Communications in Combinatorics and Optimization
Vol. xx, No. x (xxxx), pp. 1-14
https://doi.org/10.22049/cco.2025.30278.2395

CCO
Commun. Comb. Optim.

Research Article

On the s-coloring of signed graphs

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Received: 16 January 2025; Accepted: 19 April 2025
Published Online: 28 April 2025

Abstract: The *sign* of a vertex in a signed graph be defined naturally as the product of signs of edges incident to the vertex. We say that an edge is *consistent* or a *c-edge* if its end-vertices have the same sign. Over the years, different notions of vertex coloring have been defined for signed graphs. Here, we introduce a new type of coloring in which any two vertices joined by a c-edge are assigned different colors. We call this the *s-coloring* of a signed graph. The s-chromatic number $\chi_s(G)$ of a signed graph G is the minimum number of colors required to properly s-color the vertices of G. We obtain several bounds for $\chi_s(G)$. We show that the number of s-colorings of a signed graph G is a polynomial function of the number k of colors, which we call the s-chromatic polynomial $S(G,k)$ of G. We define the operations of *removal* and *compression* to develop a deletion-contraction type recursive procedure for determining $S(G,k)$. We introduce the notions of c-complete and c-full signed graphs, characterizing different classes of c-full signed graphs and determining the number of c-complete signed graphs on a given number of vertices. Furthermore, the relationship between s-coloring and other signed graph colorings is also investigated.

Keywords: signed graph, signed graph coloring, independent set, chromatic number, chromatic polynomial, deletion-contraction.

AMS Subject classification: 05C15, 05C22, 05C85

1. Introduction

A *signed graph* is a triple $G = (V, E, \sigma)$, where $G_u = (V, E)$ is a graph and $\sigma : E \to \{+, -\}$ is a sign function. The graph G_u is called the *underlying graph* of G. An edge e is said to be *positive* if $\sigma(e) = +$ and *negative* otherwise. For a vertex v in a signed graph G, let $d^+(v)$ and $d^-(v)$ respectively denote the number of positive and negative edges incident at v and call them the *positive degree* and *negative degree* of

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v. The *signed degree* of v is defined as $deg(v) = d^+(v) - d^-(v)$ and the sequence of signed degrees arranged in non-increasing or non-decreasing order is called the *signed degree sequence* of G. The *positive degree sequence* and the *negative degree sequence* are defined analogously. When listing the terms of a signed degree, positive degree or negative degree sequence, we write $x^{(m)}$ to indicate that x appears in m consecutive terms of the sequence. For other undefined notations and terminology, the readers are referred to [3, 10, 12].

There exist two types of coloring of signed graphs in the literature. Harary and Cartwright [2] define a coloring of signed graph in which no two end-vertices of a negative edge are colored the same, whereas the end vertices of a positive edge are necessarily colored the same. This process of coloring is also referred to as signed graph *partitioning* or *clustering* and is widely used in network analysis. The motivation is to partition the vertices of a network in such a way that any two actors (vertices) that have a negative relationship (edge) are partitioned into different color classes while the actors having a positive relationship are grouped together in the same color class. However, not all signed graphs can be colored using this type of coloring. For instance, consider the complete graph K_3 whose edges are alternately assigned $+$ and $-$ signs. In the sequel we refer to this coloring as the *p-coloring* and say that a signed graph is *partitionable* or p-colorable if it can be colored in this way. The corresponding chromatic number, if it exists, will be called the *p-chromatic number* and denoted by $\chi_p(G)$.

The other notion of coloring is due to Zaslavsky [13, 14], who defines a coloring of a signed graph $G = (V, E, \sigma)$ as a function $f : V \to \{0, \pm 1, \dots, \pm n\}$ such that if uv is an edge, $f(u) \neq f(v)$ and if uv is a negative edge with $f(u) = i$, then $f(v) \neq -i$. Zaslavsky obtained a chromatic polynomial for his coloring and showed that the traditional deletion-contraction recursion can be used to determine the chromatic polynomial of a signed graph. We refer to this coloring, the corresponding chromatic number and the related chromatic polynomial as *z-coloring*, *z-chromatic number* denoted by $\chi_z(G)$ and *z-chromatic polynomial* denoted by $P_z(G, k)$ respectively. Some recent work on signed graph coloring can be seen in [1, 4–9, 11].

