \\ &= \big(2n^2 - 3m\big)(1) + \big(3m + n\big)(2), \text{ by Equation (3.3).} \end{aligned}$$

Therefore, the result follows. □

By Observation 3, we find a formula for the Harary index of $\mu^c(G)$.

Theorem 7. *Let G be the graph with n vertices and m edges. Then the Harary index of $\mu^c(G)$ is given by*

$$\mathcal{H}(\mu^c(G)) = \frac{1}{2}\big[4n^2 + n - 3m\big]. \tag{3.6}$$

Proof. Note that diameter of $\mu^c(G)$ is 2. Then

$$\mathcal{H}(\mu^c(G)) = \sum_{\{x,y\}\in A_1(\mu^c(G))} \frac{1}{d_{\mu^c(G)}(x,y)} + \sum_{\{x,y\}\in A_2(\mu^c(G))} \frac{1}{d_{\mu^c(G)}(x,y)}.$$

By Observation 3,

$$\begin{aligned}\mathcal{H}(\mu^c(G)) &= p(1,\mu^c(G))(1) + p(2,\mu^c(G))\left(\frac{1}{2}\right)\\ &= \big(2n^2-3m\big)(1) + \big(3m+n\big)\left(\frac{1}{2}\right), \text{ by Equation (3.3).}\end{aligned}$$

Hence, the result follows. □

4. Hyper-Wiener index of $\mu(G)$ and $\mu^c(G)$

In this section, we determine a formula for the hyper-Wiener index of Mycielskian of G and complement of $\mu(G)$.

Theorem 8. *If G is a graph with n vertices and m edges, then the hyper-Wiener index of $\mu(G)$ is given by*

$$WW(\mu(G)) = \frac{1}{2}\big[25n^2 - 11n - 38m - 26p(2,G) - 8p(3,G)\big]. \tag{4.1}$$

In particular, if $g(G) \geq 7$, then the hyper-Wiener index of $\mu(G)$ is given by

$$WW(\mu(G)) = \frac{1}{2}\big[25n^2 - 11n - 20m - 5M_1(G) - 8M_2(G)\big]. \tag{4.2}$$

Proof. By the definition of the hyper-Wiener index of Mycielskian of G,

$$\begin{aligned}WW(\mu(G)) &= \frac{1}{2}\sum_{\{x,y\}\subseteq V(\mu(G))} d_{\mu(G)}(x,y) + \frac{1}{2}\sum_{\{x,y\}\subseteq V(\mu(G))} d^2_{\mu(G)}(x,y)\\ &= \frac{1}{2}W(\mu(G)) + \frac{1}{2}\sum_{\{x,y\}\subseteq V(\mu(G))} d^2_{\mu(G)}(x,y).\end{aligned}$$

By Theorem 4,

$$WW(\mu(G)) = \frac{1}{2}\big[6n^2 - n - 7m - 4p(2,G) - p(3,G)\big] + \frac{1}{2}\sum_{\{x,y\}\subseteq V(\mu(G))} d^2_{\mu(G)}(x,y). \tag{4.3}$$

Then,

$$\sum_{\{x,y\}\subseteq V(\mu(G))} d^2_{\mu(G)}(x,y) = \sum_{x'\in V'(G),\ y=u} d^2_{\mu(G)}(x',u) + \sum_{x\in V(G), y=u} d^2_{\mu(G)}(x,u) +$$
$$\frac{1}{2}\sum_{x',y'\in V'(G)} d^2_{\mu(G)}(x',y') + \frac{1}{2}\sum_{x,y\in V(G)} d^2_{\mu(G)}(x,y) + \sum_{x\in V(G), y'\in V'(G)} d^2_{\mu(G)}(x,y').$$

Using the similar argument as given in Theorem 5, we get

$$\sum_{\{u,v\}\subseteq V(G)} d^2_G(u,v) = \big[19n^2 - 10n - 31m - 22p(2,G) - 7p(3,G)\big].$$

By Equation (4.3),

$$WW(\mu(G)) = \frac{1}{2}\big[6n^2 - n - 7m - 4p(2,G) - p(3,G)\big] +$$
$$\frac{1}{2}\big[19n^2 - 10n - 31m - 22p(2,G) - 7p(3,G)\big].$$

Hence the result. Next, if $g(G) \geq 7$, then by Equations (2.2) and (2.3), we have $|A_2(G)| = \frac{1}{2}M_1(G) - m$ and $|A_3(G)| = M_2(G) - M_1(G) + m$. By Observation 3, we have $p(2,G) = |A_2(G)|$ and $p(3,G) = |A_3(G)|$. Therefore the result follows from the Equation (4.1). □

By Theorem 8 and Equation (2.6), we have the following.

Corollary 2. *If G be an ℓ-regular graph with $g(G) \geq 7$, then the hyper-Wiener index of $\mu(G)$, is given by*

$$WW(\mu(G)) = \frac{1}{2}\left(25n^2 - \Big[4\ell^3 + 5\ell^2 + 10\ell + 11\Big]n\right). \tag{4.4}$$

In particular, if $n \geq 7$, $WW(\mu(C_n)) = \frac{1}{2}\left(25n^2 - 83n\right)$.

Proof. As $g(G) \geq 7$, then by Equation (4.2),

$$WW(\mu(G)) = \frac{1}{2}\big[25n^2 - 11n - 20m - 5M_1(G) - 8M_2(G)\big].$$

Since G is an ℓ-regular, then by Equation (2.6), we have $M_1^{(1)}(G) = M_1(G) = n\ell^2$ and $M_2^{(1)}(G) = M_2(G) = \frac{n\ell^3}{2}$. Therefore,

$$WW(\mu(G)) = \frac{1}{2}\left[25n^2 - 11n - 20\left(\frac{n\ell}{2}\right) - 5(n\ell^2) - 8\left(\frac{n\ell^3}{2}\right)\right].$$

Hence the result follows. Next, if G is a cycle of length of n, then we have $\ell = 2$. As $g(G) \geq 7$, then by Equation (4.4),

$$WW(\mu(G)) = \frac{1}{2}\left(25n^2 - \left[4(2)^3 + 5(2)^2 + 10(2) + 11\right]n\right).$$

Hence, the result follows. □

Next, we obtain a formula for the hyper-Wiener index of the complement of the Mycielskian of G.

Theorem 9. *If G is a graph with n vertices and m edges, then the hyper-Wiener index of $\mu^c(G)$ is given by*

$$WW(\mu^c(G)) = 2n^2 + 3n + 6m. \tag{4.5}$$

Proof. By the definition of hyper-Wiener index of $\mu^c(G)$,

$$\begin{aligned} WW(\mu^c(G)) &= \tfrac{1}{2} \sum_{\{x,y\}\subseteq V(\mu^c(G))} d_{\mu^c(G)}(x,y) + \frac{1}{2} \sum_{\{x,y\}\subseteq V(\mu^c(G))} d^2_{\mu^c(G)}(x,y) \\ &= \tfrac{1}{2} W(\mu^c(G)) + \tfrac{1}{2} \sum_{\{x,y\}\subseteq V(\mu^c(G))} d^2_{\mu^c(G)}(x,y). \end{aligned}$$

By Proposition 1,

$$WW(\mu^c(G)) = \frac{1}{2}\left[2n^2 + 2n + 3m\right] + \frac{1}{2} \sum_{\{x,y\}\subseteq V(\mu^c(G))} d^2_{\mu^c(G)}(x,y). \tag{4.6}$$

It is enough to compute the sum $\sum_{\{x,y\}\subseteq V(\mu^c(G))} d^2_{\mu^c(G)}(x,y)$. Therefore,

$$\begin{aligned} \sum_{\{x,y\}\subseteq V(\mu^c(G))} d^2_{\mu^c(G)}(x,y) &= \sum_{\{x,y\}\in A_1(\mu^c(G))} d^2_{\mu^c(G)}(x,y) + \sum_{\{x,y\}\in A_2(\mu^c(G))} d^2_{\mu^c(G)}(x,y) \\ &= \left[p(1,\mu^c(G))\right](1) + \left[p(2,\mu^c(G))\right](4), \text{ by Observation 6}, \\ &= [2n^2 - 3m](1) + [3m + n](4), \text{ by Equation (3.3)}, \\ &= 2n^2 + 4n + 9m. \end{aligned}$$

By Equation (4.6),

$$WW(\mu^c(G)) = \frac{1}{2}\left[2n^2 + 2n + 3m\right] + \frac{1}{2}\left[2n^2 + 4n + 9m\right].$$

Hence, the result follows. □

Using Theorem 9, we compute the hyper-Wiener index of $\mu^c(G)$ when G is an ℓ-regular.

Corollary 3. *If G is an ℓ-regular graph with n vertices, then the hyper-Wiener index of $\mu^c(G)$ is*

$$WW(\mu^c(G)) = 2n^2 + 3n + 3n\ell.$$

Proof. The result follows from the Equation (4.5). □

As a consequence, we have the following example.

Example 1. The hyper-Wiener index of $\mu^c(C_n)$ is given by

$$WW(\mu^c(C_n)) = 2n^2 + 9n.$$

5. PI index of $\mu(G)$ and $\mu^c(G)$

In this section, we now compute the PI index of Mycielskian of G with $g(G) \geq 8$. **Notations:** Let G be a graph. For an edge $xy \in E(G)$, denote $P = N_G(y) \setminus \{x\}$, $Q = N_G\Big(N_G(y) \setminus \{x\}\Big) \setminus \{y\} = N_G(P) \setminus \{y\}$ and $R = N_G\Big(N_G\big(N_G(y) \setminus \{x\}\big) \setminus \{y\}\Big) \setminus \Big(N_G(y) \setminus \{x\}\Big) = N_G(Q) \setminus P$. For $x \in V(G)$ and $1 \leq i \leq diam(G)$, define $V_G(i,x) = \{z \in V(G) : d_G(z,x) = i\}$. Clearly, if $g(G) \geq 8$, then $\langle V_G(i,x)\rangle$ is induced subgraph of G, for $1 \leq i \leq diam(G)$, and $P \subseteq V_G(2,x)$, $Q \subseteq V_G(3,x)$ and $R \subseteq V_G(4,x)$. This can be seen in Figure 2. If $g(G) \geq 8$, then $V_G(1,x) = N_G(x)$, $V_G(2,x) = N_G\Big(N_G(x)\Big) \setminus \{x\} = N_G(V_G(1,x)) \setminus \{x\}$ and $V_G(3,x) = N_G\Big(N_G(N_G(x)) \setminus \{x\}\Big) \setminus N_G(x) = N_G(V_G(2,x)) \setminus \{N_G(x)\}$. For $xy \in E(G)$, we use n_x and n_y to denote $n_x(xy|G)$ and $n_y(xy|G)$, respectively.

