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***Research Article***

# Algorithmic complexity of three domination subdivision number problems in graphs

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**Abstract:** The paired, total, and independent domination subdivision number of a graph $G$ is the minimum number of edges that must be subdivided, where each edge can be subdivided at most once, in order to increase the paired, total, and independent domination number, respectively. In this paper, we prove that the corresponding decision problems for paired, total, and independent domination subdivision numbers are NP-hard, even when restricted to bipartite graphs. Additionally, we point out the error in the previous proof of NP-hardness of the paired domination subdivision problem by Amjadi and Chellali in "Complexity of the paired domination subdivision problem" [Commun. Comb. Optim. 7 (2022), No.2, 177–182].

**Keywords:** algorithmic complexity, NP-hardness, independent domination subdivision number, total domination subdivision number, paired domination subdivision number.

**AMS Subject classification:** Primary: 05C69, 68Q17; Secondary: 05C85, 68R10

## 1. Introduction

We consider the finite, undirected and simple graph $G$ with vertex set $V(G)$ and edge set $E(G)$. For graph theoretic terminology and notation, we refer the readers to [12], and for complexity theoretic terminology and notation, we refer to [9].
Domination number and its variants are one of the most extensively researched graph theoretical parameters due to their theoretical as well as practical importance. When a graph theoretic parameter is of interest in applications, it is often crucial to understand

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how this parameter behaves when the graph undergoes modifications. Thus it is an interesting and important problem to recognize the impact on domination number when a graph is modified through various operations, for instance, deleting an edge [6], adding an edge [15], contracting an edge [14], or subdividing an edge [11]. In these problems, given a graph $G$, a natural number $k$ and a fixed operation, the question is whether $G$ can be transformed to a graph $G'$, using at most $k$ operations, such that the domination number of $G'$ is either increased or decreased accordingly, relative to the domination number of $G$.

Velammal [16] in his Ph.D. thesis initiated the study of the effect of a graph operation called subdivision of an edge on the domination number. An edge $uv$ of a graph $G$ is *subdivided* by deleting the edge $uv$, and inserting a new vertex $w$ with edges $uw$ and $vw$. The newly added vertex $w$ is called a *subdivision vertex*. A set $D$ of vertices in a graph $G$ is called a *dominating set* if every vertex of $G$ is either in $D$ or is adjacent to a vertex in $D$. The *domination number* of $G$, denoted by $\gamma(G)$, is defined as the minimum cardinality of a dominating set of $G$. Upon subdividing an edge, the domination number of the resulting graph does not decrease relative to the domination number of $G$. The *domination subdivision number*, denoted by $\mathrm{sd}_\gamma(G)$, is the minimum number of edges of $G$ that must be subdivided (each edge can be subdivided at most once) in order to increase the domination number.

In this paper, we consider the three variants of dominating sets- paired dominating set, total dominating set, and independent dominating set. A *matching* in a graph $G$ is a set of edges without common vertices, while a *perfect matching* in $G$ is a matching such that every vertex of $G$ is incident to an edge of the matching. A vertex which is not incident to any edge of $G$ is an *isolated vertex*. A *paired (resp. total, independent) dominating set* of $G$ is a dominating set $D$ with additional property that it induces a subgraph $G[D]$ of $G$ which has a perfect matching (resp. no isolated vertex, no edge). The *paired (resp. total, independent) domination number* of $G$, denoted by $\gamma_{pr}(G)$ (resp. $\gamma_t(G)$, $i(G)$) is the minimum cardinality of a paired (resp. total, independent) dominating set. A paired (resp. total, independent) dominating set of $G$ of cardinality $\gamma_{pr}(G)$ (resp. $\gamma_t(G)$, $i(G)$) is called a *$\gamma_{pr}$-set (resp. $\gamma_t$-set, $i$-set)* of $G$. The concept of the domination subdivision number has been extended to these three variants of domination: the *paired domination subdivision number* [8], the *total domination subdivision number* [13], and the *independent domination subdivision number* [5], which are defined along similar lines of the domination subdivision number.

Determining if there exists a polynomial-time algorithm to compute the exact value of a graph parameter is a fundamental problem. If the decision problem corresponding to the computation of a certain parameter is NP-hard, then polynomial-time algorithms for this parameter do not exist unless NP = P. In this paper, we investigate the algorithmic complexity of decision problems corresponding to the determination of three aforementioned variants of domination subdivision numbers: paired domination subdivision problem, total domination subdivision problem, and independent domination subdivision problem.

Although the concept of the domination subdivision number dates back to the early 2000s, the complexity results in this area have only been studied recently. The first result concerning the NP-hardness of the decision problem of domination subdivision number is proved by Detlaff et al. [7]. The algorithmic complexity of several other variants of the domination subdivision problem have been investigated in recent years. The decision problem associated with Roman domination subdivision number is shown to be NP-hard by Amjadi et al. [2]. Amjadi and Sadeghi proved the NP-hardness of double and triple Roman domination subdivision problems in [3] and [4], respectively. Haghparast et al. [10] established a general result showing the NP-hardness of $[k]$-Roman domination subdivision number of graphs.

In this paper, we prove that the decision problem of the paired domination subdivision number is NP-hard even when restricted to bipartite graphs. Moreover, we also establish the NP-hardness of the decision problems associated with the total domination subdivision number and the independent domination subdivision number, restricted to bipartite graphs. These results are proved in Sections 3, 4, 5, respectively.

We note here that Amjadi and Chellali [1] proposed a proof of the NP-hardness of the paired domination subdivision problem. We produce a counterexample that demonstrates the error in their proof in Appendix A. Further, we provide a revised proof of the NP-hardness of the paired domination subdivision problem (Theorem 1) by introducing an alternative gadget. The proofs of our results are by a polynomial time transformation from 3-SAT problem. This problem is discussed in the next section.

## 2. 3-SAT Problem

The 3-satisfiability problem, abbreviated 3-SAT, is a well-known NP-complete problem. It serves as a benchmark for proving the NP-hardness of other problems. A problem is proven to be NP-hard by showing a polynomial time reduction from 3-SAT. In reduction, we transform an instance of 3-SAT into an instance of the considered problem. Using the techniques outlined for proving the NP-completeness presented by Garey and Johnson [9], we provide a polynomial time reduction from 3-SAT to establish the NP-hardness of the paired domination subdivision problem, the total domination subdivision problem, and the independent domination subdivision problem. Before stating the 3-SAT problem, we recall some terminology.

