Communications in Combinatorics and Optimization
Vol. xx, No. x (xxxx), pp. 1-16
https://doi.org/10.22049/cco.2025.30365.2436

CCO
Commun. Comb. Optim.

Research Article

Characterizations of additively graceful signed paths and cycles

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Received: 18 February 2025; Accepted: 22 April 2025
Published Online: 30 April 2025

Abstract: A (p, m, n) signed graph S, is a signed graph of order p with m positive edges and n negative edges. In this paper, we first prove a few basic results on vertex labelings of paths. We use these results and a sequence of lemmas to obtain a characterization of additively graceful signed paths. We prove that, apart from exactly 4 exceptions, additively graceful signed paths are characterized by the signed paths containing at most one negative section with $n \leq 2$. We also establish a characterization of additively graceful signed cycles. We prove that a (p, m, n) signed cycle S is additively graceful if and only if one among the following 4 conditions are satisfied, ① $n = 0$ and $m \equiv 0$ or $3 \pmod 4$, ② $n = 1$ and $m \equiv 1$ or $2 \pmod 4$, ③ $n = 2$, $m \equiv 1$ or $2 \pmod 4$ and S contains a single negative section, ④ S is the all negative signed cycle on C_3.

Keywords: additively graceful signed graph, signed graph, graph labeling, cycle, path.

AMS Subject classification: 05C78, 05C22, 05C38

1. Introduction

In all our discussions, a graph G shall mean a finite nonempty set of objects called *vertices* together with a set of unordered pairs of distinct vertices of G called *edges*. The vertex set and the edge set of G are denoted by $V(G)$ and $E(G)$ respectively whereas their cardinality is denoted by p and q respectively. In this case G is called a (p, q) *graph*. All basic terminology of Graph Theory that we shall use, can be found in Chartrand and Lesniak [3].

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An injective map from $V(G)$ to $\mathbb{Z}$ is called a *vertex labeling* of G while an *edge labeling* is an injective map from $E(G)$ to $\mathbb{Z}$. Although vertices and edges may be labeled independently, we will be interested in associating with a vertex labeling an edge labeling as follows. Given a vertex labeling $f : V(G) \to \mathbb{Z}$, we obtain the *induced edge labeling* $f^* : E(G) \to \mathbb{N}$ by setting $f^*(uv)$ to be $|f(u) - f(v)|$. The edge labels thus obtained are called *induced edge labels*.

Observation 1. If f is a vertex labeling in graph G, then for any integer k, the vertex labeling $g = \pm f + k$ satisfies $f^* = g^*$.

We denote $\max_{v \in V(G)} f(v)$ by $M_f(G)$. Then $M_f(G) - f$ is called the *complementary labeling* of f. Observe that the values of the vertex labels in the complementary labeling also do not exceed $M_f(G)$. In our discussions we will say that vertex labels l_1 and l_2 are *adjacent* if there exists adjacent vertices v_1 and v_2 such that $f(v_1) = l_1$ and $f(v_2) = l_2$. Similarly we can define *adjacent* edge labels. We will say that a vertex or an edge is *odd* or *even* if the label assigned to it is so.
Rosa [1] defined the concept of a β-valuation and Golomb [4] subsequently called it graceful labeling.

Definition 1. Let G be a (p, q) graph. A *graceful labeling* of G is an injection $f : V(G) \to \{0, 1, \ldots, q\}$ such that when each edge uv is assigned the label $f^*(uv) = |f(u) - f(v)|$, the resulting edge labels are all distinct. A graph which admits such a labeling is called a *graceful graph*.

Hegde [5] introduced the concept of additively graceful graphs. He gave a lower bound on the number of edges of such graphs and characterized additively graceful complete graphs. Building on this work, J. Pereira et al. [6], generalized the concept by defining additively graceful signed graphs. A (p, m, n) *signed graph* S is a (p, q) graph in which certain m edges are specified as positive and a certain n edges are specified as negative. We denote the sets of positive and negative edges in S by E^+ and E^- respectively and the corresponding edge induced subgraphs by $\langle E^+ \rangle$ and $\langle E^- \rangle$. A component of $\langle E^- \rangle$ is called a *negative section* in S while a *positive section* in S is a component of $\langle E^+ \rangle$. In diagrams of signed graphs we will use a solid line to represent a positive edge while a dashed line shall represent a negative edge.