Observation 10. If $g(G) \geq 8$, then for $xy \in E(G)$, $|P| = d_G(y) - 1$,

$$|Q| = 1 - d_G(y) + \sum_{w \in P} d_G(w), \text{ and} \tag{5.1}$$

$$|R| = d_G(y) - 1 + \sum_{z \in Q} d_G(z) - \sum_{w \in P} d_G(w). \tag{5.2}$$

Hence,

$$|Q| + |R| = \sum_{z \in Q} d_G(z). \tag{5.3}$$

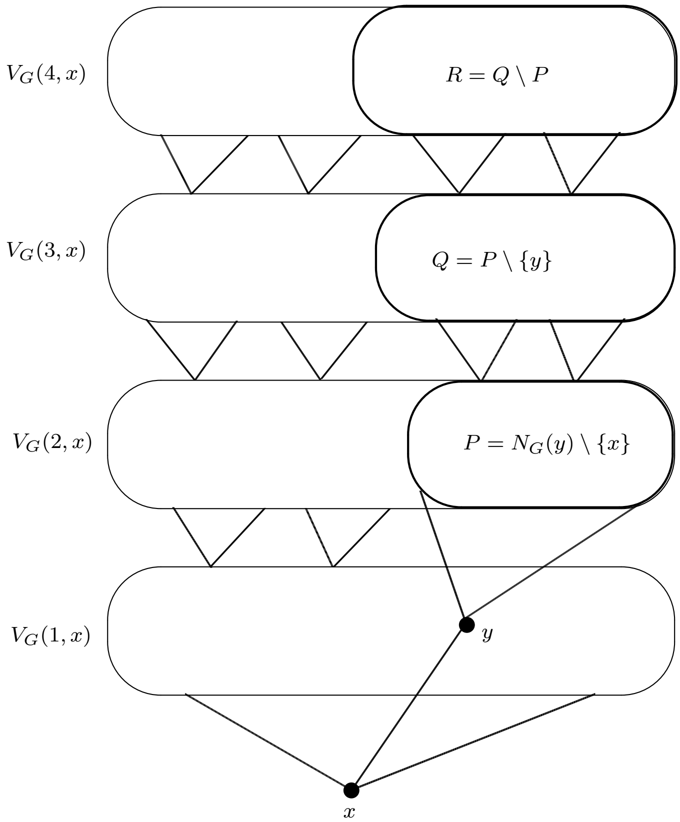

Figure 2. **The sets P, Q, R.**

For $x \in V(G)$, we can write $V(\mu(G)) = \{x, x', u\} \cup \bigcup_{i=1}^{k} V_G(i,x) \cup \bigcup_{i=1}^{k} V'_G(i,x)$, where $V'_G(i,x) = \{z' \in V'(G) : z \in V_G(i,x)\}$. Also, we define $(N_G(x))' = N'_G(x) = \{z' \in V'(G) : z \in N_G(x)\}$.

Observation 11. For $x \in V(G)$, $V_G(1,x) = N_G(x)$, $V_G(2,x) = N_G(N_G(x)) \setminus \{x\}$, $V_G(3,x) = N_G\Big(N_G(N_G(x)) \setminus \{x\}\Big) \setminus N_G(x)$ and

$$|V_G(1,x)| = d_G(x), \tag{5.4}$$

$$|V_G(2,x)| = \Big[\sum_{t \in N_G(x)} d_G(t)\Big] - d_G(x), \tag{5.5}$$

$$|V_G(3,x)| = d_G(x) + \sum_{s \in V_G(2,x)} d_G(s) - \sum_{t \in N_G(x)} d_G(t). \tag{5.6}$$

Using these observations, we prove the following result.

5.1. PI index of $\mu(G)$

In this subsection, we obtain a formula for the PI index of $\mu(G)$ when $g(G) \geq 8$. If $g(G)$ is at most 7, then $PI(\mu(G))$ becomes a complicated expression that involves many parameters in G.

Theorem 12. *If G is a connected graph on n vertices, m edges with $g(G) \geq 8$, then PI index of $\mu(G)$ is given by*

$$PI_v(\mu(G)) = 2n^2 + 6m + 4mn - \sum_{\substack{ab \in E(\mu(G)) \\ a=x' \in V' \\ b=u}} \left[\sum_{t \in N_G(x)} d_G(t) \right] + \sum_{\substack{ab \in E(\mu(G)) \\ a=x \in V \\ b=y' \in V'}} \left[\left[\sum_{w \in P} d_G(w) \right] - d_G(y) - \sum_{s \in V_G(2,x)} d_G(s) \right] +$$

$$\sum_{\substack{ab \in E(\mu(G)) \\ a=x \in V \\ b=y \in V}} \left[d_G(x) + \sum_{t \in N_G(x)} d_G(t) + \sum_{s \in V_G(2,x)} d_G(s) + \sum_{z \in Q} d_G(z) \right].$$

Proof. Let $diam(G) = k$. By the definition of PI index of $\mu(G)$,

$$PI_v(\mu(G)) = \sum_{ab \in E(\mu(G))} \Big[n_a(ab|\mu(G)) + n_b(ab|\mu(G)) \Big]. \tag{5.7}$$

For an edge $ab \in E(\mu(G))$, we have three possibilities for a and b, that is, (1) $a = x \in V(G), b = y' \in V'(G)$; (2) $a = x' \in V'(G), b = u$; (3) $a = x \in V(G), b = y \in V(G)$. Hence, the summation in Equation (5.7) can be divided into three sums as follows.

$$\begin{aligned} PI_v(\mu(G)) &= \sum_{\substack{ab \in E(\mu(G)) \\ a=x \in V \\ b=y' \in V'}} \Big[n_x + n_{y'} \Big] + \sum_{\substack{ab \in E(\mu(G)) \\ a=x' \in V' \\ b=u}} \Big[n_{x'} + n_u \Big] \\ &\quad + \sum_{\substack{ab \in E(\mu(G)) \\ a=x \in V \\ b=y \in V}} \Big[n_x + n_y \Big]. \end{aligned} \tag{5.8}$$

Case 1. $ab \in E(\mu(G))$ with $a = x$ in V(G) and $b = y'$ in $V'(G)$.

$$\text{Let } A = \{x'\} \cup V'_G(2,x) \cup \Big(V_G(3,x) \setminus Q\Big) \text{ and}$$
$$B = V(\mu(G)) \setminus A = \{x,u\} \cup \bigcup_{\substack{1\le i\le k\\ i\ne 3}} V_G(i,x) \cup \bigcup_{\substack{1\le i\le k\\ i\ne 2}} V'_G(i,x) \cup Q.$$

Then for $z \in A$,

$$d_{\mu(G)}(x,z) = d_{\mu(G)}(y',z) = \begin{cases} 2, & \text{if } z \in \{x'\} \cup V'_G(2,x);\\ 3, & \text{if } z \in \big(V_G(3,x) \setminus Q\big). \end{cases}$$

For $z \in B$,

$$d_{\mu(G)}(x,z) = \begin{cases} 0, & \text{if } z = x;\\ 2, & \text{if } z = u;\\ 1, & \text{if } z \in V_G(1,x);\\ 1, & \text{if } z \in V'_G(1,x);\\ 2, & \text{if } z \in V_G(2,x);\\ 3, & \text{if } z \in Q;\\ 4, & \text{if } z \in \bigcup_{4\le i\le k} V_G(i,x);\\ 3, & \text{if } z \in \bigcup_{3\le i\le k} V'_G(i,x), \end{cases}$$

and

$$d_{\mu(G)}(y',z) = \begin{cases} 1, & \text{if } z = x;\\ 1, & \text{if } z = u;\\ 2, & \text{if } z \in V_G(1,x);\\ 0 \text{ or } 2, & \text{if } z \in V'_G(1,x);\\ 1 \text{ or } 3, & \text{if } z \in V_G(2,x);\\ 2, & \text{if } z \in Q;\\ 3, & \text{if } z \in \bigcup_{4\le i\le k} V_G(i,x);\\ 2, & \text{if } z \in \bigcup_{3\le i\le k} V'_G(i,x). \end{cases}$$

Therefore, $d_{\mu^c}(x,z) = d_{\mu^c}(y',z)$, for every $z \in A$ and $d_{\mu^c}(x,z) \neq d_{\mu^c}(y',z)$, for every $z \in B$. Then

$$\begin{aligned} |B| &= 2 + \Big(n - 1 - |V_G(3,x)|\Big) + \Big(n - 1 - |V_G(2,x)|\Big) + |Q| \\ &= 2n - |V_G(2,x)| - |V_G(3,x)| + |Q| \\ &= 2n + 1 - d_G(y) + \sum_{w\in P} d_G(w) - \sum_{s\in V_G(2,x)} d_G(s). \end{aligned}$$

Since $n_x + n_{y'} = |B|$, then

$$\sum_{\substack{ab\in E(\mu(G))\\ x=x\in V,\\ b=y'\in V'}} \Big[n_x + n_{y'}\Big] = \sum_{\substack{ab\in E(\mu(G))\\ a=x\in V\\ b=y'\in V'}} \Bigg[2n+1-d_G(y)+\sum_{w\in P} d_G(w) - \sum_{s\in V_G(2,x)} d_G(s)\Bigg].$$