Let $U = \{u_1, u_2, \ldots, u_n\}$ be a set of variables. A *truth assignment* for $U$ is a mapping $t : U \to \{T, F\}$. If $t(u_i) = T$, then $u_i$ is said to be *true* under $t$; if $t(u_i) = F$, then $u_i$ is said to be *false* under $t$. In other words, $t$ assigns the value true $(T)$ or false $(F)$ to each variable $u_i$. The variable $u_i$ and the negated variable $\overline{u_i}$ are called *literals* over $U$. In order to extend $t$ to a truth assignment on literals, we set $t(\overline{u_i}) = T$ if $t(u_i) = F$; and $t(\overline{u_i}) = F$ otherwise. A *clause* over $U$ is a set of literals over $U$. A clause is said to be *satisfied* by a truth assignment if and only if at least one of its members is true under that assignment. A collection $\mathscr{C}$ of clauses over $U$ is *satisfiable* if and only if there exists a truth assignment for $U$ that simultaneously satisfies all the clauses in $\mathscr{C}$. Such a truth assignment is a *satisfying truth assignment* for $\mathscr{C}$. Given clauses $C_1, C_2, \ldots, C_m$ each containing exactly three literals over $U$, the 3-SAT problem is to check if there exists a truth assignment for $U$ that simultaneously satisfies all the clauses. Let us state the 3-SAT problem as follows. For a positive integer $k$, we use the standard notation $[k] = \{1, 2, \ldots, k\}$.

**3-Satisfiability Problem (3-SAT)**

*Instance:* A collection $\mathscr{C} = \{C_1, C_2, \ldots, C_m\}$ of clauses over a finite set $U$ of variables such that $|C_j| = 3$ for each $j \in [m]$.

*Question:* Is there a satisfying truth assignment for $\mathscr{C}$?

The NP-completeness of the 3-SAT problem is a cornerstone of computational complexity theory. The following theorem serves as the foundation for polynomial time reductions used to establish the NP-hardness of other problems.

**Theorem A (Theorem 3.1, [9]).** The 3-Satisfiability Problem is NP-complete.

## 3. Complexity of Paired Domination Subdivision Problem

Let us first recall the definition of the paired domination subdivision number. Let $G$ be a graph of order at least three, without isolated vertices. The paired domination subdivision number of $G$, denoted $\mathrm{sd}_{\gamma_{pr}}(G)$, is the minimum number of edges that must be subdivided (where no edge in $G$ can be subdivided more than once) in order to increase the paired domination number of $G$. Note that, if $D$ is a paired dominating set with a perfect matching $M$, then two vertices $u$ and $v$ are said to be *paired* or *partners* in $D$ if the edge $uv \in M$. Clearly, the paired domination number $\gamma_{pr}(G)$ is always an even integer.

In this section, we show that the problem of determining the paired domination subdivision number in bipartite graphs is NP-hard. Let us first state the problem as the following decision problem.

**Paired Domination Subdivision Problem**

*Instance:* A nonempty graph $G$ and a positive integer $k$.

*Question:* Is $\mathrm{sd}_{\gamma_{pr}}(G) \leq k$?

Now we prove the NP-hardness of the paired domination subdivision problem.

**Theorem 1.** *The paired domination subdivision problem is NP-hard even when restricted to bipartite graphs and $k = 1$.*

*Proof.* Let $U = \{u_1, u_2, \ldots, u_n\}$ and $\mathscr{C} = \{C_1, C_2, \ldots, C_m\}$ be an arbitrary instance of 3-SAT. We construct a bipartite graph $G$ from the instance $(U, \mathscr{C})$ of 3-SAT such that $\mathscr{C}$ is satisfiable if and only if $\mathrm{sd}_{\gamma_{pr}}(G) = 1$ as follows:

1. For each variable $u_i \in U$, associate a graph $H_i$ with the vertex set $(V(H_i) = v_i, w_i, u_i, \overline{u_i}, p_i, a_i, b_i, c_i, q_i, d_i, e_i, f_i)$ and the edge set $E(H_i) = \{a_ip_i, b_ip_i, c_ip_i, v_ip_i, w_ip_i, d_iq_i, e_iq_i, f_iq_i, u_iq_i, v_iu_i, q_i\overline{u_i}, w_i\overline{u_i}\}$
2. For each clause $C_j$, associate a single vertex $C_j$. We add the edge $C_ju_i$ if $u_i \in C_j$, and the edge $C_j\overline{u_i}$ if $\overline{u_i} \in C_j$ to the edge set of $G$.
3. Add a path $L$ with the vertex set $V(L) = \{l_1, l_2, l_3, l_4\}$ and the edge set $E(L) = \{l_1l_2, l_2l_3, l_3l_4\}$. We denote this path by $l_1l_2l_3l_4$.
4. Finally, for each $j \in [m]$, join the vertex $l_1$ to the clause vertex $C_j$.

It is easy to see that the graph $G$ contains $12n + m + 4$ vertices and $12n + 4m + 3$ edges. Hence, the construction of $G$ can be accomplished in polynomial time. It is clear that the constructed graph $G$ is a bipartite graph with bipartition

$X = \{a_i, b_i, c_i, w_i, v_i, q_i, C_j, l_2, l_4 \mid 1 \leq i \leq n, 1 \leq j \leq m\}$ and

$Y = \{u_i, \overline{u_i}, p_i, d_i, e_i, f_i, l_1, l_3 \mid 1 \leq i \leq n\}$.

Figure 1 illustrates the construction of the graph $G$, for the instance $(U, \mathscr{C})$ where $U = \{u_1, u_2, u_3, u_4\}$, $\mathscr{C} = \{C_1, C_2, C_3\}$, with $C_1 = \{u_1, \overline{u_3}, \overline{u_4}\}, C_2 = \{\overline{u_1}, u_2, u_3\}, C_3 = \{u_1, u_2, u_4\}$.

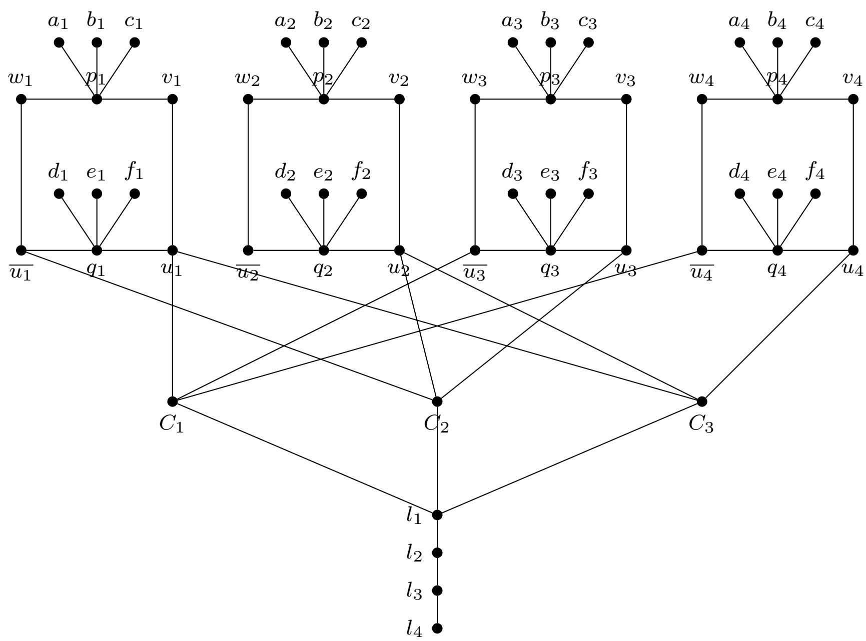


**Figure 1.** **An illustration of the construction of $G$ in the proof of Theorem 1**

We show that, the collection $\mathscr{C}$ is satisfiable if and only if $\mathrm{sd}_{\gamma_{pr}}(G) = 1$, through the following four claims.