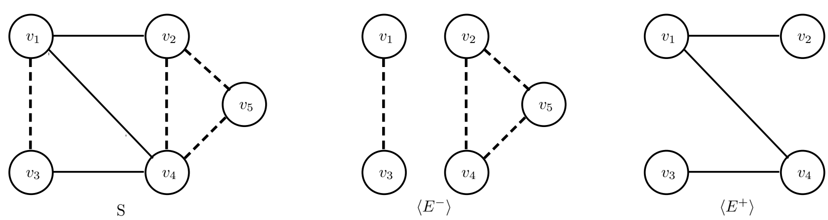

Figure 1. **Signed graph S with 2 negative sections and 1 positive section.**

Definition 2. Let $S = (V, E)$ be a (p, m, n)-signed graph. An *additively graceful labeling* of S is an injective mapping $f : V \to \{0, 1, \ldots, m + \lceil \frac{(n+1)}{2} \rceil\}$ with the induced edge function defined as $f^*(uv) = f(u) + f(v)$,$\forall$ $uv \in E^-$ and $f^*(uv) = |f(u) - f(v)|$,$\forall$ $uv \in E^+$; satisfying $\{f^*(uv) : uv \in E^-\} = \{1, 2, \ldots, n\}$ and $\{f^*(uv) : uv \in E^+\} = \{1, 2, \ldots, m\}$. The signed graph which admits such a labeling is called an *additively graceful signed graph.*

In this paper we prove a few basic results on vertex labelings of paths. We use these results along with Theorem 8 as building blocks to construct additively graceful labelings of signed paths, which lead to a characterization of additively graceful signed paths. We subsequently establish a characterization of additively graceful signed cycles.

We state below a few results from the literature on graceful graphs followed by two theorems obtained in [6], which we will use.

Theorem 2. [4] *Suppose $f : V(G) \to \mathbb{Z}$ is a vertex labeling of a graph G and suppose the induced edge labels on each edge uv are given by $f^*(uv) = |f(u) - f(v)|$ then, the sum of edge labels around any circuit in G is even.*

Theorem 3. [1, 4] *If G is a graceful eulerian graph of size q, then $q \equiv 0$ or $3 \pmod 4$.*

Theorem 4. [1, 4] *The cycle C_p is graceful if and only if $p \equiv 0$ or $3 \pmod 4$.*

Theorem 5. [1, 4] *Every non-trivial path is graceful.*

Theorem 6. [6] *If a signed graph S is additively graceful, then the sum of all edge labels of any circuit C in S is even.*

Theorem 7. [6] *If a (p, m, n) eulerian signed graph is additively graceful then $m^2 + n^2 + m + n \equiv 0 \pmod 4$.*

While constructing labelings on graphs we can often reduce the problem at hand to that of finding a graceful labeling with some additional conditions. Working on π-representations and α-labelings Cattell [2] gained the following control on graceful labelings of paths.

Theorem 8. [2] *Given a path P_p and any vertex v in P_p, there exists a graceful labeling of P_p in which v has label i, for any $i \in \{0, 1, \ldots, p-1\}$, whenever at least one of the following conditions is met:*

- *p is even,*
- *$p \equiv 5$ or $9 \pmod{12}$,*
- *v is in the larger of the two partite sets of vertices,*

- $i \neq \frac{p-1}{2}$.

It follows immediately from Theorem 8 that for any $i \in \{0, 1, \ldots, p-1\}$, a graceful labeling of path P_p can be obtained such that label i is assigned to an end vertex, a fact which we will use in Sections 2 and 3.

2. Some Basic Results

Throughout this paper, for a signed graph S, with a vertex labeling f, we will use the notation $f(w_1, w_2, \ldots, w_k)$ to denote the ordered tuple of vertex labels $(f(w_1), f(w_2), \ldots, f(w_k))$ where $w_1, w_2, \ldots, w_k$ are some distinct vertices in S. In particular for the path $S = v_1, v_2, \ldots, v_p$, with a vertex labeling f, we will use the notation $f(S)$ to denote the ordered tuple of vertex labels $(f(v_1), f(v_2), \ldots, f(v_p))$. The following are a couple of very basic observations, but we state them here since they will be used repeatedly in our discussions. Observation 9 follows directly from the definition of graceful graph and additively graceful signed graph.

Observation 9. If a graph G is graceful then as a signed graph with $n = 0$ it is an additively graceful signed graph.