As there are $2m$ edges between $V(G)$ and $V'(G)$, we get

$$\sum_{\substack{ab\in E(\mu(G))\\ x=x\in V\\ b=y'\in V'}} \Big[n_x + n_{y'}\Big] = 4mn + 2m + \sum_{\substack{ab\in E(\mu(G))\\ a=x\in V\\ b=y'\in V'}} \Bigg[\Big[\sum_{w\in P} d_G(w)\Big] - d_G(y) - \sum_{s\in V_G(2,x)} d_G(s)\Bigg]. \quad (5.9)$$

Case 2. $ab \in E(\mu(G))$ with $a = x'$ in $V'(G)$ and $b = u$.
Let $A = \{x\} \cup V_G(2,x)$ and $B = V(\mu(G)) \setminus A = \{u\} \cup \bigcup_{\substack{1\le i\le k\\ i\neq 2}} V_G(i,x) \cup V'(G)$. Then for $z \in \{x\} \cup V_G(2,x)$, we have $d_{\mu(G)}(x,z) = 2 = d_{\mu(G)}(u,z)$.
For $z \in B$,

$$d_{\mu(G)}(x',z) = \begin{cases} 0, & \text{if } z = x'; \\ 1, & \text{if } z = u; \\ 1, & \text{if } z \in V_G(1,x); \\ 2, & \text{if } z \in V'(G)\setminus\{x'\}; \\ 3, & \text{if } z \in \bigcup_{3\le i\le k} V_G(i,x), \end{cases}$$

and

$$d_{\mu(G)}(u,z) = \begin{cases} 1, & \text{if } z = x'; \\ 0, & \text{if } z = u; \\ 2, & \text{if } z \in V_G(1,x); \\ 1, & \text{if } z \in V'(G)\setminus\{x'\}; \\ 2, & \text{if } z \in \bigcup_{3\le i\le k} V_G(i,x). \end{cases}$$

Therefore, $d_{\mu(G)}(x',z) = d_{\mu(G)}(u,z)$ for every $z \in A$ and $d_{\mu(G)}(x',z) \neq d_{\mu(G)}(u,z)$, for every $z \in B$. Then

$$\begin{aligned} |B| &= 1 + \Big((n-1) - |V_G(2,x)|\Big) + n \\ &= 2n + d_G(x) - \sum_{t\in N_G(x)} d_G(t), \text{ by Equation (5.5).} \end{aligned}$$

Thus,

$$\begin{aligned}\sum_{\substack{ab\in E(\mu(G))\\ a=x'\in V'\\ b=u}}\Big[n_{x'}+n_u\Big] &= \sum_{\substack{ab\in E(\mu(G))\\ a=x'\in V'\\ b=u}}\Big[2n+d_G(x)-\sum_{t\in N_G(x)} d_G(t)\Big]\\ &= \sum_{\substack{ab\in E(\mu(G))\\ a=x'\in V'\\ b=u}} 2n + \sum_{\substack{ab\in E(\mu(G))\\ a=x'\in V'\\ b=u}} d_G(x)\\ &- \sum_{\substack{ab\in E(\mu(G))\\ a=x'\in V'\\ b=u}}\sum_{t\in N_G(x)} d_G(t).\end{aligned}$$

As there are n edges between V' and $\{u\}$ and $\sum_{\substack{ab\in E(\mu(G))\\ a=x'\in V'(G)\\ b=u}} d_G(x) = \sum_{x\in V(G)} d_G(x) = 2m$,

we get

$$\sum_{\substack{ab\in E(\mu(G))\\ a=x'\in V'(G)\\ b=u}}\Big[n_{x'}+n_u\Big] = 2n^2+2m-\sum_{\substack{ab\in E(\mu(G))\\ a=x'\in V'(G)\\ b=u}}\sum_{t\in N_G(x)} d_G(t). \tag{5.10}$$

Case 3. $ab\in E(\mu(G))$ with $a=x$ in $V(G)$ and $b=y$ in $V(G)$.

$$\text{Let } A=\{u\}\cup\Big(V_G(4,x)\setminus R\Big)\cup\Big(V'_G(3,x)\setminus Q'\Big)\cup\bigcup_{i=5}^{k} V_G(i,x)\cup\bigcup_{i=4}^{k} V'_G(i,x)$$
$$\text{and } B=V(\mu(G))\setminus A=\{x,x'\}\cup\bigcup_{i=1}^{3} V_G(i,x)\cup\bigcup_{i=1}^{2} V'_G(i,x)\cup R\cup Q'.$$

Clearly, $y\in V_G(1,x)$ and $y'\in V'_G(1,x)$.
For $z\in A$,

$$d_{\mu(G)}(x,z)=d_{\mu(G)}(y,z)=\begin{cases}4, & \text{if } z\in V_G(4,x)\setminus R;\\ 3, & \text{if } z\in V'_G(3,x)\setminus Q';\\ 4, & \text{if } z\in \bigcup_{5\le i\le k} V_G(i,x);\\ 3, & \text{if } z\in \bigcup_{4\le i\le k} V'_G(i,x);\\ 2, & \text{if } z=u.\end{cases}$$

For $z \in B$,

$$d_{\mu(G)}(x,z) = \begin{cases} 0, & \text{if } z = x; \\ 1, & \text{if } z = y; \\ 2, & \text{if } z = x'; \\ 1, & \text{if } z = y'; \\ 1, & \text{if } z \in \big(V_G(1,x) \setminus \{y\}\big) \cup \big(V'_G(1,x) \setminus \{y'\}\big); \\ 2, & \text{if } z \in P \cup P'; \\ 2, & \text{if } z \in \big(V_G(2,x) \setminus P\big) \cup \big(V_G(2,x) \setminus P\big)'; \\ 3, & \text{if } z \in V_G(3,x); \\ 4, & \text{if } z \in R; \\ 3, & \text{if } z \in Q', \end{cases}$$

and

$$d_{\mu(G)}(y,z) = \begin{cases} 1, & \text{if } z = x; \\ 0, & \text{if } z = y; \\ 1, & \text{if } z = x'; \\ 2, & \text{if } z = y'; \\ 2, & \text{if } z \in \big(V_G(1,x) \setminus \{y\}\big) \cup \big(V'_G(1,x) \setminus \{y'\}\big); \\ 1, & \text{if } z \in P \cup P'; \\ 3, & \text{if } z \in \big(V_G(2,x) \setminus P\big) \cup \big(V_G(2,x) \setminus P\big)'; \\ 2 \text{ or } 4, & \text{if } z \in V_G(3,x); \\ 3, & \text{if } z \in R; \\ 2, & \text{if } z \in Q'. \end{cases}$$

Hence, $d_{\mu(G)}(x,z) = d_{\mu(G)}(y',z)$, for every $z \in A$ and $d_{\mu(G)}(x,z) \neq d_{\mu(G)}(y',z)$, for every $z \in B$. Then,

$$|B| = 2 + \sum_{i=1}^{3} |V_G(i,x)| + \sum_{i=1}^{2} |V'_G(i,x)| + |R| + |Q'|.$$

As $|V_G(1,x)| = d_G(x)$, $|V_G(i,x)| = |V'_G(i,x)|$ and $|(N_G(x))'| = |N_G(x)|$, we have

$$\begin{aligned} |B| &= 2 + 2d_G(x) + 2|V_G(2,x)| + |V_G(3,x)| + |R| + |Q| \\ &= 2 + d_G(x) + \sum_{t \in N_G(x)} d_G(t) + \sum_{s \in V_G(2,x)} d_G(s) + \sum_{z \in Q} d_G(z). \end{aligned}$$

As there are m edges in G, we have

$$\sum_{\substack{ab\in E(\mu(G))\\ a=x\in V\\ b=y\in V}}\Big[n_x+n_y\Big]=2m+\sum_{\substack{ab\in E(\mu(G))\\ a=x\in V\\ b=y\in V}}\Bigg[d_G(x)+\sum_{t\in N_G(x)}d_G(t)+\sum_{s\in V_G(2,x)}d_G(s)+\sum_{z\in Q}d_G(z)\Bigg]. \tag{5.11}$$

Thus by Equations (5.9)−(5.11), we get

$$\begin{aligned}PI_v(\mu(G))&=4mn+2m+\sum_{\substack{ab\in E(\mu(G))\\ a=x\in V\\ b=y'\in V'}}\Bigg[\Big[\sum_{w\in P}d_G(w)\Big]-d_G(y)-\sum_{s\in V_G(2,x)}d_G(s)\Bigg]\\&\quad+2n^2+2m-\sum_{\substack{ab\in E(\mu(G))\\ a=x'\in V'\\ b=u}}\Bigg[\sum_{t\in N_G(x)}d_G(t)\Bigg]\\&\quad+2m+\sum_{\substack{ab\in E(\mu(G))\\ a=x\in V\\ b=y\in V}}\Bigg[d_G(x)+\sum_{t\in N_G(x)}d_G(t)+\sum_{s\in V_G(2,x)}d_G(s)+\sum_{z\in Q}d_G(z)\Bigg].\end{aligned}$$

Hence, the result follows. □

As a consequence of Theorem 12, we calculate the PI index of $\mu(G)$ when G is an ℓ-regular with $g(G)\geq 8$. First, we observe the following.