**Claim 1.1.** $\gamma_{pr}(G) \geq 4n+2$. Moreover, if $\gamma_{pr}(G) = 4n+2$, then for any $\gamma_{pr}$-set $D$ of $G$, we have, $D \cap L = \{l_2, l_3\}$, $|D \cap V(H_i)| = 4$ and $|D \cap \{u_i, \overline{u_i}\}| \leq 1$ for each $i \in [n]$, and $C_j \notin D$ for each $j \in [m]$.

*Proof.* Let $D$ be a $\gamma_{pr}$-set of $G$. It is clear from the construction of $G$ that the pendant vertices $a_i, b_i, c_i$ and $d_i, e_i, f_i$ are covered only by the support vertices $p_i$ and $q_i$, respectively. Also, the vertices $p_i$ and $q_i$ are not adjacent. So at least four vertices from each gadget $H_i$ must be present in $D$. Hence, $|D \cap V(H_i)| \geq 4$ for each $i \in [n]$. Further, to paired dominate the pendant vertex $l_4$, the set $D$ must include at least two vertices from $L$. Hence, $\gamma_{pr}(G) \geq 4n+2$.
Suppose $\gamma_{pr}(G) = 4n+2$. Then $|D \cap V(H_i)| = 4$ for each $i \in [n]$ and $|D \cap V(L)| = 2$. This shows that, $D \cap L = \{l_2, l_3\}$ and the clause vertex $C_j \notin D$ for each $j \in [m]$. Observe that $u_i$ and $\overline{u_i}$ along with their possible partners can not dominate $H_i$. If both $u_i$ and $\overline{u_i}$ are in $D$ for some $i \in [n]$, then four additional vertices (other than $u_i$ and $\overline{u_i}$) from $H_i$ must be included in the set $D$, which is a contradiction to $|D \cap V(H_i)| = 4$. Therefore, $|D \cap \{u_i, \overline{u_i}\}| \leq 1$ for each $i \in [n]$. This proves Claim 1.1.

**Claim 1.2.** The collection $\mathscr{C}$ is satisfiable if and only if $\gamma_{pr}(G) = 4n+2$.

*Proof.* Suppose $D$ is a $\gamma_{pr}$-set of $G$ with cardinality $4n+2$. By Claim 1.1, $|D \cap \{u_i, \overline{u_i}\}| \leq 1$ for each $i \in [n]$. Let us define a mapping

$$t : U \to \{T, F\} \quad \text{by} \quad t(u_i) = \begin{cases} T & \text{if } u_i \in D; \\ F & \text{otherwise.} \end{cases}$$

Now we show that $t$ is a satisfying truth assignment for the collection $\mathscr{C}$. Choose an arbitrary clause $C_j$ in $\mathscr{C}$. Since $l_1 \notin D$ and the corresponding clause vertex $C_j \notin D$, there exists at least one $k \in [n]$ such that vertex $C_j$ is dominated either by $u_k \in D$ or by $\overline{u_k} \in D$. Now, by definition of the mapping $t$, the literal belonging to $D$ assumes value $T$. It follows that the clause $C_j$ is satisfied by $t$. Since $C_j$ is arbitrary, we get that $t$ satisfies all the clauses in $\mathscr{C}$. Hence, the collection $\mathscr{C}$ is satisfiable.
Conversely, suppose the collection $\mathscr{C}$ is satisfiable. If $t : U \to \{T, F\}$ is a satisfying truth assignment for $\mathscr{C}$, then we can construct a paired dominating set $D$ of $V(G)$ with cardinality $4n+2$ as follows. If $t(u_i) = T$, put $u_i, q_i, p_i, w_i$ in $D$ and if $t(u_i) = F$, put $\overline{u_i}, q_i, p_i, w_i$ in $D$. Now $\{u_i, q_i, p_i, w_i\}$ as well as $\{\overline{u_i}, q_i, p_i, w_i\}$ are paired dominating sets for the gadget $H_i$. As $t$ is a satisfying truth assignment for $\mathscr{C}$, the clause vertex $C_j$ in $G$ is adjacent to at least one vertex in $D$. Therefore, each clause vertex $C_j$ with $j \in [m]$ is dominated by a vertex in $D$. Finally, add the vertices $l_2, l_3$ to $D$ to form a paired dominating set of the path $L$. Thus, we have constructed a paired dominating set $D$ of $G$ with cardinality $4n+2$. But by Claim 1.1, $\gamma_{pr}(G) \geq 4n+2$, and hence, $\gamma_{pr}(G) = 4n+2$. This proves Claim 1.2.

**Claim 1.3.** Let $G'$ be a graph obtained by subdividing any edge of the graph $G$. Then $\gamma_{pr}(G') \leq 4n+4$.

*Proof.* Observe that, the vertex $l_1$ is adjacent to each clause vertex $C_j$ in graph $G$. We will make use of this fact to construct a paired dominating set $D'$ of $G'$ with cardinality $4n+4$.
Suppose an edge in the gadget $L$ consisting of path $l_1l_2l_3l_4$ is subdivided by introducing a subdivision vertex $w$. Now, we construct a set $D'$ that dominates clause vertices and path vertices as follows. Select the vertex $l_1$ and its next vertex, which is either $l_2$ or $w$, call it $t_1$. Select vertex $l_4$ and its preceding vertex, which is either $l_3$ or $w$, call it $t_4$. It is easy to observe that the set $D' = \{l_1, t_1, l_4, t_4\} \cup \{p_i, b_i, d_i, q_i \mid i \in [n]\}$ is a paired dominating set of $G'$.
Assume that an edge incident to a clause vertex $C_j$ is subdivided. Note that, the vertex $C_j$ is adjacent to the vertex $l_1$ and three literals over $U$. If $C_j$ is adjacent to $u_k$ for some $k \in [n]$, and the edge $C_ju_k$ is subdivided, then $D' = \{l_1, l_2, l_3, l_4\} \cup \{b_i, p_i, q_i, u_i \mid i \in [n]\}$ is a paired dominating set of $G'$. Similarly, if $C_j$ is adjacent to some $\overline{u_k}$, we can obtain a paired dominating set by replacing $u_k$ with $\overline{u_k}$ in $D'$.
Consider the case where an edge in the gadget $H_k$ is subdivided for some $k \in [n]$. If an edge incident to the vertex $p_k$ is subdivided with a subdivision vertex $w$, then $D' = \{l_1, l_2, l_3, l_4, p_k, w, d_k, q_k\} \cup \{b_i, p_i, d_i, q_i \mid i \in [n], i \neq k\}$ is a paired dominating set of $G'$. Similarly, if an edge incident to the vertex $q_k$ is subdivided with a subdivision vertex $w$, then $D' = \{l_1, l_2, l_3, l_4, q_k, w, p_k, b_k\} \cup \{b_i, p_i, d_i, q_i, \mid i \in [n], i \neq k\}$ is a paired dominating set of $G'$. If the edge $u_iv_i$ or $\overline{u_i}w_i$ is subdivided, then $D' = \{l_1, l_2, l_3, l_4\} \cup \{q_i, u_i, p_i, w_i \mid i \in [n]\}$ is the required paired dominating set.
Therefore, in each case, we get a paired dominating set $D'$ of $G'$ with cardinality $4n+4$. This proves Claim 1.3.