In this paper, we define a new type of vertex coloring for signed graphs. Given a signed graph G, two vertices u and v of G are said to be *c-adjacent* if uv is an edge and $d^-(u)$ and $d^-(v)$ are either both even or both odd. In this case, we call the edge uv a *consistent edge* or a *c-edge*. For a vertex v, let $Sgn(v)$ denote the product of the signs of the edges incident at v. Clearly, two adjacent vertices u and v are c-adjacent if and only if $Sgn(u) = Sgn(v)$. We define an *s-coloring* of a signed graph G as a coloring of its vertices such that no two c-adjacent vertices receive the same color. The *s-chromatic number*, $\chi_s(G)$, is the minimum number of colors required for s-coloring G. Figure 1 compares the three notions of coloring for a signed graph G. Clearly, *s-coloring* of a signed graph G is based on the parity of the negative degree of the end vertices of an edge. That is, end vertices of an edge uv receive different colors if $d^-(u) \equiv d^-(v) (mod\ 2)$. This will help to understand the structure of the graph and make connections between graphs and number theory.

This paper is organized as follows. In Section 2, we define the concepts of c-

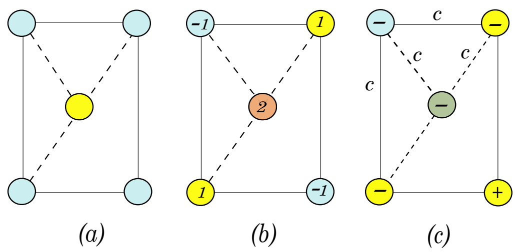

Figure 1. **Comparison of (a) p-coloring, (b) z-coloring and (c) s-coloring.**

completeness and c-fullness. We characterize different classes of c-full signed graphs and also determine the number of simple c-complete signed graphs on a given number of vertices. In Section 3, we obtain bounds for $\chi_s(G)$. Furthermore, we study the relationship between the three chromatic numbers $\chi_p(\cdot)$, $\chi_s(\cdot)$ and $\chi_z(\cdot)$ of signed graphs. In Section 4, we define the *s-chromatic polynomial*, $S(G,k)$, of a signed graph G as the polynomial that counts the number of s-colorings of G as a function of the number of colors k. We define new operations of *removal* and *compression* that are modifications of the classical operations of deletion and contraction and show that $S(G,k)$ can be determined by a removal-compression recursion. Some other properties of $S(G,k)$ are also studied.

2. Complete, full and regular signed graphs

We begin by introducing some new terminology. A path P in a signed graph $G = (V,E,\sigma)$ is a *c-path* if all edges in P are c-edges. Clearly, a path is a c-path if and only if $d^-(v)$ is either even or odd for all vertices v in P. A signed graph G is *c-connected* if any two vertices in G are connected by a c-path. A set $S \subset V$ is *c-independent* if no two vertices in S are c-adjacent. The *c-independence number* $\beta_c(G)$ of G is the maximum cardinality of a c-independent set. Thus $\chi_s(G)$ is the minimum order of a partition of V into c-independent sets. Given a signed graph $G = (V,E,\sigma)$, the c-graph of G is an unsigned graph given by $G_c = (V, E_c)$, where E_c is the set of c-edges of G. If $\chi(\cdot)$ and $\beta(\cdot)$ respectively denote the chromatic number and the independence number for unsigned graphs, then clearly for any signed graph G, we have $\chi_s(G) = \chi(G_c)$ and $\beta_c(G) = \beta(G_c)$.
We say that a signed graph G is *complete* if any two vertices in G are adjacent and *c-complete* if any two vertices in G are c-adjacent. Thus G is a c-complete signed graph of order n if and only if $\chi_s(G) = n$. A signed graph G is *c-full* if every edge in G is a c-edge. For simple signed graphs the notions of c-completeness and c-fullness coincide.

Proposition 1. *For a signed graph G of order n, the following statements are equivalent.*
(i) G is c-complete.

(ii) G is complete and c-full.

(iii) $\chi_s(G) = n$.

(iv) $\beta_c(G) = 1$.

Proof. *(i)* $\Longrightarrow$ *(ii)*: Follows from the above definitions.
(ii) $\Longrightarrow$ *(iii)*: Assume that G is complete and c-full. So every edge of G is a c-edge. This means any arbitrary vertex u is c-adjacent to all other vertices. In other words, any two vertices are c-adjacent, which implies $\chi_s(G) = n$.
(iii) $\Longrightarrow$ *(iv)*: Since the order of G is n and $\chi_s(G) = n$, there are n color classes with cardinality of each class equal to one. Hence $\beta_c(G) = 1$.
(iv) $\Longrightarrow$ *(i)*: Let $\beta_c(G) = 1$. So there are n color classes each of cardinality one so that $\chi_s(G) = n$. Therefore any two vertices in G are c-adjacent and thus G is c-complete. □

The following result characterizes c-full signed paths.