Observation 13. If G is an ℓ-regular with $g(G)\geq 8$, then for $xy\in E(G)$,

$$|P|=\ell-1, |Q|=\ell^2-2\ell+1 \text{ and } |V_G(2,x)|=\ell^2-\ell. \tag{5.12}$$

Corollary 4. *If G is an ℓ-regular graph on n vertices with $g(G)\geq 8$, then the PI index of $\mu(G)$ is given by*

$$PI_v(\mu(G))=2n^2\big(\ell+1\big)+n\ell\big(\ell^2-2\ell+3\big).$$

Proof. By Theorem 12,

$$\begin{aligned}PI_v(\mu(G))&=2n^2+6m+4mn-\sum_{\substack{ab\in E(\mu(G))\\ a=x'\in V'\\ b=u}}\Bigg[\sum_{t\in N_G(x)}d_G(t)\Bigg]\\&\quad+\sum_{\substack{ab\in E(\mu(G))\\ a=x\in V\\ b=y'\in V'}}\Bigg[\Big[\sum_{w\in P}d_G(w)\Big]-d_G(y)-\sum_{s\in V_G(2,x)}d_G(s)\Bigg]\end{aligned}$$

$$+\sum_{\substack{ab\in E(\mu(G))\\ a=x\in V\\ b=y\in V}}\left[d_G(x)+\sum_{t\in N_G(x)}d_G(t)+\sum_{s\in V_G(2,x)}d_G(s)+\sum_{z\in Q}d_G(z)\right].$$

As $d_G(v)=\ell$, for every $v\in V(G)$ and $m=\frac{n\ell}{2}$, we have

$$\begin{aligned}
PI_v(\mu(G)) = 2n^2+6\left(\frac{n\ell}{2}\right)+4n\left(\frac{n\ell}{2}\right) &- \sum_{\substack{ab\in E(\mu(G))\\ a=x'\in V'\\ b=u}}\left[\sum_{t\in N_G(x)}\ell\right]\\
&+\sum_{\substack{ab\in E(\mu(G))\\ a=x\in V\\ b=y'\in V'}}\left[\Big[\sum_{w\in P}\ell\Big]-\ell-\sum_{s\in V_G(2,x)}\ell\right]\\
&+\sum_{\substack{ab\in E(\mu(G))\\ a=x\in V\\ b=y\in V}}\left[\ell+\sum_{t\in N_G(x)}\ell+\sum_{s\in V_G(2,x)}\ell+\sum_{z\in Q}\ell\right]\\
&= 2n^2+3n\ell+2n^2\ell-\sum_{\substack{ab\in E(\mu(G))\\ a=x'\in V'\\ b=u}}\Big[|N_G(x)|\big(\ell\big)\Big]\\
&+\sum_{\substack{ab\in E(\mu(G))\\ a=x\in V\\ b=y'\in V'}}\left[\Big[|P|\big(\ell\big)\Big]-\ell-|V_G(2,x)|\big(\ell\big)\right]\\
&+\sum_{\substack{ab\in E(\mu(G))\\ a=x\in V\\ b=y\in V}}\left[\ell+|N_G(x)|\big(\ell\big)+|V_G(2,x)|\big(\ell\big)+|Q|\big(\ell\big)\right].
\end{aligned}$$

By Equation (5.12) in Observation 13, we get

$$\begin{aligned}
PI_v(\mu(G)) = 2n^2+3n\ell+2n^2\ell &- \sum_{\substack{ab\in E(\mu(G))\\ a=x'\in V'\\ b=u}}\Big[\ell\big(\ell\big)\Big]\\
&+\sum_{\substack{ab\in E(\mu(G))\\ a=x\in V\\ b=y'\in V'}}\left[\Big[\big[\ell-1\big]\big(\ell\big)\Big]-\ell-\big[\ell^2-\ell\big]\big(\ell\big)\right]\\
&+\sum_{\substack{ab\in E(\mu(G))\\ a=x\in V\\ b=y\in V}}\left[\ell+\big[\ell\big]\big(\ell\big)+\big[\ell^2-\ell\big]\big(\ell\big)+\big[\ell^2-2\ell+1\big]\big(\ell\big)\right]
\end{aligned}$$

$$= 2n^2 + 3n\ell + 2n^2\ell - \sum_{\substack{ab\in E(\mu(G))\\ a=x'\in V'\\ b=u}} \left[\ell^2\right] + \sum_{\substack{ab\in E(\mu(G))\\ a=x\in V\\ b=y'\in V'}} \left[2\ell^2 - 2\ell - \ell^3\right]$$

$$+ \sum_{\substack{ab\in E(\mu(G))\\ a=x\in V\\ b=y\in V}} \left[2\ell^3 - 2\ell^2 + 2\ell\right].$$

As there are n edges between V and $\{u\}$, there are $2m = n\ell$ edges between V and V' and G has m edges, we get

$$PI_v(\mu(G)) = 2n^2 + 3n\ell + 2n^2\ell - \big(n\big)\left[\ell^2\right] + \big(n\ell\big)\left[2\ell^2 - 2\ell - \ell^3\right]$$

$$+ \left(\frac{n\ell}{2}\right)\left[2\ell^3 - 2\ell^2 + 2\ell\right].$$

Hence, the result follows. □

By Corollary 4, we calculate the PI index of $\mu(G)$ if G is a cycle of length $n \geq 8$.

Example 2. For $n \geq 8$, PI index of $\mu(C_n)$ is given by

$$PI_v(\mu(C_n)) = 6\big(n^2 + n\big).$$

5.2. PI index of $\mu^c(G)$

In this subsection, we find a formula for the PI index of the complement of Mycielskian of G.

Theorem 14. *If G is a graph with n vertices and m edges, then the PI index of complement of $\mu(G)$ is given by*

$$\begin{aligned} PI_v(\mu^c(G)) &= 6n^2 - 6nm + 4M_1(G) \\ &+ 6 \sum_{\substack{ab\in E(\mu^c(G))\\ a=x\in V(G)\\ b=y\in V(G)}} \left[d_G(x) + d_G(y) - d_G(x\wedge y)\right] \\ &+ \sum_{\substack{ab\in E(\mu^c(G))\\ a=x\in V(G)\\ b=y'\in V'(G)\\ x\neq y}} \left[2d_G(x) + d_G(y) - 2d_G(x\wedge y)\right] \\ &- 2 \sum_{\substack{ab\in E(\mu^c(G))\\ a=x'\in V'(G)\\ b=y'\in V'(G)\\ xy\notin E(G^c)}} d_G(x\wedge y). \end{aligned}$$

Proof. By the definition of PI index of $\mu^c(G)$,

$$PI_v(\mu^c(G)) = \sum_{ab\in E(\mu^c(G))} \Big[n_a(ab|\mu^c(G)) + n_a(ab|\mu^c(G))\Big]. \tag{5.13}$$

For an edge $ab \in E(\mu^c(G))$, we have six possibilities for a and b, namely, (1) $a = u, b = x \in V(G)$; (2) $a = x \in V(G), b = y' \in V'(G)$, with $x = y$; (3) $a = x \in V(G), b = y' \in V'(G)$, with $x \neq y$; (4) $a = x \in V(G), b = y \in V(G)$; (5) $a = x' \in V'(G), b = y' \in V'(G)$, with $xy \in E(G^c)$; (6) $a = x' \in V'(G), b = y' \in V'(G)$, with $xy \notin E(G^c)$. Hence, the summation in Equation (5.13) can be divided into six sums as follows.

$$\begin{aligned}PI_v(\mu^c(G)) \;=\; & \sum_{\substack{ab\in E(\mu^c(G))\\ a=u,\\ b=x\in V(G)}} [n_u + n_x] + \sum_{\substack{ab\in E(\mu^c(G))\\ a=x\in V(G)\\ b=y'\in V'(G)\\ x=y}} [n_y + n_{y'}] + \sum_{\substack{ab\in E(\mu^c(G))\\ a=x\in V(G)\\ b=y'\in V'(G)\\ x\neq y}} [n_x + n_{y'}]\\ & + \sum_{\substack{ab\in E(\mu^c(G))\\ a=x\in V(G)\\ b=y\in V(G)}} [n_x + n_y] + \sum_{\substack{ab\in E(\mu^c(G))\\ a=x'\in V'(G)\\ b=y'\in V'(G)\\ xy\in E(G^c)}} [n_{x'} + n_{y'}] + \sum_{\substack{ab\in E(\mu^c(G))\\ a=x'\in V'(G)\\ b=y'\in V'(G)\\ xy\notin E(G^c)}} [n_{x'} + n_{y'}].\end{aligned} \tag{5.14}$$

It is easy to observe that if $x, y \in V(G)$, then

$$d_{G^c}(x \wedge y) = \begin{cases} [n-2] - d_G(x) - d_G(y) + d_G(x \wedge y), & \text{if } xy \in E(G^c);\\ n - d_G(x) - d_G(y) + d_G(x \wedge y), & \text{if } xy \notin E(G^c). \end{cases} \tag{5.15}$$

By the definition of first Zagreb index of G^c, $M_1(G^c)$, we have

$$\begin{aligned} M_1(G^c) &= \sum_{uv\in E(G^c)} \Big[d_{G^c}(u) + d_{G^c}(v)\Big] = \sum_{v\in V(G^c)} d^2_{G^c}(v) \\ &= \sum_{v\in V(G)} \Big[(n-1) - d_G(v)\Big]^2, \text{ by } |V(G)| = |V(G^c)| = n,\\ &= \sum_{v\in V(G)} \Big[(n-1)^2 + d^2_G(v) - 2(n-1)d_G(v)\Big]. \end{aligned} \tag{5.16}$$

As $\sum\limits_{v\in V(G)} d_G(v) = 2m$ and $\sum\limits_{v\in V(G)} d^2_G(v) = M_1(G)$, we have

$$M_1(G^c) = n^3 - 2n^2 + n - 4nm + 4m + M_1(G). \tag{5.17}$$

Case 1. $ab \in E(\mu^c(G))$ with $a = u$ and $b = x \in V(G)$.

$$\text{Let } A = N_{G^c}(x) \cup \big(V'(G)\backslash N'_{G^c}[x]\big) \text{ and}$$

$$B = V(\mu^c(G)) \setminus A = \{u, x, x'\} \cup \Big(V(G) \backslash N_{G^c}[x]\Big) \cup N'_{G^c}(x).$$

Then for $z \in A$,

$$d_{\mu^c(G)}(u,z) = d_{\mu^c(G)}(x,z) = \begin{cases} 1, & \text{if } z \in N_{G^c}(x); \\ 2, & z \in V'(G) \backslash N'_{G^c}[x]. \end{cases}$$

For $z \in B$,

$$d_{\mu^c(G)}(u,z) = \begin{cases} 2, & \text{if } z \in N'_{G^c}(x); \\ 1, & \text{if } z \in V(G) \setminus N_{G^c}[x]; \\ 0, & \text{if } z = u; \\ 1, & \text{if } z = x; \\ 2, & \text{if } z = x', \end{cases}$$

and

$$d_{\mu^c(G)}(x,z) = \begin{cases} 1, & \text{if } z \in N'_{G^c}(x); \\ 2, & \text{if } z \in V(G) \setminus N_{G^c}[x]; \\ 1, & \text{if } z = u; \\ 0, & \text{if } z = x; \\ 1, & \text{if } z = x'. \end{cases}$$