**Claim 1.4.** $\gamma_{pr}(G) = 4n+2$ if and only if $\mathrm{sd}_{\gamma_{pr}}(G) = 1$.

*Proof.* Assume that $\gamma_{pr}(G) = 4n+2$. We subdivide the edge $l_3l_4$ in the gadget $L$ to obtain a new graph $G'$ with subgraph $L'$ being the path $l_1l_2l_3wl_4$. By Claim 1.3, $\gamma_{pr}(G') \le 4n+4$. Also, if $D'$ is a paired dominating set of $G'$, then $|D' \cap V(H_i)| \ge 4$ for each $i$ and no vertex of $L'$ is dominated by $D' \cap V(H_i)$ for any $i$. Hence, at least four vertices are required to paired dominate the vertices of $L'$, leading to $\gamma_{pr}(G') \ge 4n+4$, and the equality follows from Claim 1.3. Therefore, $\gamma_{pr}(G) < \gamma_{pr}(G')$ and thus $\mathrm{sd}_{\gamma_{pr}}(G) = 1$.
Conversely, suppose that $\mathrm{sd}_{\gamma_{pr}}(G) = 1$, and note that the paired domination number is always an even integer. By Claim 1.1, $\gamma_{pr}(G) \ge 4n+2$. Let $G'$ be obtained from $G$ by subdividing any edge of $G$ such that $\gamma_{pr}(G) < \gamma_{pr}(G')$. By Claim 1.3, $\gamma_{pr}(G') \le 4n+4$ and thus $4n+2 \le \gamma_{pr}(G) < \gamma_{pr}(G') \le 4n+4$. Therefore, $\gamma_{pr}(G) = 4n+2$, as desired.

By Claims 1.2 and 1.4, it follows that $\mathrm{sd}_{\gamma_{pr}}(G) = 1$ if and only if the collection $\mathscr{C}$ is satisfiable. The construction of the paired domination subdivision instance is straightforward from the 3-SAT instance and the size of the paired domination subdivision instance is bounded above by a polynomial function of the size of the 3-SAT instance. It follows that this is a polynomial reduction and thus the NP-hardness of the paired domination subdivision problem in bipartite graphs is established. $\square$

As noted in the introduction, a proof of the NP-hardness of the paired domination subdivision problem was given by Amjadi and Chellali [1], but the proof contains an error which will be explained in Appendix A.

## 4. Complexity of Total Domination Subdivision Problem

Recall that the total domination subdivision number $\mathrm{sd}_{\gamma_t}(G)$ of an isolate-free graph $G$ of order at least 3, is the minimum number of edges that must be subdivided, each at most once, in order to increase the total domination number of $G$. In this section, we show that the problem of determining the total domination subdivision number in bipartite graphs is NP-hard by a slight modification of the proof of Theorem 1. We first state the problem as the decision problem, followed by the proof of the NP-hardness.

**Total Domination Subdivision Problem**

*Instance:* A nonempty graph $G$ and a positive integer $k$.

*Question:* Is $\mathrm{sd}_{\gamma_t}(G) \leq k$?

**Theorem 2.** *The total domination subdivision problem is NP-hard even when restricted to bipartite graphs and $k = 1$.*

*Proof.* Let $U = \{u_1, u_2, \ldots, u_n\}$ and $\mathscr{C} = \{C_1, C_2, \ldots, C_m\}$ be an arbitrary instance of 3-SAT. We construct a bipartite graph $G$ from the instance $(U, \mathscr{C})$ of 3-SAT such that $\mathscr{C}$ is satisfiable if and only if $\mathrm{sd}_{\gamma_t}(G) = 1$ as follows. The graph required for this proof is a slight modification of the graph used in the proof of the paired domination subdivision problem by introducing an additional vertex $l_5$ adjacent to vertex $l_4$. Clearly the graph remains bipartite and the construction can be accomplished in polynomial time.
Figure 2 illustrates the construction of the graph $G$ from the instance $(U, \mathscr{C})$ where $U = \{u_1, u_2, u_3, u_4\}$, $\mathscr{C} = \{C_1, C_2, C_3\}$, and
$C_1 = \{\overline{u_1}, \overline{u_3}, \overline{u_4}\}, C_2 = \{u_1, u_3, u_4\}, C_3 = \{\overline{u_2}, u_3, u_4\}$.

In the next, we prove that the collection $\mathscr{C}$ is satisfiable if and only if $\mathrm{sd}_{\gamma_t}(G) = 1$, through the following four claims.

**Claim 2.1.** $\gamma_t(G) \geq 4n + 3$. Moreover, if $\gamma_t(G) = 4n + 3$, then for any $\gamma_t$-set $D$ of $G$, we have $\{l_3, l_4\} \subseteq D$ and $|D \cap V(H_i)| = 4$, $|D \cap \{u_i, \overline{u_i}\}| \leq 1$ for each $i \in [n]$, and $l_1 \notin D$.

*Proof.* Let $D$ be a $\gamma_t$-set of $G$. By the construction of $G$, the pendant vertices $a_i, b_i, c_i$ and $d_i, e_i, f_i$ are covered only by vertices $p_i$ and $q_i$, respectively. Since the support vertices $p_i$ and $q_i$ are neither adjacent nor have common neighbors, at least four vertices from each gadget $H_i$ must be present in the total dominating set $D$.

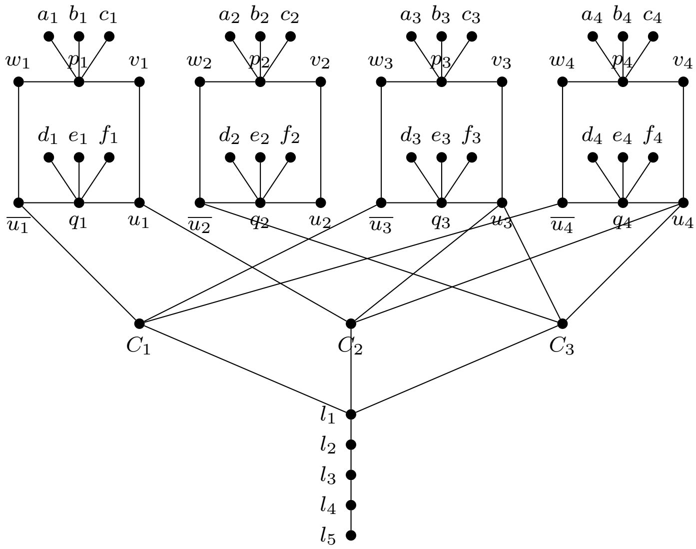


**Figure 2.** **An illustration of the construction of $G$ in the proof of Theorem 2**

Moreover, to totally dominate the vertices of path $L$ at least three vertices are required from $V(G) \setminus \bigcup_{i=1}^{n} V(H_i)$. Hence, $\gamma_t(G) \geq 4n+3$.