Theorem 1. *Let P_n denote the signed path of order n.*

(i) For n even, P_n is c-full if and only if all the edges in P_n are positive or the edges of P_n are alternately negative and positive, starting and ending with negative edges.

(ii) For n odd, P_n is c-full if and only if all the edges in P_n are positive.

Proof. *(i)* Let n be even and P_n be c-full. Assume that P_n is the path $v_1v_2\cdots v_n$. Two cases arise (1) v_1v_2 is positive, (2) v_1v_2 is negative.
Case 1. If v_1v_2 is positive, then $d^-(v_1) = 0$. Since each edge is c-edge, so $d^-(v_2) = 0$. This requires that v_2v_3 should be positive. Continuing in this way, we observe that all the edges in P_n are positive.
Case 2. If v_1v_2 is negative, then $d^-(v_1) = 1$. Now $d^-(v_2) = 1$, so that v_2v_3 is positive. Now since $d^-(v_3) = 1$, we have v_3v_4 to be negative. If v_iv_j and v_jv_k are arbitrary edges, we have $d^-(v_i) = d^-(v_j) = d^-(v_k) = 1$, and so if v_iv_j is negative then v_jv_k is positive. In particular, since n is even, the edge $v_{n-1}v_n$ is negative.
Conversely, if all the edges of P_n are positive then $d^-(v_i) = 0$, for $i = 1, 2, \dots, n$. Thus P_n is c-full. If the edges of P_n are negative and positive, then $d^-(v_i) = 1$, for all $i = 1, 2, \dots, n$, which again implies that P_n is c-full.
(ii) This can be proved on the same lines as the proof of (i). □

From the above result, we have the following observations which determines the number of c-full signed paths.

Corollary 1. *Among all signed paths of order $n \geq 2$ and n even, there are exactly two signed paths which are c-full.*

Corollary 2. *Among all signed paths of order $n \geq 2$ and n odd, there is exactly one signed path which is c-full.*

The following result characterizes c-full signed cycles.

Theorem 2. *Let C_n be a signed cycle of order n.*

(i) For n even, C_n is c-full if and only if either all the edges in C_n are either positive or all edges in C_n are negative or (c) the edges in C_n are alternatively positive and negative.

(ii) For n odd, C_n is c-full if and only if (a) all the edges in C_n are positive or (b) all the edges are in C_n are negative.

Proof. Let C_n be a signed cycle and let n be either even or odd. Assume that C_n is c-full. Let u and v be consecutive vertices of C_n. So uv is a c-edge and therefore $d^-(u)$ and $d^-(v)$ are both even or both odd. If uv is positive then the edge xu entering u is either positive or negative. If xu is positive, then $d^-(u) = 0$. This implies that $d^-(v) = 0$ as $d^-(v) \geq 1$ is not possible ($d^-(v) \neq 1$, since uv is c-full and $d^-(v) \neq 2$ because uv is positive and C_n is a cycle). Thus the edge vw leaving v is also positive. Continuing in this way, we see that all the edges of C_n are positive. If uv is negative, then the edge xu entering u is either negative or positive. If xu is negative, then $d^-(u) = 2$. So $d^-(v) = 2$ implying that the edge vw is also negative. Continuing in this way, all the edges of C_n are negative.
If xu is positive, then $d^-(u) = 1$ so that $d^-(v) = 1$, because $d^-(v) \geq 2$ is not possible. Thus the edge vw leaving v is positive. Continuing in this way, for n even, we observe that the edges of C_n are alternatively positive and negative. If n is odd the edges to be alternatively positive and negative is not possible, for otherwise if we start with uv to be positive, then the last edge xu is also positive, and this contradicts the fact that C_n is c-full. Conversely, if all the edges of C_n are all positive or are all negative, then $d^-(v_i) = 0$ or $d^-(v_i) = 2$ for all $i = 1, 2, \cdots, n$. Thus C_n is c-full. In case n is even and edges are alternatively positive and negative, then $d^-(v_i) = 1$, for all i and so again C_n is c-full. If n is odd, then the edges cannot be alternatively positive and negative, so that C_n is not c-full. □

In a signed graph G, if $d^-(v)$ is even for all v in G, then no two adjacent vertices receive the same color in an s-coloring of G. This is also true when $d^-(v)$ is odd for all v in G. This implies the following.