Therefore, every vertex of A is equidistant from the edge ux, and every vertex in B is non-equidistant from the edge ux. Then,

$$|B| = 3 + \Big(n - \big(d_{G^c}(x) + 1\big)\Big) + d_{G^c}(x) = n + 2.$$

Therefore,

$$\sum_{\substack{ab \in E(\mu^c(G)) \\ a=u, \\ b=x \in V(G)}} \Big[n_u + n_x\Big] = \sum_{\substack{ab \in E(\mu^c(G)) \\ a=u, \\ b=x \in V(G)}} \Big[n + 2\Big].$$

As there are n edges between V and $\{u\}$, we get

$$\sum_{\substack{ab \in E(\mu^c(G)) \\ a=u, \\ b=x \in V(G)}} \Big[n_u + n_x\Big] = \sum_{\substack{ab \in E(\mu^c(G)) \\ a=u, \\ b=x \in V(G)}} \Big[n + 2\Big] = n[n+2] = n^2 + 2n. \tag{5.18}$$

Case 2. $ab \in E(\mu^c(G))$ with $a = x \in V(G)$, $b = y' \in V'(G)$ and $x = y$.

$$\text{Let } A = N_{G^c}(y) \cup N'_{G^c}(y) \cup \big(V(G) \setminus \big(N_{G^c}[y]\big) \text{ and}$$
$$B = V(\mu^c(G)) \setminus A = \{u, y, y'\} \cup V'(G) \setminus N'_{G^c}[y].$$

For $z \in A$,

$$d_{\mu^c(G)}(y,z) = d_{\mu^c(G)}(y',z) = \begin{cases} 1, & \text{if } z \in N_{G^c}(y) \cup N'_{G^c}(y); \\ 2, & \text{if } z \in V(G) \setminus N_{G^c}[y]. \end{cases}$$

For $z \in B$,

$$d_{\mu^c(G)}(y,z) = \begin{cases} 2, & \text{if } z \in V'(G) \setminus N'_{G^c}[y]; \\ 1, & \text{if } z = u; \\ 0, & \text{if } z = y; \\ 1, & \text{if } z = y', \end{cases}$$

and

$$d_{\mu^c(G)}(y',z) = \begin{cases} 1, & \text{if } z \in V'(G) \setminus N'_{G^c}[y]; \\ 2, & \text{if } z = u; \\ 1, & \text{if } z = y; \\ 0, & \text{if } z = y'. \end{cases}$$

Therefore, every vertex of A is equidistant from the edge yy' and every vertex in B is non-equidistant from the edge yy'. Then

$$|B| = 3 + \Big(n - [d_{G^c}(y) + 1]\Big) = 3 + n - d_{G^c}(y) - 1 = 2 + n - d_{G^c}(y).$$

Thus,

$$\begin{aligned} \sum_{\substack{ab \in E(\mu^c(G)) \\ a = x \in V(G) \\ b = y' \in V'(G) \\ x = y}} \Big[n_y + n_{y'}\Big] &= \sum_{\substack{ab \in E(\mu^c(G)) \\ a = x \in V(G) \\ b = y' \in V'(G) \\ x = y}} \Big[2 + n - d_{G^c}(y)\Big] \\ &= \sum_{\substack{ab \in E(\mu^c(G)) \\ a = x \in V(G) \\ b = y' \in V'(G) \\ x = y}} \Big[2 + n\Big] - \sum_{\substack{ab \in E(\mu^c(G)) \\ a = x \in V(G) \\ b = y' \in V'(G) \\ x = y}} d_{G^c}(y). \end{aligned}$$

As there are n edges between V and V' with $yy' \in E(\mu^c(G))$ and $\sum\limits_{y \in V(G^c)} d_{G^c}(y) = 2|E(G^c)| = 2\Big[\binom{n}{2} - m\Big] = n^2 - n - 2m$, we obtain

$$\sum_{\substack{ab \in E(\mu^c(G)) \\ a = x \in V(G) \\ b = y' \in V'(G) \\ x = y}} \Big[n_y + n_{y'}\Big] = n[2 + n] - \Big[n^2 - n - 2m\Big] = 3n + 2m. \tag{5.19}$$

Case 3. $ab \in E(\mu^c(G))$ with $a = x \in V(G)$, $b = y' \in V'(G)$ and $x \neq y$.
Clearly, $xy, x'y, yy', xx' \in E(\mu^c(G))$. Let

$$A = \{x', y\} \cup N_{G^c}(x \wedge y) \cup \Big(V(G) \setminus \big[N_{G^c}(x) \cup N_{G^c}(y)\big]\Big) \cup \Big(N'_{G^c}(x) \ \setminus \{y'\}\Big) \text{ and}$$

$$\begin{aligned} B = V(\mu^c(G)) \setminus A = \{u, x, y'\} &\cup \Big(N_{G^c}(x) \setminus \big[N_{G^c}(x \wedge y) \cup \{y\}\big]\Big) \\ &\cup \Big(N_{G^c}(y) \setminus \big[N_{G^c}(x \wedge y) \cup \{x\}\big]\Big) \cup \Big(V'(G) \setminus N'_{G^c}[x]\Big). \end{aligned}$$

For $z \in A$,

$$d_{\mu^c(G)}(x, z) = d_{\mu^c(G)}(y', z) = \begin{cases} 1, & \text{if } z \in N_{G^c}(x \wedge y) \cup \big(N'_{G^c}(x) \setminus \{y'\}\big) \cup \{x', y\}; \\ 2, & \text{if } z \in V(G) \setminus \big[N_{G^c}(x) \cup N_{G^c}(y)\big]. \end{cases}$$

For $z \in B$,

$$d_{\mu^c(G)}(x, z) = \begin{cases} 1, & \text{if } z \in N_{G^c}(x) \setminus \big[N_{G^c}(x \wedge y) \cup \{y\}\big]; \\ 2, & \text{if } z \in N_{G^c}(y) \setminus \big[N_{G^c}(x \wedge y) \cup \{x\}\big]; \\ 2, & \text{if } \ z \in V'(G) \setminus N'_{G^c}[x]; \\ 1, & \text{if } z = u; \\ 0, & \text{if } z = x; \\ 1, & \text{if } z = y', \end{cases}$$

and

$$d_{\mu^c(G)}(y', z) = \begin{cases} 2, & \text{if } z \in N_{G^c}(x) \setminus \big[N_{G^c}(x \wedge y) \cup \{y\}\big]; \\ 1, & \text{if } z \in N_{G^c}(y) \setminus \big[N_{G^c}(x \wedge y) \cup \{x\}\big]; \\ 1, & \text{if } z \in V'(G) \setminus N'_{G^c}[x]; \\ 2, & \text{if } z = u; \\ 1, & \text{if } z = x; \\ 0, & \text{if } z = y'. \end{cases}$$

Therefore, every vertex in A is equidistant from the edge xy' and every vertex in B is a non-equidistant vertex from the edge xy'. Then

$$\begin{aligned} |B| = 3 &+ \Big[d_{G^c}(x) - (d_{G^c}(x \wedge y) + 1)\Big] + \Big[d_{G^c}(y) - (d_{G^c}(x \wedge y) + 1)\Big] \\ &+ \Big[n - (d_{G^c}(x) + 1)\Big] \\ &= n + d_{G^c}(y) - 2d_{G^c}(x \wedge y). \end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{\substack{ab\in E(\mu^c(G))\\ a=x\in V(G)\\ b=y'\in V'(G)\\ x\neq y}}[n_x+n_{y'}] &= \sum_{\substack{ab\in E(\mu^c(G))\\ a=x\in V(G)\\ b=y'\in V'(G)\\ x\neq y}}\Big[n+d_{G^c}(y)-2d_{G^c}(x\wedge y)\Big]\\ &= \sum_{\substack{ab\in E(\mu^c(G))\\ a=x\in V(G)\\ b=y'\in V'(G)\\ x\neq y}} n+\sum_{\substack{ab\in E(\mu^c(G))\\ a=x\in V(G)\\ b=y'\in V'(G)\\ x\neq y}}[d_{G^c}(y)]-2\sum_{\substack{ab\in E(\mu^c(G))\\ a=x\in V(G)\\ b=y'\in V'(G)\\ x\neq y}}\Big[d_{G^c}(x\wedge y)\Big].\end{aligned}$$

As there are $2\big(\binom{n}{2}-m\big)$ edges between V and V' in $\mu^c(G)$ with respect to this case and $d_{G^c}(y)=n-1-d_G(y)$,

$$\begin{aligned}\sum_{\substack{ab\in E(\mu^c(G))\\ a=x\in V(G)\\ b=y'\in V'(G)\\ x\neq y}}[n_x+n_{y'}] &= \Big(n^2-n-2m\Big)[n]+\sum_{\substack{ab\in E(\mu^c(G))\\ a=x\in V(G)\\ b=y'\in V'(G)\\ x\neq y}}\Big[n-1-d_G(y)\Big]\\ &\quad -2\sum_{\substack{ab\in E(\mu^c(G))\\ a=x\in V(G)\\ b=y'\in V'(G)\\ x\neq y}}\Big[d_{G^c}(x\wedge y)\Big]\\ &= n^3-n^2-2mn+\Big(n^2-n-2m\Big)[n-1]-\sum_{\substack{ab\in E(\mu^c(G))\\ a=x\in V(G)\\ b=y'\in V'(G)\\ x\neq y}}\Big[d_G(y)\Big]\\ &\quad -2\sum_{\substack{ab\in E(\mu^c(G))\\ a=x\in V(G)\\ b=y'\in V'(G)\\ x\neq y}}\Big[d_{G^c}(x\wedge y)\Big].\end{aligned}$$