The converse of Observation 9 may not hold since the definition of additively graceful signed graph with $n = 0$ allows $m + 1$ to be a vertex label. Also since $0 + 1$ and $0 + 2$ are the only ways in which 1 and 2 can be expressed as the sum of two distinct, non-negative integers, we obtain Observation 10.

Observation 10. In an additively graceful labeling of a signed graph S having two or more negative edges, the negative edges labeled 1 and 2 must be adjacent to the vertex labeled 0.

We now prove that there is a natural bijective correspondence between signed graphs with exactly one negative edge, which admit an additively graceful labeling and a particular sub-class of signed graphs with exactly two negative edges, which admit an additively graceful labeling. This means that if we can obtain a characterization of one of these classes of labeled signed graphs then we immediately have a characterization of the other.

Theorem 11. *A signed graph S, with exactly one negative edge (a, b) is additively graceful if and only if the signed graph T, obtained from S by performing an elementary subdivision of the edge (a, b) into an all negative path (a, c, b) is additively graceful.*

Proof. Suppose S admits an additively graceful labeling f. Then $f(a)$ and $f(b)$ must be 0 and 1. Now define the required additively graceful labeling g of T as follows:

$$g(v) = \begin{cases} 0 & \text{if } v = c. \\ f(v) + 1 & \text{otherwise.} \end{cases}$$

Conversely, if T admits an additively graceful labeling g, then by Observation 10, $g(a)$ and $g(b)$ must be 1 and 2 while $g(c) = 0$. Now obtain the required additively graceful labeling f of S by defining $f(v) = g(v) - 1, \forall\, v \in V(S)$. □

The next result in this section is about graphs rather than signed graphs but is essential to our proof of the characterization of additively graceful signed paths in Section 3. Besides the ability to fix a particular vertex label as in Theorem 8 we may want to avoid a particular vertex label. Theorem 12 allows us to do so in certain cases.

Theorem 12. *For each $i = 0, 1, 2, p-2, p-1$, except for $(p, i) = (4, 2)$, the path P_p, $p \geq 3$ admits a vertex labeling $f : V(P_p) \to R_p^i$, where $R_p^i = \{0, 1, \ldots, p\} - \{i\}$, satisfying,*

1. *$f^*(E(P_p)) = \{1, 2, \ldots, p-1\}$.*
2. *The smallest vertex label is assigned to an end vertex.*

Proof. Let $P_p = v_1, v_2, \ldots, v_p$ with $p \geq 3$.

Case 1. $i = 0$.
By Theorem 8, P_p admits a graceful labeling g with $g(v_1) = 0$. Define $f(v_j) = g(v_j)+1$ for $j = 1, 2, \ldots, p$. It can be easily verified that $f(V(P_p)) = R_p^0$ and conditions (1) and (2) are satisfied.

Case 2. $i = 1$.
By applying Theorem 8 to $P_{p-1} = v_1, v_2, \ldots, v_{p-1}$, we can get a graceful labeling $g : V(P_{p-1}) \to \{0, 1, \ldots, p-2\}$ such that $g(v_{p-1}) = p-3$. Now define $f : V(P_p) \to R_p^1$ by, $f(v_p) = 0$ and $f(v_j) = g(v_j) + 2$ for $j = 1, 2 \ldots, p-1$. It can be easily verified that $f(V(P_p)) = R_p^1$ and conditions (1) and (2) are satisfied.

Case 3. $i = 2$.
We first show that the result is impossible for the case $(p, i) = (4, 2)$. Suppose P_4 with $i = 2$, admits a vertex labeling with codomain $R_4^2 = \{0, 1, 3, 4\}$ and satisfying condition (1). The vertex labels 1 and 3 must be adjacent to obtain edge label 2. We cannot have adjacent vertex labels 0 and 4 as it would result in an edge label 4. Hence one among the vertex labels 0 and 4, must be adjacent to 3 and the other to 1. In either case we would obtain repeated edge labels, violating condition (1) and hence giving a contradiction.
Now consider P_p, $p \neq 4$. We prove the result by induction on p. For the cases $p = 3, 5, 6, 7, 8$ and 10, the required labeling $f(P_p)$ is given by (0, 1, 3), (0, 3, 1, 5, 4), (0, 4, 1, 6, 5, 3), (0, 6, 1, 5, 3, 4, 7), (0, 7, 1, 6, 3, 5, 4, 8) and (0, 9, 1, 8, 3, 7, 10,