Corollary 3. *For a connected signed graph G, the following statements are equivalent.*

(i) G is c-full.

(ii) G is c-connected.

(iii) The negative degree $d^-(v)$ is even (or odd) for all $v \in G$.

(iv) For each edge uv in G, $Sgn(u) = Sgn(v)$.

(v) $\chi_s(G) = \chi(G_u)$.

We now determine the number of c-complete signed graphs on a given number of vertices. There are exactly two c-complete signed graphs on three vertices, namely the one in which all the edges are signed positive and its s-complement. Also, it is easy to check that there are six c-complete signed graphs on four vertices with two c-complete signed graphs K_4 with all edges having positive sign and all edges having negative sign, and the other four c-complete signed graphs. In general, the number is given by the following result.

Theorem 3. *There are exactly $2n+2$ simple c-complete signed graphs on $2n$ vertices, while the number of simple c-complete signed graphs on $2n+1$ vertices is $n+1$.*

Proof. We first consider the case of $2n$ vertices. We note that the complete signed graph K_{2n} is c-complete if and only if for all vertices in K_{2n}, d^- is either even or odd. In other words, every term in the negative degree sequence is either even or odd. So it follows that the number of non-isomorphic c-complete signed graphs on $2n$ vertices is the same as the number of distinct negative degree sequences in each of which all the terms are either even or odd. Now there are exactly $2n+2$ non-isomorphic c-complete signed graphs on $2n$ vertices. They are given by their negative degree sequences as follows.
(i) $0,(2n-2)^{(2n-1)}$: This is the case in which all the edges at one vertex are signed positive and the other remaining edges are signed negative.
(ii) $1^{(2n-1)},(2n-1)$: This is the case when all the edges at one vertex are signed negative and the remaining others are signed positive.
(iii) r^{2n}, where $r=1,3,\cdots,(2n-1)$: This is the case in which r edges at each vertex are signed negative and the remaining others are signed positive. These are n in number.
(iv) r^{2n}, where $r=2,4,\cdots,(2n-2)$: This is the case in which r edges at each vertex are signed negative and the remaining others are signed positive. These are $n-1$ in number.
(v) 0^{2n}: This is the case in which all edges are signed negative. Thus there are $2n+2$ c-complete signed graphs on $2n$ vertices.
We now address the case of $2n+1$ vertices. The following negative degree sequences arise in this case:
(i) 0^{2n+1}: This is the one in which all edges are signed positive.
(ii) $(2n)^{(2n+1)}$: This is the one in which all the edges are signed negative.
(iii) $r^{(2n+1)}$, where $r=2,4,\cdots,(2n-2)$: This is the case when r edges at each vertex are signed negative and the other edges are signed positive. These are $n-1$ in number.
Hence there are $n+1$ c-complete signed graphs on $2n+1$ vertices. $\square$

Given some integer $r\geq 0$, G is *r-regular* if $deg(v)=r$ for all vertices v in G. Moreover, G is *r^+-regular* (resp. *r^--regular*) if $d^+(v)=r$ (resp. $d^-(v)=r$), for all vertices v in G. The *s-complement* G^s of a signed graph G is obtained by interchanging the positive signs and negative signs on the edges of G.

Proposition 2. *Let G be a connected signed graph which is r-regular. Then*
(i) G is c-connected if and only if G^s is c-connected.

(ii) G is c-complete if and only if G^s is c-complete.

Proof. Let r be even and G is c-connected. Then every edge uv in G is a c-edge and both $d^-(u)$ and $d^-(v)$ are either even or odd. The negative degrees of u and v in G^s are $r-d^-(u)$ and $r-d^-(v)$. Since r is even, these are both even or odd according as $d^-(u)$ and $d^-(v)$ are either even or odd. This implies that uv is a c-edge in G^s and hence G^s is c-connected. Similarly if r is odd, we can find that uv is a c-edge in G^s. Thus G^s is also c-connected. The converse can be proved in the same way. $\square$

Proposition 3. *If G is r^--regular, then G is c-connected. The converse is not true.*

We note that a c-complete signed graph need not be r^+-regular or r^--regular for any $r \geq 0$. Also the s-complement G^s need not be c-connected.