Suppose $\gamma_t(G) = 4n+3$. Then $|D \cap V(H_i)| = 4$ for each $i \in [n]$. If both $u_i$ and $\overline{u_i}$ are in $D$, then it is not possible to totally dominate all vertices of $H_i$ by selecting at most two additional vertices from $H_i$. Therefore, $|D \cap \{u_i, \overline{u_i}\}| \leq 1$ for each $i \in [n]$. Now we show that $\{l_3, l_4\} \subseteq D$. Since $l_5$ is a pendant vertex with support vertex $l_4$, we must have either $\{l_4, l_5\} \subseteq D$ or $\{l_3, l_4\} \subseteq D$. We show that the former is not possible. The vertices $l_1$ and $l_2$ are not dominated by $l_4, l_5$ nor by the vertices of gadgets $H_i$. Hence, if $\{l_4, l_5\} \subseteq D$, then it is not possible to choose a single vertex of $G$ to totally dominate both $l_1$ and $l_2$. Therefore, $\{l_3, l_4\} \subseteq D$. Now, if $l_1 \in D$, then no vertex in open neighborhood of $l_1$ belongs to $D$, which is a contradiction to the fact that $D$ is a total dominating set of $G$. Hence, $l_1 \notin D$. This completes the proof of Claim 2.1.

**Claim 2.2.** The collection $\mathscr{C}$ is satisfiable if and only if $\gamma_t(G) = 4n+3$.

*Proof.* Let $D$ be a $\gamma_t$-set of $G$ with cardinality $4n+3$. By Claim 2.1, $l_1 \notin D$ and for each $i \in [n]$, $|D \cap \{u_i, \overline{u_i}\}| \leq 1$. Let us define a mapping

$$t : U \to \{T, F\} \quad \text{by} \quad t(u_i) = \begin{cases} T & \text{if } u_i \in D; \\ F & \text{otherwise.} \end{cases}$$

Choose an arbitrary clause $C_j$ in the collection $\mathscr{C}$. Since $l_1 \notin D$ and the fact that $D$ is a total dominating set, the clause vertex $C_j$ must be adjacent to either $u_k \in D$

or $\overline{u_k} \in D$ for some $k \in [n]$. By definition of the mapping $t$, the literal present in $D$ assumes value $T$. It follows that the clause $C_j$ is satisfied by $t$. By the arbitrariness of $C_j$ with $j \in [m]$, we get that $t$ is a satisfying truth assignment for $\mathscr{C}$. Thus, the collection $\mathscr{C}$ is satisfiable.
Conversely, suppose the collection $\mathscr{C}$ is satisfiable. If $t : U \to \{T, F\}$ is a satisfying truth assignment for $\mathscr{C}$, then we construct a total dominating set $D$ of $G$ with cardinality $4n + 3$ as follows. If $t(u_i) = T$, put $u_i, q_i, p_i, w_i$ in $D$ and if $t(u_i) = F$, put $\overline{u_i}, q_i, p_i, w_i$ in $D$. Now $\{u_i, q_i, p_i, w_i\}$ as well as $\{\overline{u_i}, q_i, p_i, w_i\}$ are total dominating sets for the gadget $H_i$. As $t$ is a satisfying truth assignment for $\mathscr{C}$, the clause vertex $C_j$ is adjacent to at least one vertex in $D$. Therefore, each clause vertex $C_j$ with $j \in [m]$ is dominated by at least one vertex in $D$. Finally, add the vertices $l_2, l_3, l_4$ to $D$ to totally dominate the vertices of the path $L$. Thus, $D$ is a total dominating set of $G$ with cardinality $4n + 3$. But by Claim 2.1, $\gamma_t(G) \geq 4n + 3$, and hence, $\gamma_t(G) = 4n + 3$. This completes the proof of Claim 2.2.

**Claim 2.3.** Let $G'$ be a graph obtained by subdividing any edge of the graph $G$. Then $\gamma_t(G') \leq 4n + 4$.

*Proof.* The proof is similar to the proof of Claim 1.3.

**Claim 2.4.** $\gamma_t(G) = 4n + 3$ if and only if $\mathrm{sd}_{\gamma_t}(G) = 1$.

*Proof.* The proof is similar to the proof of Claim 1.4.

By Claims 2.2 and 2.4, we proved that $\mathrm{sd}_{\gamma_t}(G) = 1$ if and only if the collection $\mathscr{C}$ is satisfiable. The construction of the total domination subdivision instance is polynomial-time from a 3-SAT instance, since the graph $G$ contains $12n + m + 5$ vertices and $12n + 4m + 4$ edges. Thus the NP-hardness of the total domination subdivision problem in bipartite graphs follows. $\square$

## 5. Complexity of Independent Domination Subdivision Problem

In this section, we study the complexity of the independent domination subdivision number problem. Recall that the independent domination subdivision number, $\mathrm{sd}_i(G)$, of a connected graph $G$ of order at least 3, is the minimum number of edges that must be subdivided, each at most once, in order to increase the independent domination number of $G$. We show that the problem of determining the independent domination subdivision number in bipartite graphs is NP-hard. The corresponding decision problem is stated below, followed by the proof of NP-hardness.

**Independent Domination Subdivision Problem**

*Instance:* A nonempty graph $G$ and a positive integer $k$.

*Question:* Is $\mathrm{sd}_i(G) \leq k$?

**Theorem 3.** *The independent domination subdivision problem is NP-hard even when restricted to bipartite graphs and $k = 1$.*

*Proof.* Let $U = \{u_1, u_2, \ldots, u_n\}$ and $\mathscr{C} = \{C_1, C_2, \ldots, C_m\}$ be an arbitrary instance of 3-SAT. We construct a bipartite graph $G$ from the instance $(U, \mathscr{C})$ of 3-SAT such that $\mathscr{C}$ is satisfiable if and only if $\mathrm{sd}_i(G) = 1$ as follows.

1. For each variable $u_i \in U$, with $i \in 1, 2, \ldots n$, associate a graph $H_i$ with the vertex set $V(H_i) = u_i, \overline{u_i}, a_i, b_i, c_i, a_{i1}, a_{i2}, a_{i3}, b_{i1}, b_{i2}, b_{i3}, c_{i1}, c_{i2}, c_{i3}$ and the edge set $E(H_i) = \{u_i a_i, \overline{u}_i c_i, a_i b_{i1}, a_i b_{i2}, a_i b_{i3}, a_i c_{i1}, a_i c_{i2}, a_i c_{i3}, b_i a_{i1}, b_i a_{i2}, b_i a_{i3}, b_i c_{i1}, b_i c_{i2}, b_i c_{i3}, c_i a_{i1}, c_i a_{i2}, c_i a_{i3}, c_i b_{i1}, c_i b_{i2}, c_i b_{i3}\}$.
2. For each clause $C_j$, associate a single vertex $C_j$. We add the edge $C_j u_i$ if $u_i \in C_j$, and the edge $C_j \overline{u_i}$ if $\overline{u_i} \in C_j$ to the edge set for each $j \in [m]$.
3. Finally, add a path $L$, with vertex set $V(L) = \{l_1, l_2, l_3\}$ and edge set $E(L) = \{l_1 l_2, l_2 l_3\}$, by joining the vertex $l_1$ to each vertex $C_j$.