4, 6, 5) respectively. Now assume that the result is true for a path with p vertices and consider the path $P_{p+6} = v_1, v_2, \ldots, v_{p+6}$. By the induction hypothesis, the subpath $P_p = v_1, v_2, \ldots, v_p$ has a labeling $f : V(P_p) \to R_p^2$ satisfying conditions (1) and (2). Without loss of generality assume $f(v_p) = 0$. Define $h : V(P_{p+6}) \to R_{p+6}^2$ by $h(v_{p+1}, v_{p+2}, \ldots, v_{p+6}) = (p+5, 5, p+6, 1, p+4, 0)$ and $h(v_j) = f(v_j) + 3$ for $j = 1, 2, \ldots, p$. It can be easily verified that h has range R_{p+6}^2 and satisfies conditions (1) and (2).

Case 4. $i = p - 2$.
We have seen that $(p, i) = (4, p-2)$ is an exception. For P_p, $p \neq 4$, we again proceed by induction on p. For the cases $p = 3, 5, 6, 7, 8$ and 10, the required labeling $f(P_p)$ is given by (0, 2, 3), (0, 1, 5, 2, 4), (0, 1, 6, 2, 5, 3), (0, 3, 4, 2, 6, 1, 7), (0, 4, 3, 5, 2, 7, 1, 8) and (0, 5, 6, 4, 7, 3, 9, 2, 10, 1) respectively. Now assume that the result is true for a path with p vertices and consider path $P_{p+6} = v_1, v_2, \ldots, v_{p+6}$. By the induction hypothesis, the subpath $P_p = v_1, v_2, \ldots, v_p$ has a labeling $f : V(P_p) \to R_p^{p-2}$ satisfying conditions (1) and (2). Without loss of generality assume $f(v_p) = 0$. Define $h : V(P_{p+6}) \to R_{p+6}^{p+4}$ by $h(v_{p+1}, v_{p+2}, \ldots, v_{p+6}) = (p+5, 2, p+6, 1, p+1, 0)$ and $h(v_j) = f(v_j) + 3$ for $j = 1, 2, \ldots, p$. It can be easily verified that h indeed has range R_{p+6}^{p+4} and satisfies conditions (1) and (2).

Case 5. $i = p - 1$.
Here again we proceed by induction on p. For the cases $p = 3, 4, \ldots, 8$ the required labeling $f(P_p)$ is given by (0, 1, 3), (0, 1, 4, 2), (0, 3, 5, 1, 2), (0, 3, 4, 2, 6, 1), (0, 4, 3, 5, 2, 7, 1) and (0, 4, 5, 3, 6, 1, 8, 2) respectively. Now assume that the result is true for a path with p vertices and consider path $P_{p+6} = v_1, v_2, \ldots, v_{p+6}$. By the induction hypothesis, the subpath $P_p = v_1, v_2, \ldots, v_p$ has a labeling $f : V(P_p) \to R_p^{p-1}$ satisfying conditions (1) and (2). Without loss of generality assume $f(v_p) = 0$. Define $h : V(P_{p+6}) \to R_{p+6}^{p+5}$ by $h(v_{p+1}, v_{p+2}, \ldots, v_{p+6}) = (p+4, 1, p+6, 2, p+2, 0)$ and $h(v_j) = f(v_j) + 3$ for $j = 1, 2, \ldots, p$. It can be easily verified that h indeed has range R_{p+6}^{p+5} and satisfies conditions (1) and (2). □

Corollary 1. *For each $i = 0, 1, 2, p-2, p-1$, except for $(p, i) = (4, 2)$, the path P_p, $p \geq 3$ admits a vertex labeling $f : V(P_p) \to R_p^i$, where $R_p^i = \{0, 1, \ldots, p\} - \{i\}$ satisfying,*

1. *$f^*(E(P_p)) = \{1, 2, \ldots, p-1\}$.*
2. *The largest vertex label is assigned to an end vertex.*

Proof. By Theorem 8, P_p admits a graceful labeling g with $g(v_1) = 0$. Define $f(v_j) = p - g(v_j)$ for $j = 1, 2, \ldots, p$. It can be verified that $f(V(P_p)) = R_p^0$ and conditions (1) and (2) are satisfied. This proves the case $i = 0$.
The cases $i = 1, 2, p-2$ and $p-1$ are obtained by taking the complementary labelings of cases $i = p-1, p-2, 2$ and 1 respectively, in Theorem 12. □

The technique of proof used to establish the case $i = 1$ in Theorem 12 can be modified to obtain Theorem 13.