Proposition 4. *Let G be a c-connected signed graph. Then its s-complement G^s is c-connected if and only if $d^+(v)$ is even or odd for all vertices v in G.*

3. The s-chromatic number

Let P be a coloring of the vertices of a graph G. A set $S \subset V$ is a color class (with respect to P) if all the vertices of S receive the same color in some P coloring of G. For example, in the usual coloring of a graph any independent set of vertices is a color class. Motivated by this definition, we define a color class in a s-coloring as follows.
A set of vertices in a signed graph is an *$s-$ color class* if all the vertices in S receive the same color in some s-coloring of G. Note that an s-color class is a c-independent set. Thus the s-chromatic number $\chi_s(G)$ is the minimum order of a partition of the vertex set of G into s-color classes or c-independent sets. Also note that an independent set is c-independent. The c-independence number $\beta_{0c}(G)$ of a signed graph is the maximum cardinality of a c-independent set. Clearly, since any independent set is c-independent, we have for any signed graph $\beta_0(G) \leq \beta_{0c}$, where β_0 is the independence number of the underlying graph G_u of G. As in graphs we have the following bounds for $\chi_s(G)$.

Proposition 5. *For any signed graph of order n,*

$$n\beta_{0c} \leq \chi_s(G) \leq n - \beta_{0c}(G) + 1.$$

A set S of vertices in a signed graph G is a c-clique if the subgraph $< S >$ induced by S is c-complete.

The $c - graph$ G_c of a signed graph G is a graph having the same vertex set as G and two vertices u and v are adjacent in G_c if and only if uv is a c-edge in G. Clearly the degree of a vertex v in G_c is equal to its c-degree in G and $\chi_s(G) = \chi(G_c)$. Also $\Delta_c(G) = \Delta(G_c)$, $\delta_c(G) = \delta(G_c)$ and $\beta_{s0}(G) = \beta_0(G_c)$. It is well known that the determination of chromatic number of a graph is NP-hard and we have the following.

Proposition 6. *Determination of the s-chromatic number $\chi_s(G)$ of a signed graph is NP-hard.*

In view of the above remarks and the fact $\chi_s(G) = \chi(G_c)$, we deduce many bounds for $\chi_s(G)$ from the well known bounds of the chromatic number $\chi(G_c)$ of the graph G_c.

Proposition 7. *For a signed graph G, $\chi_s(G) \leq 1 + max\ \delta_c(G')$, where the maximum is taken over all induced sub-signed graphs G' of G.*

Corollary 4. *For any signed graph G, $\chi_s(G) \leq 1 + \Delta_s(G)$.*

Proposition 8. *For a signed graph G, $\chi_s(G) \leq \chi(G_u)$ and equality holds if and only if G is c-full.*

Proof. This follows from the following facts. For a set $S \subset V$ in a signed graph $G = (V, E)$, (i) S is independent implies S is c-independent, but not conversely, and (ii) if G is c-full, then S is independent if and only if S is c-independent. □

Proposition 9. *The chromatic number of a signed tree is less or equal to two.*

Proof. If the signed tree T is c-full, then $\chi_s(T) = 2$. In case T is not c-full and for every edge uv $d^-(u)$ and $d^-(v)$ are of different parity, that is one is odd and the other is even, then $\chi_s(T) = 2$. □

Theorem 4. *Given a signed graph $G = (V, E, \sigma)$, let $G^- = (V, E^c, \sigma^-)$ be the signed graph obtained by deleting all the non c-edges from G and assigning negative sign to all c-edges of G, then $\chi_s(G) = \chi_p(G^-)$.*

Proof. The p-chromatic number $\chi_p(G^-)$ equals the minimum number of colors required to color the vertices of G^- such that no two vertices joined by a negative edge receive the same color. Since all the edges of G^- are negative and correspond to the c-edges of G, the result follows immediately. □

Theorem 5. *Let $G = (V, E, \sigma)$ be a signed graph and let $G^+ = (V, E^c, \sigma^+)$ be the signed graph obtained by deleting all non c-edges from G and assigning positive sign to all c-edges of G. Then $\chi_s(G) = 2\chi_z(G^+) + 1$.*

Proof. Since G^+ does not have any negative egdes, the z-chromatic number $\chi_z(G^+)$ equals the minimum n such that the vertices of G^+ can be colored by $\{0, \pm 1, \ldots, \pm n\}$ with no two vertices joined by a positive edge receiving the same color. As the positive edges of G^+ exactly correspond to the c-edges of G, therefore if $\chi_z(G^+) = n$ then we can minimally s-color the vertices of G with $2n + 1$ colors $\{0, \pm 1, \ldots, \pm n\}$. This completes the proof. □