The construction of the graph $G$ can be accomplished in polynomial time as its order is polynomially bounded in terms of $m$ and $n$. Observe that the graph $G$ constructed above is a bipartite graph with bipartition

$X = \{a_i, b_i, c_i, C_j, l_2 \mid i \in [n], j \in [m]\}$ and

$Y = \{u_i, \overline{u_i}, a_{i1}, a_{i2}, a_{i3}, b_{i1}, b_{i2}, b_{i3}, c_{i1}, c_{i2}, c_{i3}, l_1, l_3 \mid i \in [n]\}$.

Figure 3 illustrates this construction for the instance $U, \mathscr{C}$ with $U = \{u_1, u_2, u_3\}$ and $\mathscr{C} = \{C_1, C_2, C_3, C_4\}$, where $C_1 = \{u_1, u_2, u_3\},\ C_2 = \{\overline{u_1}, u_2, u_3\},\ C_3 = \{u_1, \overline{u_2}, u_3\},\ C_4 = \{u_1, u_2, \overline{u_3}\}$.

We establish that the collection $\mathscr{C}$ is satisfiable if and only if $\mathrm{sd}_i(G) = 1$ through the following four claims.

**Claim 3.1.** $i(G) \geq 3n + 1$. Moreover, if $i(G) = 3n + 1$, then for any $i$-set $D$ of $G$, $D \cap V(L) = \{l_2\}$, $|D \cap V(H_i)| = 3$, $|D \cap \{u_i, \overline{u_i}\}| \leq 1$ for each $i \in [n]$, and $C_j \notin D$ for each $j \in [m]$.

*Proof.* Let $D$ be an $i$-set of $G$. By the construction of $G$, the vertex $l_2$ can be dominated only by vertices in $\{l_1, l_2, l_3\}$. This implies $|D \cap V(L)| \geq 1$. Also, it is easy to observe that at least three vertices from each gadget $H_i$ must be present in any dominating set. Therefore, $i(G) \geq 3n + 1$.

Suppose $i(G) = 3n + 1$. Then $|D \cap V(L)| = 1$ and $|D \cap V(H_i)| = 3$ for each $i \in [n]$. As a result, the clause vertex $C_j \notin D$ for each $j \in [m]$. Moreover, it is not possible

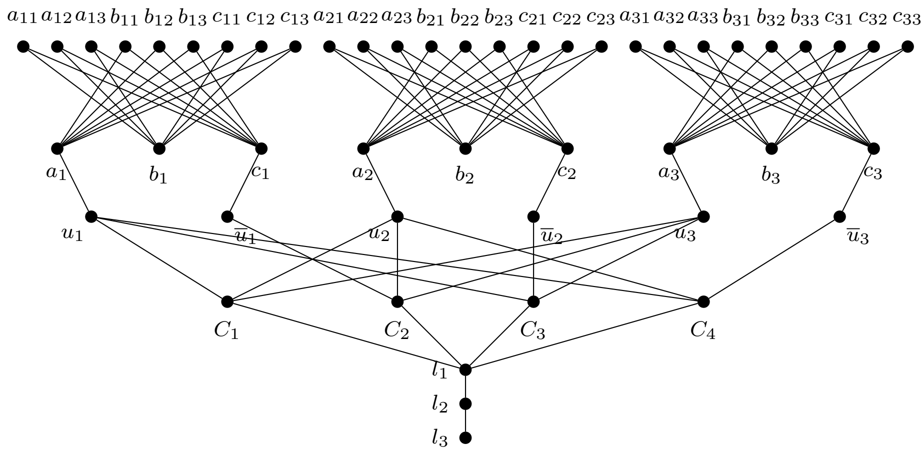


**Figure 3.** **An illustration of the construction of $G$ in the proof of Theorem 3**

to construct an independent dominating set for $H_i$ of cardinality 3 containing both $u_i$ and $\overline{u_i}$, leading to $|D \cap \{u_i, \overline{u_i}\}| \leq 1$. Also, as $l_2$ is dominated only by the vertices of $L$ and $|D \cap V(L)| = 1$, we obtain $D \cap V(L) = \{l_2\}$. This completes the proof of Claim 3.1.

**Claim 3.2.** The collection $\mathscr{C}$ is satisfiable if and only if $i(G) = 3n + 1$.

*Proof.* Suppose that $i(G) = 3n + 1$ and let $D$ be an $i$-set of $G$. By Claim 3.1, we obtain for each $i \in [n]$, $|D \cap \{u_i, \overline{u_i}\}| \leq 1$, and $l_2 \in D$. Let us define a mapping

$$t : U \to \{T, F\} \quad \text{by} \quad t(u_i) = \begin{cases} T & \text{if } u_i \in D; \\ F & \text{otherwise.} \end{cases}$$

Now, we show that $t$ is a satisfying truth assignment for $\mathscr{C}$. Let $C_j$ be an arbitrary clause in the collection $\mathscr{C}$. Then the clause vertex $C_j$ is dominated by a literal in $H_k$, say $u_k$ or $\overline{u_k}$, for some $k \in [n]$. The literal belonging to $D$ assumes value $T$ under the mapping $t$. It follows that the clause $C_j$ is satisfied by $t$. By the arbitrariness of $C_j$ with $j \in [m]$, we get that $t$ satisfies all the clauses in $\mathscr{C}$, that is, $\mathscr{C}$ is satisfiable.
Conversely, suppose $\mathscr{C}$ is satisfiable. We construct a dominating set $D$ of $G$ as follows. Let $t : U \to \{T, F\}$ be a satisfying truth assignment for $\mathscr{C}$. For each $i \in [n]$, if $t(u_i) = T$, put $u_i, b_i, c_i$ in $D$; otherwise put $\overline{u_i}, a_i, b_i$ in $D$. Add the vertex $l_2$ to $D$, and observe that $D$ is an independent set of cardinality $3n + 1$.
Now, $\{u_i, b_i, c_i\}$ as well as $\{\overline{u_i}, a_i, b_i\}$ are dominating sets of the gadget $H_i$. Since $t$ is a satisfying truth assignment for $\mathscr{C}$, the clause vertex $C_j$ in $G$ is adjacent to at least one vertex in $D$. So, each clause vertex $C_j$ is dominated by at least one vertex in $D$. The vertex $l_2$ dominates the path $L$. Thus we have constructed an independent

dominating set $D$ of graph $G$ with cardinality $3n+1$. But by Claim 3.1, $i(G) \geq 3n+1$. Hence, $i(G) = 3n+1$, and this completes the proof of Claim 3.2.