By Equation (5.15), we have

$$\begin{aligned}\sum_{\substack{ab\in E(\mu^c(G))\\ a=x\in V(G)\\ b=y'\in V'(G)\\ x\neq y}}[n_x+n_{y'}] &= 2n^3-3n^2-4mn+n+2m-\sum_{\substack{ab\in E(\mu^c(G))\\ a=x\in V(G)\\ b=y'\in V'(G)\\ x\neq y}}\Big[d_G(y)\Big]\\ &\quad -2\sum_{\substack{ab\in E(\mu^c(G))\\ a=x\in V(G)\\ b=y'\in V'(G)\\ x\neq y}}\Big[n-2-d_G(x)-d_G(y)+d_G(x\wedge y)\Big]\end{aligned}$$

$$= 2n^3 - 3n^2 - 4mn + n + 2m - \sum_{\substack{ab \in E(\mu^c(G)) \\ a=x \in V(G) \\ b=y' \in V'(G) \\ x \neq y}} \Big[d_G(y)\Big]$$

$$- 2\big(n^2 - 2m - n\big)\big[n-2\big] + \sum_{\substack{ab \in E(\mu^c(G)) \\ a=x \in V(G) \\ b=y' \in V'(G) \\ x \neq y}} \Big[2d_G(x) + 2d_G(y) - 2d_G(x \wedge y)\Big].$$

Hence,

$$\sum_{\substack{ab \in E(\mu^c(G)) \\ a=x \in V(G) \\ b=y' \in V'(G) \\ x \neq y}} [n_x + n_{y'}] = 3n^2 - 3n - 6m + \sum_{\substack{ab \in E(\mu^c(G)) \\ a=x \in V(G) \\ b=y' \in V'(G) \\ x \neq y}} \Big[2d_G(x) + d_G(y) - 2d_G(x \wedge y)\Big]. \tag{5.20}$$

Case 4. $ab \in E(\mu^c(G))$ with $a = x \in V(G)$ and $b = y \in V(G)$.
By the definition of $\mu^c(G)$, $xy', x'y, xx', yy' \in E(\mu^c(G))$. Let

$$A = \{u, x', y'\} \cup N_{G^c}(x \wedge y) \cup N'_{G^c}(x \wedge y) \cup \Big(V(G) \setminus \big[N_{G^c}(x) \cup N_{G^c}(y)\big]\Big) \cup \Big(V'(G) \setminus \big[N'_{G^c}(x) \cup N'_{G^c}(y)\big]\Big) \text{ and}$$

$$B = V(\mu^c(G)) \setminus A = \Big(N_{G^c}(x) \setminus N_{G^c}(x \wedge y)\Big) \cup \Big(N_{G^c}(y) \setminus N_{G^c}(x \wedge y)\Big) \cup \Big(N'_{G^c}(x) \setminus \big[N'_{G^c}(x \wedge y) \cup \{y'\}\big]\Big) \cup \Big(N'_{G^c}(y) \setminus \big[N'_{G^c}(x \wedge y) \cup \{x'\}\big]\Big).$$

For $z \in A$,

$$d_{\mu^c(G)}(x, z) = d_{\mu^c(G)}(y, z) = \begin{cases} 1, & \text{if } z \in N_{G^c}(x \wedge y) \cup N'_{G^c}(x \wedge y); \\ 2, & \text{if } z \in V(G) \setminus \big(N_{G^c}(x) \cup N_{G^c}(y)\big); \\ 2, & \text{if } z \in V'(G) \setminus \big(N'_{G^c}(x) \cup N'_{G^c}(y)\big); \\ 1, & \text{if } z \in \{u, x', y'\}. \end{cases}$$

For $z \in B$,

$$d_{\mu^c(G)}(x, z) = \begin{cases} 1, & \text{if } z \in N_{G^c}(x) \setminus \big[N_{G^c}(x \wedge y) \cup \{y\}\big]; \\ 1, & \text{if } z \in N'_{G^c}(x) \setminus \big[N'_{G^c}(x \wedge y) \cup \{y'\}\big]; \\ 2, & \text{if } z \in N_{G^c}(y) \setminus \big[N_{G^c}(x \wedge y) \cup \{x\}\big]; \\ 2, & \text{if } z \in N'_{G^c}(y) \setminus \big[N'_{G^c}(x \wedge y) \cup \{x'\}\big]; \\ 0 & \text{if } z = x; \\ 1, & \text{if } z = y, \end{cases}$$

and

$$d_{\mu^c(G)}(y,z) = \begin{cases} 2, & \text{if } z \in N_{G^c}(x) \setminus \big[N_{G^c}(x \wedge y) \cup \{y\}\big]; \\ 2, & \text{if } z \in N'_{G^c}(x) \setminus \big[N'_{G^c}(x \wedge y) \cup \{y'\}\big]; \\ 1, & \text{if } z \in N_{G^c}(y) \setminus \big[N_{G^c}(x \wedge y) \cup \{x\}\big]; \\ 1, & \text{if } z \in N'_{G^c}(y) \setminus \big[N'_{G^c}(x \wedge y) \cup \{x'\}\big]; \\ 1, & \text{if } z = x; \\ 0, & \text{if } z = y. \end{cases}$$

Therefore, every vertex of A is equidistant from the edge xy and every vertex in B is non-equidistant from the edge xy. Then

$$\begin{aligned} |B| = {} & \Big[d_{G^c}(x) - d_{G^c}(x \wedge y)\Big] + \Big[d_{G^c}(y) - d_{G^c}(x \wedge y)\Big] + \Big[d_{G^c}(x) - d_{G^c}(x \wedge y) - 1\Big] \\ & + \Big[d_{G^c}(y) - d_{G^c}(x \wedge y) - 1\Big] \\ & = 2d_{G^c}(x) + 2d_{G^c}(y) - 4d_{G^c}(x \wedge y) - 2. \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{\substack{ab \in E(\mu^c(G)) \\ a=x \in V(G) \\ b=y \in V(G)}} [n_x + n_y] &= \sum_{\substack{ab \in E(\mu^c(G)) \\ a=x \in V(G) \\ b=y \in V(G)}} \Big[2d_{G^c}(x) + 2d_{G^c}(y) - 4d_{G^c}(x \wedge y) - 2\Big] \\ &= 2 \sum_{\substack{ab \in E(\mu^c(G)) \\ a=x \in V(G) \\ b=y \in V(G)}} \Big[d_{G^c}(x) + d_{G^c}(y) - 2d_{G^c}(x \wedge y)\Big] - \sum_{\substack{ab \in E(\mu^c(G)) \\ a=x \in V(G) \\ b=y \in V(G)}} [2]. \end{aligned}$$

As there are $\Big[\binom{n}{2} - m\Big]$ edges in G^c, $\displaystyle\sum_{\substack{ab \in E(\mu^c(G)) \\ a=x \in V(G) \\ b=y \in V(G)}} [2] = n^2 - n - 2m$. Therefore,

$$\begin{aligned} \sum_{\substack{ab \in E(\mu^c(G)) \\ a=x \in V(G) \\ b=y \in V(G)}} [n_x + n_y] &= 2 \sum_{\substack{ab \in E(\mu^c(G)) \\ a=x \in V(G) \\ b=y \in V(G)}} \Big[d_{G^c}(x) + d_{G^c}(y) - 2d_{G^c}(x \wedge y)\Big] - \Big[(n^2 - n - 2m)\Big] \\ &= n + 2m - n^2 + 2 \sum_{\substack{ab \in E(\mu^c(G)) \\ a=x \in V(G) \\ b=y \in V(G)}} \Big[d_{G^c}(x) + d_{G^c}(y)\Big] - 4 \sum_{\substack{ab \in E(\mu^c(G)) \\ a=x \in V(G) \\ b=y \in V(G)}} d_{G^c}(x \wedge y) \\ &= n + 2m - n^2 + 2M_1(G^c) - 4 \sum_{\substack{ab \in E(\mu^c(G)) \\ a=x \in V(G) \\ b=y \in V(G)}} d_{G^c}(x \wedge y), \text{ by Equation (5.16)}. \end{aligned}$$

By Equations (5.15) and (5.17), we get

$$\sum_{\substack{ab\in E(\mu^c(G))\\ a=x\in V(G)\\ b=y\in V(G)}}[n_x+n_y]=n+2m-n^2+2\Big[n^3-2n^2+n-4nm+4m+M_1(G)\Big]$$

$$-4\sum_{\substack{ab\in E(\mu^c(G))\\ a=x\in V(G)\\ b=y\in V(G)}}\Big[n-2-d_G(x)-d_G(y)+d_G(x\wedge y)\Big]$$

$$=2n^3-5n^2+3n+10m-8mn+2M_1(G)-4\sum_{\substack{ab\in E(\mu^c(G))\\ a=x\in V(G)\\ b=y\in V(G)}}[n-2]+$$

$$4\sum_{\substack{ab\in E(\mu^c(G))\\ a=x\in V(G)\\ b=y\in V(G)}}\Big[d_G(x)+d_G(y)-d_G(x\wedge y)\Big]$$

$$=2n^3-5n^2+3n+10m-8mn+2M_1(G)-4\bigg(\frac{1}{2}[n^2-n-2m]\bigg)[n-2]+$$

$$4\sum_{\substack{ab\in E(\mu^c(G))\\ a=x\in V(G)\\ b=y\in V(G)}}\Big[d_G(x)+d_G(y)-d_G(x\wedge y)\Big],\ \text{since } |E(G^c)|=\bigg[\binom{n}{2}-m\bigg].$$

Hence,

$$\sum_{\substack{ab\in E(\mu^c(G))\\ a=x\in V(G)\\ b=y\in V(G)}}[n_x+n_y]=n^2-n-4mn+2m+2M_1(G)$$

$$+4\sum_{\substack{ab\in E(\mu^c(G))\\ a=x\in V(G)\\ b=y\in V(G)}}\big[d_G(x)+d_G(y)-d_G(x\wedge y)\big]. \tag{5.21}$$