4. The s-chromatic polynomial

Let $G = (V, E, \sigma)$ be a signed graph with underlying graph $G_u = (V, E)$. Assume that G_u has possible multiple edges but no loops. As we have noted earlier, $\chi_s(G)$ is well-defined despite the multiple edges. Define the c-graph, $G_c = (V, E_c)$, of G as the unsigned graph consisting of the same vertex set as G but only containing the c-edges of G. Let $S(G, x)$ denote the number of s-colorings of G as a function of the number of colors x, while $P(H, x)$ denotes the classical chromatic polynomial of a graph H. Clearly, $S(G, x) = P(G_u, x)$ and so $S(G, x)$ is indeed a polynomial in x. We call it the s-chromatic polynomial of G.
The aim is to develop a deletion-contraction type recursive procedure for determining $S(G, x)$ for any signed graph G. It turns out that the classical operations of deletion and contraction do not work for counting s-colorings. For instance, deletion of an edge does not distinguish between the positive and negative edges. However, deleting a negative edge changes the signs of its end vertices and can potentially change the consistency of other edges incident to those vertices. Therefore, in order to calculate $S(G, x)$ recursively we must first define new operations for signed graphs that remove all the c-edges from G one by one in such a way that the values of $Sgn(v)$ are preserved for all vertices of G and so no new c-edges are created. At the same time we want these operations to mirror the traditional deletion and contraction operations as closely as possible. We define *removal* and *compression* operations that play the role of deletion and contraction respectively for the s-coloring of signed graphs. Repeating the removal and the compression operations eventually produces signed graphs without any c-edges that can be s-colored trivially.

Definition 1. Let $e = uv$ be a c-edge of a signed graph $G = (V, E, \sigma)$, so that $Sgn(u) = Sgn(v)$. The removal of e from G produces a signed graph $G \ominus e$ as follows.
(1) If $\sigma(e) = +$ and $Sgn(v) = +$, remove e. This removes a c-edge while preserving the values of $Sgn(.)$ for all the vertices of G.
(2) If $\sigma(e) = +$ and $Sgn(v) = -$, remove e.
(3) If $\sigma(e) = -$ and $Sgn(v) = +$, remove e from G, insert two new vertices x and y and insert two new edges $e' = ux$ and $e'' = yv$ such that $\sigma(e') = \sigma(e'') = -$. This removes a

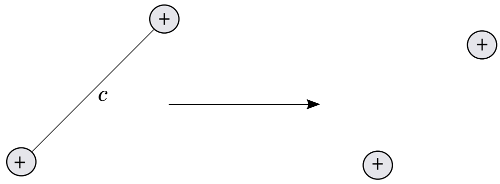

Figure 2. **Removal of a positive c-edge joining positive vertices**

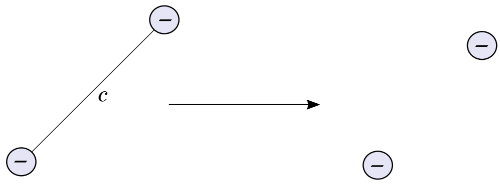

Figure 3. **Removal of a positive c-edge joining negative vertices**

c-edge and adds two vertices and two non c-edges to G, while preserving the values of $Sgn(\cdot)$ for all the vertices of G.

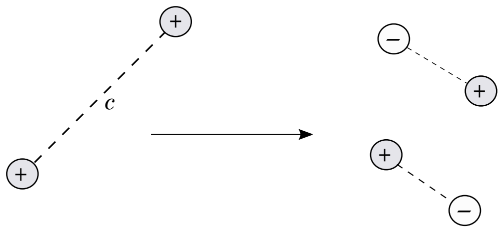

Figure 4. Removal of a negative c-edge joining positive vertices. The dashed lines represent negative edges.

(4) If $\sigma(e) = -$ and $Sgn(v) = -$, remove e and insert four new vertices w,x,y and z and four new edges $e' = uw$, $e'' = wx$, $f' = vy$ and $f'' = yz$ such that $\sigma(e') = \sigma(e'') = \sigma(f') = \sigma(f'') = -$. This removes a c-edge and adds four vertices and four non c-edges to G, while preserving the values of $Sgn(\cdot)$ for all existing vertices of G.