**Claim 3.3.** Let $G'$ be a graph obtained by subdividing any edge of the graph $G$. Then $i(G') \leq 3n+2$.

*Proof.* Observe that, the vertex $l_1$ is adjacent to each vertex $C_j$ in graph $G$. We will make use of this fact to construct an independent dominating set $D'$ of $G'$ with cardinality $3n+2$ in each of the following cases.

If an edge in the gadget $L$ consisting of path $l_1l_2l_3$ is subdivided, then $D' = \{l_1, l_3\} \cup \{a_i, b_i, c_i \mid i \in [n]\}$ is an independent dominating set of $G'$.

Consider an arbitrary clause vertex $C_j$. Note that, $C_j$ is adjacent to the vertex $l_1$ and three literals over $U$. Suppose an edge incident to $C_j$ is subdivided. If the edge $C_ju_k$ is subdivided, then $D' = \{l_1, l_3\} \cup \{u_i, b_i, c_i \mid i \in [n]\}$ is an independent dominating set of $G'$. Similarly, if the edge $C_j\overline{u_k}$ is subdivided then $D' = \{l_1, l_3\} \cup \{\overline{u_i}, a_i, b_i \mid i \in [n]\}$ is an independent dominating set of $G'$. If the edge $C_jl_1$ is subdivided and $C_j$ is adjacent to literal vertex $u_k$ for some $k$, then $D' = \{l_1, l_3\} \cup \{u_i, b_i, c_i \mid i \in [n]\}$ is an independent dominating set of $G'$. Similarly, if the edge $C_jl_1$ is subdivided and $C_j$ is adjacent to literal vertex $\overline{u_k}$ for some $k$, then $D' = \{l_1, l_3\} \cup \{\overline{u_i}, a_i, b_i \mid i \in [n]\}$ is an independent dominating set of $G'$.

Finally, suppose an edge in the gadget $H_k$, for some $k \in [n]$ is subdivided. If the edge $a_ku_k$ is subdivided with a subdivision vertex $w$, then $D' = \{l_1, l_3, w, b_k, c_k\} \cup \{a_i, b_i, c_i \mid i \in [n], i \neq k\}$ is an independent dominating set of $G'$. If the edge $c_k\overline{u_k}$ is subdivided with a subdivision vertex $w$, then $D' = \{l_1, l_3, w, a_k, b_k\} \cup \{a_i, b_i, c_i \mid i \in [n], i \neq k\}$ is an independent dominating set of $G'$. If an edge other than $a_ku_k$ and $c_k\overline{u_k}$ is subdivided, then $D' = \{l_1, l_3\} \cup \{a_i, b_i, c_i \mid i \in [n]\}$ is an independent dominating set of $G'$.

Therefore, in each case, we have constructed an independent dominating set $D'$ of $G'$ with cardinality $3n+2$. Hence, if a graph $G'$ is obtained from the graph $G$ by subdividing any edge of $G$, then $i(G') \leq 3n+2$. This completes the proof of Claim 3.3.

**Claim 3.4.** $i(G) = 3n+1$ if and only if $\mathrm{sd}_i(G) = 1$.

*Proof.* Assume that, $i(G) = 3n+1$. Subdivide the edge $l_2l_3$ with a subdivision vertex $w$, to obtain a new graph $G'$ with subgraph $L'$ being the path $l_1l_2wl_3$. From Claim 3.3 that $i(G') \leq 3n+2$. To prove the equality we first observe that, if $D'$ is an independent dominating set of $G'$, then $|D' \cap H_i| \geq 3$ for each $i \in [n]$. The path $L'$ is not dominated by vertices in $H_i$. Also, at least two vertices are required to dominate $L'$. So, $i(G') \geq 3n+2$. Using Claim 3.3, we obtain $i(G') = 3n+2$, leading to $i(G) < i(G')$, and therefore, $\mathrm{sd}_i(G) = 1$.

Conversely, suppose $\mathrm{sd}_i(G) = 1$. By Claim 3.1, $i(G) \geq 3n+1$. Let $G'$ be obtained from $G$ by subdividing an edge of $G$ such that $i(G) < i(G')$. By Claim 3.3, $i(G') \leq 3n+2$.

Hence, we obtain $3n+1 \leq i(G) < i(G') \leq 3n+2$, and thus $i(G) = 3n+1$, as desired. This completes the proof of Claim 3.4.

By Claims 3.2 and 3.4, we proved that $\mathrm{sd}_i(G) = 1$ if and only if the collection $\mathscr{C}$ is satisfiable. Since the graph $G$ contains $14n+m+3$ vertices and $20n+4m+2$ edges, the size of the independent domination subdivision instance is bounded above by a polynomial function of the size of the 3-SAT instance. It follows that this is a polynomial reduction. Thus, we obtain the NP-hardness of independent domination subdivision problem in bipartite graphs. This completes the proof. □

## 6. Concluding Remarks

In this paper we proved that the decision problems for paired, total, and independent domination subdivision numbers are NP-hard, even when restricted to bipartite graphs. It would be interesting to study the complexity of these problems for other classes of graphs such as chordal graphs and planar graphs. Similarly, it would be interesting to study the complexity of these problems for subclasses of bipartite graphs such as planar bipartite graphs, chordal bipartite graphs and star-convex bipartite graphs.

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**Declaration of Data Sharing:** Data sharing not applicable to this article as no data sets were generated or analysed during the current study.

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## A. Appendix

Amjadi and Chellali [1] proposed a proof of the NP-hardness of paired domination

subdivision problem. We hereby point out the error in the proof. Throughout, we follow the notations of [1]. We refer to Claims 1, 2, 3, 4 as stated in [1]. One can verify that Claims 1 and 2 are correct. However, we show that the Claims 3 and 4 are incorrect by providing the following counterexample.
Consider the following instance of 3-SAT, where $U = \{u_1, u_2, \dots, u_6\}$ and $\mathscr{C} = \{C_1, C_2, \dots, C_9\}$, with

$$\begin{array}{lll} C_1 = \{u_1, u_2, u_3\}, & C_2 = \{\overline{u_1}, u_2, u_3\}, & C_3 = \{u_1, \overline{u_2}, u_3\}, \\ C_4 = \{u_1, u_2, \overline{u_3}\}, & C_5 = \{\overline{u_1}, \overline{u_2}, u_3\}, & C_6 = \{\overline{u_1}, u_2, \overline{u_3}\}, \\ C_7 = \{u_1, \overline{u_2}, \overline{u_3}\}, & C_8 = \{\overline{u_1}, \overline{u_2}, \overline{u_3}\}, & C_9 = \{u_4, u_5, u_6\}. \end{array}$$

Note that, the collection $\mathscr{C}$ is not satisfiable. Observe that regardless of how we assign the truth values $T, F$ to the literals $u_1, u_2, u_3$, there always exists a clause that is not satisfied. Let $\alpha_i$ is the literal which is assigned the value $T$ among $u_i$ and $\overline{u}_i$ for each $i = 1, 2, 3$. Then all the literals in the clause $\{\overline{\alpha_1}, \overline{\alpha_2}, \overline{\alpha_3}\} \in \mathscr{C}$ assume truth value $F$, making the clause unsatisfied.
We construct a graph $G$ as per the construction given in [1] for above instance $(U, \mathscr{C})$ (Refer Figure 4). This graph $G$ is a counterexample to Claims 3 and 4. For the sake of convenience of the reader, we state these Claims 1 to 4 here.