Case 5. $ab\in E(\mu^c(G))$ with $a=x'\in V'(G)$, $b=y'\in V'(G)$ and $xy\in E(G^c)$.
Let $A=\{u,x,y\}\cup N_{G^c}(x\wedge y)\cup\big(V(G)\setminus\big[N_{G^c}(x)\cup N_{G^c}(y)\big]\big)\cup\big(V'(G)\setminus\{x',y'\}\big)$ and

$$B=V(\mu^c(G))\setminus A=\{x',y'\}\cup\bigg(N_{G^c}(x)\setminus\big[N_{G^c}(x\wedge y)\cup\{y\}\big]\bigg)\cup\bigg(N_{G^c}(y)\setminus\big[N_{G^c}(x\wedge y)\cup\{x\}\big]\bigg).$$

Then for $z \in A$,

$$d_{\mu^c(G)}(x', z) = d_{\mu^c(G)}(y', z) = \begin{cases} 1, & \text{if } z \in N_{G^c}(x \wedge y); \\ 2, & \text{if } z \in \big(V(G) \setminus \big[N_{G^c}(x) \cup N_{G^c}(y)\big]\big); \\ 1, & \text{if } z \in V'(G) \setminus \{x', y'\}; \\ 2, & \text{if } z = u; \\ 1, & \text{if } z \in \{x, y\}. \end{cases}$$

For $z \in B$,

$$d_{\mu^c(G)}(x', z) = \begin{cases} 1, & \text{if } z \in N_{G^c}(x) \setminus \big[N_{G^c}(x \wedge y) \cup \{y\}\big]; \\ 2, & \text{if } z \in N_{G^c}(y) \setminus \big[N_{G^c}(x \wedge y) \cup \{x\}\big]; \\ 0, & \text{if } z = x'; \\ 1, & \text{if } z = y', \end{cases}$$

and

$$d_{\mu^c(G)}(y', z) = \begin{cases} 2, & \text{if } z \in N_{G^c}(x) \setminus \big[N_{G^c}(x \wedge y) \cup \{y\}\big]; \\ 1, & \text{if } z \in N_{G^c}(y) \setminus \big[N_{G^c}(x \wedge y) \cup \{x\}\big]; \\ 1, & \text{if } z = x'; \\ 0, & \text{if } z = y'. \end{cases}$$

Therefore, $d_{\mu^c(G)}(x', z) = d_{\mu^c(G)}(y', z)$ for every $z \in A$ and $d_{\mu^c(G)}(x', z) \neq d_{\mu^c(G)}(y', z)$ for every $z \in B$. Then

$$\begin{aligned} |B| &= 2 + \Big(d_{G^c}(x) - \big[d_{G^c}(x \wedge y) + 1\big]\Big) + \Big(d_{G^c}(y) - \big[d_{G^c}(x \wedge y) + 1\big]\Big) \\ &= d_{G^c}(x) + d_{G^c}(y) - 2d_{G^c}(x \wedge y). \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{\substack{ab \in E(\mu^c(G)) \\ a = x' \in V'(G) \\ b = y' \in V'(G) \\ xy \in E(G^c)}} \Big[n_{x'} + n_{y'}\Big] &= \sum_{\substack{ab \in E(\mu^c(G)) \\ a = x' \in V'(G) \\ b = y' \in V'(G) \\ xy \in E(G^c)}} \Big[d_{G^c}(x) + d_{G^c}(y) - 2d_{G^c}(x \wedge y)\Big] \\ &= \sum_{\substack{ab \in E(\mu^c(G)) \\ a = x' \in V'(G) \\ b = y' \in V'(G) \\ xy \in E(G^c)}} \Big[d_{G^c}(x) + d_{G^c}(y)\Big] - 2 \sum_{\substack{ab \in E(\mu^c(G)) \\ a = x' \in V'(G) \\ b = y' \in V'(G) \\ xy \in E(G^c)}} d_{G^c}(x \wedge y) \\ &= M_1(G^c) - 2 \sum_{\substack{ab \in E(\mu^c(G)) \\ a = x' \in V'(G) \\ b = y' \in V'(G) \\ xy \in E(G^c)}} d_{G^c}(x \wedge y), \text{ by Equation (5.16).} \end{aligned}$$

By Equations (5.15) and (5.17), we have

$$\sum_{\substack{ab\in E(\mu^c(G))\\ a=x'\in V'(G)\\ b=y'\in V'(G)\\ xy\in E(G^c)}} \Big[n_{x'}+n_{y'}\Big] = M_1(G^c) - 2\sum_{\substack{ab\in E(\mu^c(G))\\ a=x'\in V'(G)\\ b=y'\in V'(G)\\ xy\in E(G^c)}} d_{G^c}(x\wedge y)$$
$$= \big[n^3-2n^2+n-4nm+4m+M_1(G)\big] - 2\sum_{\substack{ab\in E(\mu^c(G))\\ a=x'\in V'(G)\\ b=y'\in V'(G)\\ xy\in E(G^c)}} \Big[n-2\Big]$$
$$+2\sum_{\substack{ab\in E(\mu^c(G))\\ a=x'\in V'(G)\\ b=y'\in V'(G)\\ xy\in E(G^c)}} \big[d_G(x)+d_G(y)-d_G(x\wedge y)\big].$$

As there are $\binom{n}{2}-m$ edges in G^c,

$$\sum_{\substack{ab\in E(\mu^c(G))\\ a=x'\in V'(G)\\ b=y'\in V'(G)\\ xy\in E(G^c)}} \Big[n_{x'}+n_{y'}\Big] = \big[n^3-2n^2+n-4nm+4m+M_1(G)\big]$$
$$-2\Big[\frac{1}{2}[n^2-n-2m]\Big]\Big[n-2\Big]$$
$$+2\sum_{\substack{ab\in E(\mu^c(G))\\ a=x'\in V'(G)\\ b=y'\in V'(G)\\ xy\in E(G^c)}} \big[d_G(x)+d_G(y)-d_G(x\wedge y)\big].$$

Hence,

$$\sum_{\substack{ab\in E(\mu^c(G))\\ a=x'\in V'(G)\\ b=y'\in V'(G)\\ xy\in E(G^c)}} \Big[n_{x'}+n_{y'}\Big] = n^2-n-2nm+M_1(G)$$
$$+2\sum_{\substack{ab\in E(\mu^c(G))\\ a=x'\in V'(G)\\ b=y'\in V'(G)\\ xy\in E(G^c)}} \big[d_G(x)+d_G(y)-d_G(x\wedge y)\big]. \tag{5.22}$$

Case 6. $ab\in E(\mu^c(G))$ with $a=x'\in V'(G)$, $b=y'\in V'(G)$ and $xy\notin E(G^c)$. Let $A=\{u\}\cup N_{G^c}(x\wedge y)\cup\Big(V(G)\setminus\big[N_{G^c}(x)\cup N_{G^c}(y)\cup\{x,y\}\big]\Big)\cup\Big(V'(G)\setminus\{x',y'\}\Big)$

and $B = V(\mu^c(G))\backslash A = \{x, y, x', y'\} \cup \Big(N_{G^c}(x)\backslash N_{G^c}(x\wedge y)\Big) \cup \Big(N_{G^c}(y)\backslash N_{G^c}(x\wedge y)\Big)$.
For $z \in A$,

$$d_{\mu^c(G)}(x', z) = d_{\mu^c(G)}(y', z) = \begin{cases} 1, & \text{if } z \in N_{G^c}(x \wedge y); \\ 2, & \text{if } z \in V(G) \setminus \big[N_{G^c}(x) \cup N_{G^c}(y) \cup \{x, y\}\big]; \\ 1, & \text{if } z \in V'(G) \setminus \{x', y'\}; \\ 2, & \text{if } z = u. \end{cases}$$

For $z \in B$,

$$d_{\mu^c(G)}(x', z) = \begin{cases} 1, & \text{if } z \in N_{G^c}(x) \setminus N_{G^c}(x \wedge y); \\ 2, & \text{if } z \in N_{G^c}(y) \setminus N_{G^c}(x \wedge y); \\ 0, & \text{if } z = x'; \\ 1, & \text{if } z = y'; \\ 1, & \text{if } z = x; \\ 2, & \text{if } z = y, \end{cases}$$

and

$$d_{\mu^c(G)}(y', z) = \begin{cases} 2, & \text{if } z \in N_{G^c}(x) \setminus N_{G^c}(x \wedge y); \\ 1, & \text{if } z \in N_{G^c}(y) \setminus N_{G^c}(x \wedge y); \\ 1, & \text{if } z = x'; \\ 0, & \text{if } z = y'; \\ 2, & \text{if } z = x; \\ 1, & \text{if } z = y. \end{cases}$$

Therefore, $d_{\mu^c(G)}(x', z) = d_{\mu^c(G)}(y', z)$ for every $z \in A$ and $d_{\mu^c(G)}(x', z) \neq d_{\mu^c(G)}(y', z)$, for every $z \in B$. Then,

$$\begin{aligned} |B| &= 4 + \Big[d_{G^c}(x) - d_{G^c}(x \wedge y)\Big] + \Big[d_{G^c}(y) - d_{G^c}(x \wedge y)\Big] \\ &= 4 + d_{G^c}(x) + d_{G^c}(y) - 2d_{G^c}(x \wedge y). \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{\substack{ab\in E(\mu^c(G)) \\ a=x'\in V'(G) \\ b=y'\in V'(G) \\ xy\notin E(G^c)}} [n_{x'} + n_{y'}] &= \sum_{\substack{ab\in E(\mu^c(G)) \\ a=x'\in V'(G) \\ b=y'\in V'(G) \\ xy\notin E(G^c)}} \Big[4 + d_{G^c}(x) + d_{G^c}(y) - 2d_{G^c}(x \wedge y)\Big] \\ &= \sum_{\substack{ab\in E(\mu^c(G)) \\ a=x'\in V'(G) \\ b=y'\in V'(G) \\ xy\notin E(G^c)}} [4] + \sum_{\substack{ab\in E(\mu^c(G)) \\ a=x'\in V'(G) \\ b=y'\in V'(G) \\ xy\notin E(G^c)}} \big[d_{G^c}(x) + d_{G^c}(y)\big] - 2 \sum_{\substack{ab\in E(\mu^c(G)) \\ a=x'\in V'(G) \\ b=y'\in V'(G) \\ xy\notin E(G^c)}} d_{G^c}(x \wedge y) \end{aligned}$$