Definition 2. Let $e = uv$ be a c-edge of a signed graph $G = (V, E, \sigma)$, so that $Sgn(u) = Sgn(v)$. The compression of G by e produces a signed graph $G \triangleleft e$ as follows.
(1) If $\sigma(e) = +$ and $Sgn(v) = +$, contract e. This removes a c-edge and a vertex from G while preserving the values of $Sgn(\cdot)$ for all remaining vertices.
(2) If $\sigma(e) = +$ and $Sgn(v) = -$, contract e to v, add two new vertices w and x and two new edges $e = vw$ and $f = wx$ to G such that $\sigma(e) = \sigma(f) = -$.
(3) If $\sigma(e) = -$ and $Sgn(v) = +$, contract e.
(4) If $\sigma(e) = -$ and $Sgn(v) = -$, contract e to v, add two new vertices w and x and two new edges $e = vw$ and $f = wx$ to G such that $\sigma(e) = \sigma(f) = -$.

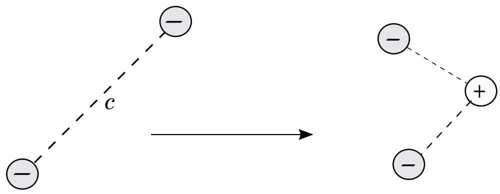

Figure 5. Removal of a negative c-edge joining negative vertices. The dashed lines represent negative edges.

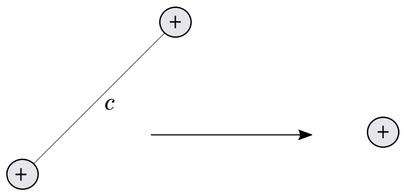

Figure 6. Compression of a positive c-edge joining positive vertices

The following gives a recursive procedure based on performing removal-compression opertations to determine $S(G, k)$ for any simple signed graph G.

Theorem 6. *Let $G = (V, E, \sigma)$ be a simple signed graph. Given a c-edge $e = uv$ of G,*

$$S(G,x) = \begin{cases} S(G \ominus e, x) - S(G \triangleleft e, x), & \text{if } \sigma(e) = +, Sgn(v) = + \\ S(G \ominus e, x) - S(G \triangleleft e, x)/x^2, & \text{if } \sigma(e) = +, Sgn(v) = - \\ S(G \ominus e, x)/x^2 - S(G \triangleleft e, x), & \text{if } \sigma(e) = -, Sgn(v) = + \\ S(G \ominus e, x)/x - S(G \triangleleft e, x)/x^2, & \text{if } \sigma(e) = -, Sgn(v) = -. \end{cases}$$

Proof. Based on the definition of removal and compression, four cases arise naturally.
Case 1. Let $\sigma(e) = +$ and $Sgn(v) = +$. In this case, the operations of removal and compression coincide with the operations of deletion and contraction respectively, as deleting or contracting e preserves the signs of all remaining vertices and hence the consistency of all remaining edges. In the graph $G \ominus e$, there is no edge joining u and v. The s-colorings of $G \ominus e$ using x colors can be classified into two types, the ones in which u and v receive distinct colors and the ones in which they receive the same color. Thus the number $S(G \ominus e, x)$ is the sum of $S(G, x)$ and $S(G \triangleleft e, x)$.
Case 2. If $\sigma(e) = +$ and $Sgn(v) = -$. In this case, the operations of removal coincides with deletion but compression differs from contraction as we add vertices and edges. In the graph $G \ominus e$, there is no edge joining u and v. The s-colorings of $G \ominus e$ using x colors can be classified into two types, the ones in which u and v receive distinct colors and the ones in which they receive the same color. Thus the number $S(G \ominus e, x)$ is the sum of $S(G, x)$ and $S(G \triangleleft e, x)/x^2$.
Case 3. If $\sigma(e) = -$ and $Sgn(v) = +$. In this case, the operations of removal and compression coincide with the operations of deletion and contraction respectively. In

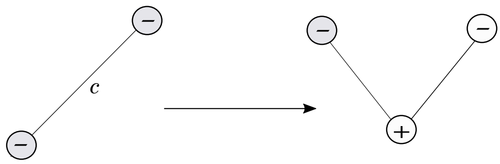

Figure 7. Compression of a positive c-edge joining negative vertices. The dashed lines represent negative edges.