**Claim 1.** $\gamma_{pr}(G) \geq 2n + 4$. Moreover, if $\gamma_{pr}(G) = 2n + 4$, then for every $\gamma_{pr}(G)$-set $S$ of $G$, $|V(H_i) \cap S| = 2$ and $(V(H) \cup \{c_1, c_2, \dots, c_m\}) \cap S = \{s_3, s_4, s_5, s_6\}$.

**Claim 2.** $\mathscr{C}$ is satisfiable if and only if $\gamma_{pr}(G) = 2n + 4$.

**Claim 3.** Let $G'$ be obtained from $G$ by subdividing any edge $e$ of $E(G)$, then $\gamma_{pr}(G') \leq 2n + 6$.

**Claim 4.** $\gamma_{pr}(G) = 2n + 4$ if and only if $\mathrm{sd}_{\gamma_{pr}}(G) = 1$.

Note that for the graph $G$ constructed from the above instance $(U, \mathscr{C})$, we have $n = 6, m = 9$. Since $\mathscr{C}$ is not satisfiable, by Claims 1 and 2, we obtain that $\gamma_{pr}(G) > 2n+4$, that is, $\gamma_{pr}(G) \geq 18$. Observe that $S = \{s_1, s_3, s_5, s_7, s_4, s_6, y_i, v_i \mid i = 1, 2, \dots, 6\}$ is a paired dominating set of $G$ with cardinality 18. Hence, $\gamma_{pr}(G) = 18$. We now show that the Claim 3 does not hold for the graph $G$. Consider the graph $G'$ obtained by subdividing the edge $z_6r_6$ in $G$ with a subdivision vertex $w$. We show that $\gamma_{pr}(G') = 20$, that is, $\gamma_{pr}(G') > 2n + 6$ contradicting Claim 3.
We use the following notations. For each $i \in \{1, 2, 3, 4, 5\}$, the variable gadget consisting of complete bipartite graph $K_{3,5}$ is denoted by $H_i$. Let $H'_6$ denote the graph obtained by subdividing the edge $z_6r_6$ in $K_{3,5}$ by introducing a subdivision vertex $w$. The clause gadget, composed of two disjoint paths $s_1s_3s_5s_5$ and $s_2s_4s_6s_8$, is denoted by $H$.

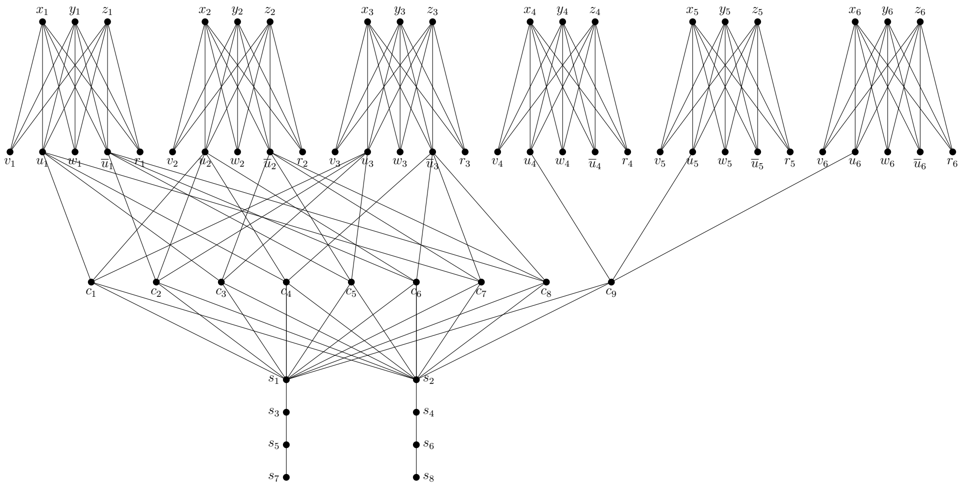


**Figure 4.** **Counterexample to Claims 3 and 4 in Theorem 1 of [1]**

Let $S'$ be a $\gamma_{pr}$-set of $G'$. Since, $s_7$ and $s_8$ are pendant vertices with support vertices $s_5$ and $s_6$, respectively, it is trivial to note that $s_5, s_6$ along with their possible partners must be present in $S'$ and thus $|S' \cap V(H)| \geq 4$. For each $i = 1, 2, \ldots, 5$, it is clear that $|S' \cap V(H_i)| \geq 2$. Furthermore, note that at least 4 vertices from $H_6' \cup \{c_9\}$ are required to paired dominate the vertices of $H_6'$. Hence, we get that $\gamma_{pr}(G') \geq 18$.

We claim that $\gamma_{pr}(G') = 20$. Let, if possible, $\gamma_{pr}(G') = 18$. Then we get $|S' \cap V(H)| = 4$, $|S' \cap H_i| = 2$ for each $i = 1, 2, \ldots, 5$ and $|S' \cap (V(H_6') \cup \{c_9\})| = 4$. It is clear that $c_j \notin S'$ for each $j = 1, 2, \ldots, 8$. Since $|S' \cap V(H)| = 4$, we obtain $s_1, s_2 \notin S'$. The vertices in $S' \cap (V(H_4) \cup V(H_5) \cup V(H_6') \cup \{c_9\})$ collectively dominate the vertices in the gadgets $H_4, H_5, H_6'$ and the clause vertex $c_9$ alone. Observe that, $|S' \cap \{u_i, \overline{u_i}\}| \leq 1$ for each $i = 1, 2, 3$. If the literal $u_i$ or $\overline{u_i}$ belongs to $S'$ then let $\alpha_i$ be negation of the literal belonging to $S'$, otherwise set $\alpha_i = u_i$. Now, it is easy to see that the clause vertex $\{\alpha_1, \alpha_2, \alpha_3\} \in \mathscr{C} \setminus \{c_9\}$ is not dominated by $S'$, which is a contradiction to $\gamma_{pr}(G') = 18$. Therefore, $\gamma_{pr}(G') \geq 20$. Moreover, the set $\{s_1, s_3, s_5, s_7, s_4, s_6, z_6, w\} \cup \{x_i, v_i \mid 1 \leq i \leq 6\}$ is a paired dominating set of $G'$ with cardinality 20. Hence, $\gamma_{pr}(G') = 20$.

Since $\gamma_{pr}(G) = 18$ and $\gamma_{pr}(G') = 20$, we get $\mathrm{sd}_{\gamma_{pr}}(G) = 1$. However, the underlying collection of clauses $\mathscr{C}$ in the construction of the graph $G$ is not satisfiable. This shows that Claim 4 does not hold.