As $|\{ab \in E(\mu^c(G)) : a = x' \in V', b = y' \in V', xy \in E(G)\}| = m$ and $d_{G^c}(x) = n - 1 - d_G(x)$, we have

$$\begin{aligned}\sum_{\substack{ab\in E(\mu^c(G))\\ a=x'\in V'(G)\\ b=y'\in V'(G)\\ xy\notin E(G^c)}} [n_{x'} + n_{y'}] = m[4] + \sum_{\substack{ab\in E(\mu^c(G))\\ a=x'\in V'(G)\\ b=y'\in V'(G)\\ xy\in E(G)}} \Big[\big(n-1-d_G(x)\big) + \big(n-1-d_G(y)\big)\Big] \\ - 2\sum_{\substack{ab\in E(\mu^c(G))\\ a=x'\in V'(G)\\ b=y'\in V'(G)\\ xy\notin E(G^c)}} d_{G^c}(x\wedge y) \\ = 4m + \sum_{\substack{ab\in E(\mu^c(G))\\ a=x'\in V'(G)\\ b=y'\in V'(G)\\ xy\in E(G)}} \big[2n-2-d_G(x)-d_G(y)\big] - 2\sum_{\substack{ab\in E(\mu^c(G))\\ a=x'\in V'(G)\\ b=y'\in V'(G)\\ xy\notin E(G^c)}} d_{G^c}(x\wedge y).\end{aligned}$$

By Equation (5.15),

$$\begin{aligned}\sum_{\substack{ab\in E(\mu^c(G))\\ a=x'\in V'(G)\\ b=y'\in V'(G)\\ xy\notin E(G^c)}} [n_{x'} + n_{y'}] = 4m + m[2n-2] - \sum_{\substack{ab\in E(\mu^c(G))\\ a=x'\in V'(G)\\ b=y'\in V'(G)\\ xy\in E(G)}} \big[d_G(x) + d_G(y)\big] \\ - 2\sum_{\substack{ab\in E(\mu^c(G))\\ a=x'\in V'(G)\\ b=y'\in V'(G)\\ xy\in E(G)}} \big[n - d_G(x) - d_G(y) + d_G(x\wedge y)\big] \\ = 4m + m[2n-2] - \sum_{\substack{ab\in E(\mu^c(G))\\ a=x'\in V'(G)\\ b=y'\in V'(G)\\ xy\in E(G)}} \big[d_G(x) + d_G(y)\big] \\ - 2\sum_{\substack{ab\in E(\mu^c(G))\\ a=x'\in V'(G)\\ b=y'\in V'(G)\\ xy\in E(G)}} \big[n\big] + 2\sum_{\substack{ab\in E(\mu^c(G))\\ a=x'\in V'(G)\\ b=y'\in V'(G)\\ xy\in E(G)}} \big[d_G(x) + d_G(y)\big] \\ - 2\sum_{\substack{ab\in E(\mu^c(G))\\ a=x'\in V'(G)\\ b=y'\in V'(G)\\ xy\in E(G)}} d_G(x\wedge y).\end{aligned}$$

By the definition of the first Zagreb index of G,

$$\sum_{\substack{ab\in E(\mu^c(G))\\ a=x'\in V'(G)\\ b=y'\in V'(G)\\ xy\notin E(G^c)}} [n_{x'}+n_{y'}] = 4m + m[2n-2] - M_1(G) - 2\big(m\big)\big[n\big] + 2\Big[M_1(G)\Big] - 2\sum_{\substack{ab\in E(\mu^c(G))\\ a=x'\in V'(G)\\ b=y'\in V'(G)\\ xy\in E(G)}} d_G(x\wedge y).$$

Hence,

$$\sum_{\substack{ab\in E(\mu^c(G))\\ a=x'\in V'(G)\\ b=y'\in V'(G)\\ xy\notin E(G^c)}} [n_{x'}+n_{y'}] = 2m + M_1(G) - 2\sum_{\substack{ab\in E(\mu^c(G))\\ a=x'\in V'(G)\\ b=y'\in V'(G)\\ xy\in E(G)}} d_G(x\wedge y). \tag{5.23}$$

By Equations (5.18)–(5.23) and

$$\sum_{\substack{ab\in E(\mu^c(G))\\ a=x\in V(G)\\ b=y\in V(G)}} \Big[d_G(x)+d_G(y)-d_G(x\wedge y)\Big] = \sum_{\substack{ab\in E(\mu^c(G))\\ a=x'\in V'(G)\\ b=y'\in V'(G)\\ xy\in E(G^c)}} \Big[d_G(x)+d_G(y)-d_G(x\wedge y)\Big],$$

then the result follows. □

As a consequence, we calculate the PI index of $\mu^c(G)$ when G is an ℓ-regular.

Corollary 5. *If G is an ℓ-regular graph on n vertices, then PI index of $\mu^c(G)$ is given by*

$$\begin{aligned} PI_v(\mu^c(G)) &= [12\ell+6]n^2 - [5\ell+9]n\ell - 6\sum_{\substack{ab\in E(\mu^c(G))\\ a=x\in V(G)\\ b=y\in V(G)}} \Big[d_G(x\wedge y)\Big] \\ &\quad - 2\sum_{\substack{ab\in E(\mu^c(G))\\ a=x\in V(G)\\ b=y'\in V'(G)\\ x\neq y}} \Big[d_G(x\wedge y)\Big] - 2\sum_{\substack{ab\in E(\mu^c(G))\\ a=x'\in V'(G)\\ b=y'\in V'(G)\\ xy\notin E(G^c)}} \Big[d_G(x\wedge y)\Big]. \end{aligned} \tag{5.24}$$

In particular, if G is an ℓ-regular graph on n vertices and triangle-free, then

$$PI_v(\mu^c(G)) = [12\ell+6]n^2 - [5\ell+9]n\ell.$$

Proof. By Theorem 14,

$$PI_v(\mu^c(G)) = 6n^2 - 6nm + 4M_1(G) + 6 \sum_{\substack{ab\in E(\mu^c(G))\\ a=x\in V(G)\\ b=y\in V(G)}} \Big[d_G(x) + d_G(y) - d_G(x\wedge y)\Big]$$
$$+ \sum_{\substack{ab\in E(\mu^c(G))\\ a=x\in V(G)\\ b=y'\in V'(G)\\ x\neq y}} \Big[2d_G(x) + d_G(y) - 2d_G(x\wedge y)\Big] - 2 \sum_{\substack{ab\in E(\mu^c(G))\\ a=x'\in V'(G)\\ b=y'\in V'(G)\\ xy\notin E(G^c)}} d_G(x\wedge y).$$

As $d_G(v) = \ell$, for every $v \in V(G)$, $m = \frac{n\ell}{2}$ and $M_1(G) = n\ell^2$,

$$PI_v(\mu^c(G)) = 6n^2 - 6n\left(\frac{n\ell}{2}\right) + 4\left(n\ell^2\right) + 6 \sum_{\substack{ab\in E(\mu^c(G))\\ a=x\in V(G)\\ b=y\in V(G)}} \Big[[\ell] + [\ell] - d_G(x\wedge y)\Big]$$
$$+ \sum_{\substack{ab\in E(\mu^c(G))\\ a=x\in V(G)\\ b=y'\in V'(G)\\ x\neq y}} \Big[2[\ell] + [\ell] - 2d_G(x\wedge y)\Big] - 2 \sum_{\substack{ab\in E(\mu^c(G))\\ a=x'\in V'(G)\\ b=y'\in V'(G)\\ xy\notin E(G^c)}} d_G(x\wedge y)$$
$$= 6n^2 - 3n^2\ell + 4n\ell^2 + 6 \sum_{\substack{ab\in E(\mu^c(G))\\ a=x\in V(G)\\ b=y\in V(G)}} [2\ell] - 6 \sum_{\substack{ab\in E(\mu^c(G))\\ a=x\in V(G)\\ b=y\in V(G)}} [d_G(x\wedge y)]$$
$$+ \sum_{\substack{ab\in E(\mu^c(G))\\ a=x\in V(G)\\ b=y'\in V'(G)\\ x\neq y}} [3\ell] - 2 \sum_{\substack{ab\in E(\mu^c(G))\\ a=x\in V(G)\\ b=y'\in V'(G)\\ x\neq y}} [d_G(x\wedge y)] - 2 \sum_{\substack{ab\in E(\mu^c(G))\\ a=x'\in V'(G)\\ b=y'\in V'(G)\\ xy\notin E(G^c)}} [d_G(x\wedge y)].$$

As G^c has $\left(\binom{n}{2} - m\right)$ edges, there are $n^2 - 2m - n$ edges between V and V' such that $xy' \in E(\mu^c(G))$ with $x \neq y$,

$$PI_v(\mu^c(G)) = 6n^2 - 3n^2\ell + 4n\ell^2 + 6\left(\frac{1}{2}[n^2 - n - 2m]\right)[2\ell] - 6 \sum_{\substack{ab\in E(\mu^c(G))\\ a=x\in V(G)\\ b=y\in V(G)}} [d_G(x\wedge y)]$$
$$+ \left(n^2 - 2m - n\right)[3\ell] - 2 \sum_{\substack{ab\in E(\mu^c(G))\\ a=x\in V(G)\\ b=y'\in V'(G)\\ x\neq y}} [d_G(x\wedge y)] - 2 \sum_{\substack{ab\in E(\mu^c(G))\\ a=x'\in V'(G)\\ b=y'\in V'(G)\\ xy\notin E(G^c)}} [d_G(x\wedge y)].$$

Next, if G is an ℓ-regular with triangle-free, then we have $d_G(x\wedge y) = 0$. The result follows from Equation 5.24. □

By Corollary 5, we calculate the PI index of $\mu^c(G)$ if G is a cycle of length $n \geq 3$.

Example 3. For $n \geq 3$, PI index of $\mu^c(C_n)$ is given by

$$PI_v(\mu^c(C_n)) = 30n^2 - 38n.$$

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