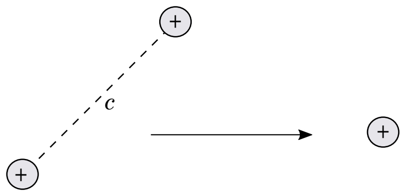

Figure 8. Compression of a negative c-edge joining positive vertices. The dashed lines represent negative edges.

the graph $G \ominus e$, there is no edge joining u and v. The s-colorings of $G \ominus e$ using x colors can be classified into two types, the ones in which u and v receive distinct colors and the ones in which they receive the same color. Thus the number $S(G \ominus e, x)$ is the sum of $S(G, x)$ and $S(G \triangleleft e, x)$.

Case 4. If $\sigma(e) = -$ and $Sgn(v) = -$. In this case, the operations of removal and compression coincide with the operations of deletion and contraction respectively. In the graph $G \ominus e$, there is no edge joining u and v. The s-colorings of $G \ominus e$ using x colors can be classified into two types, the ones in which u and v receive distinct colors and the ones in which they receive the same color. Thus the number $S(G \ominus e, x)$ is the sum of $S(G, x)$ and $S(G \triangleleft e, x)$.

□

Theorem 7. *Given a signed graph $G = (V, E, \sigma)$, the following holds for the polynomial $S(G, k)$.*
(i) The degree of $S(G, k)$ is $|V|$.
(ii) The leading coefficient of $S(G, k)$ is 1.
(iii) The coefficient of $k^{|V|-1}$ in $S(G, k)$ is equal to $-|E_c|$.
(iv) The constant term of $S(G, k)$ equals 0.
(v) The coefficient of k in $S(G, k)$ is nonzero if and only if G is c-connected.
(vi) If the lowest nonzero coefficient of $S(G, k)$ is of k^p then the number of c-connected components of G is equal to p.

Proof. First we observe that the number of s-colorings of $G = (V, E, \sigma)$ using k colors is the same as the number of vertex colorings of the c-graph G_c of G. Thus $S(G, k) = P(G_c, k)$ and so $S(G, k)$ is a polynomial in k. Furthermore, the following

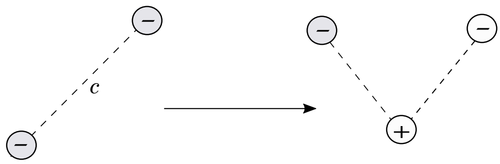

Figure 9. Compression of a negative c-edge joining negative vertices. The dashed lines represent negative edges.

result shows that several properties of chromatic polynomial are also satisfied by the s-chromatic polynomial.
(i): We can color the vertices of G with at the most $|V|$ colors. The number of ways this can be done is a polynomial of degree $|V|$, while for coloring with lesser all other partitions it has a degree less than n. The sum of such polynomials is one of degree n
(ii-vi): These can be established. □

Let $G = (V, E, \sigma)$ be a signed graph, where $G_u = (V, E)$ is a graph with possible multiple edges but no loops, such that for any multiple edges e and f, $\sigma(e) = \sigma(f)$. The aim of this section is to develop a chromatic polynomial for s-coloring of signed graphs on the same lines as the chromatic polynomial of unsigned graphs. Let $S(G, k)$ denote the number of distinct s-colorings of G as a function of the number of colors k. Let $P(H, k)$ denote the classical chromatic polynomial of an unsigned graph H. First we observe that the number of s-colorings of $G = (V, E, \sigma)$ using k colors is the same as the number of vertex colorings of the c-graph G_c of G. Thus $S(G, k) = P(G_c, k)$ and so $S(G, k)$ is a polynomial in k. Furthermore, the following result shows that several properties of chromatic polynomial are also satisfied by the s-chromatic polynomial.

Theorem 8. *Given a signed graph $G = (V, E, \sigma)$, the following holds for the polynomial $S(G, k)$.*
(i) The degree of $S(G, k)$ is $|V|$.
(ii) The leading coefficient of $S(G, k)$ is 1.
(iii) The coefficient of $k^{|V|-1}$ in $S(G, k)$ is equal to $-|E_c|$.
(iv) The constant term of $S(G, k)$ equals 0.
(v) The coefficient of k in $S(G, k)$ is nonzero if and only if G is c-connected.
(vi) If the lowest nonzero coefficient of $S(G, k)$ is of k^p then the number of c-connected components of G is equal to p.

Proof. *(i)*: We can color the vertices of G with at the most $|V|$ colors. The number of ways this can be done is a polynomial of degree $|V|$.
(ii-vi): These can be readily established. □

Acknowledgements: The authors are thankful to the anonymous referee for his useful comments. The research of the first author is supported by the NBHM-DAE research project number NBHM/02011/20/2022.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

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