Theorem 13. *For $0 < s < p$, the path P_{p+1} admits a vertex labeling $f : V(P_{p+1}) \to R^{[1,s]}_{p+s}$, where $R^{[1,s]}_{p+s} = \{0, 1+s, 2+s, \ldots, p+s\}$, satisfying,*

1. *$f^*(E(P_{p+1})) = \{1, 2, \ldots, p\}$.*
2. *One of the end vertices is labeled 0.*

Proof. Let $P_{p+1} = v_1, v_2, \ldots, v_{p+1}$ and observe that $0 \leq (p-1-s) < p-1$. By Theorem 8, the subpath $P_p = v_1, v_2, \ldots, v_p$ has a graceful labeling $g : V(P_p) \to \{0, 1, \ldots, p-1\}$ such that $g(v_p) = p-1-s$. Define $f : V(P_{p+1}) \to R^{[1,s]}_{p+s}$ by $f(v_{p+1}) = 0$ and $f(v_j) = g(v_j) + s + 1$ for $j = 1, 2, \ldots, p$. It can be easily verified that $f(V(P_{p+1})) = R^{[1,s]}_{p+s}$ and conditions (1) and (2) are satisfied. □

3. Characterization of Additively Graceful Signed Paths

In this section, we characterize additively graceful signed paths. We first identify a class of signed paths, which are the only ones capable of admitting an additively graceful labeling. Of these, we show that exactly 4 signed paths do not admit an additively graceful labeling.

Lemma 1. *If a signed path S with n negative edges is additively graceful, then $n \leq 2$ and S has at most one negative section.*

Proof. Let S be a (p, m, n) additively graceful signed path. If $n > 2$ then by Observation 10, the negative edges labeled 1 and 2 must both be incident with the vertex labeled 0. Further $0+3$ and $1+2$ are the only ways in which 3 can be expressed as the sum of two distinct, non-negative integers. Both of these are not possible in S, hence $n \leq 2$. Again using Observation 10, we conclude that S has at most one negative section. □

In general, additively graceful signed graphs may have more than one negative section, as seen in Figure 2.

Lemma 2. *Every signed path S, with no negative edges is an additively graceful signed graph.*

Proof. The result follows from Observation 9. □

Lemma 3. *The signed path $S = u_1, v_1, v_2, \ldots, v_r$, $r \geq 1$ with one negative edge (u_1, v_1) is an additively graceful signed graph.*

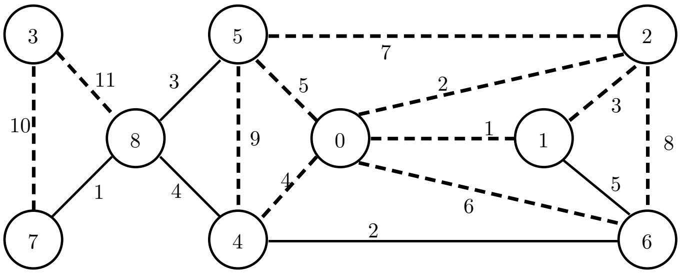

Figure 2. **An additively graceful signed graph with two negative sections**

Proof. By Theorem 8, $P_r = v_1, v_2, \ldots, v_r$ admits a graceful labeling $g : V(P_r) \to \{0, 1, \ldots, r-1\}$ such that $g(v_1) = 0$. Let $f : V(S) \to \{0, 1, \ldots, r\}$ be defined by $f(u_1) = 0$ and $f(v_j) = g(v_j) + 1$ for $j = 1, 2, \ldots, r$. It can be easily verified that f is an additively graceful labeling of S. $\square$

Lemma 4. *Let $P_p = v_1, v_2, \ldots, v_p$. The following four signed paths are not additively graceful.*

The signed path on P_4 with one negative edge (v_2, v_3). (3.1)

The signed path on P_6 with one negative edge (v_3, v_4). (3.2)

The signed path on P_5 with two negative edges (v_2, v_3) and (v_3, v_4). (3.3)

The signed path on P_7 with two negative edges (v_3, v_4) and (v_4, v_5). (3.4)

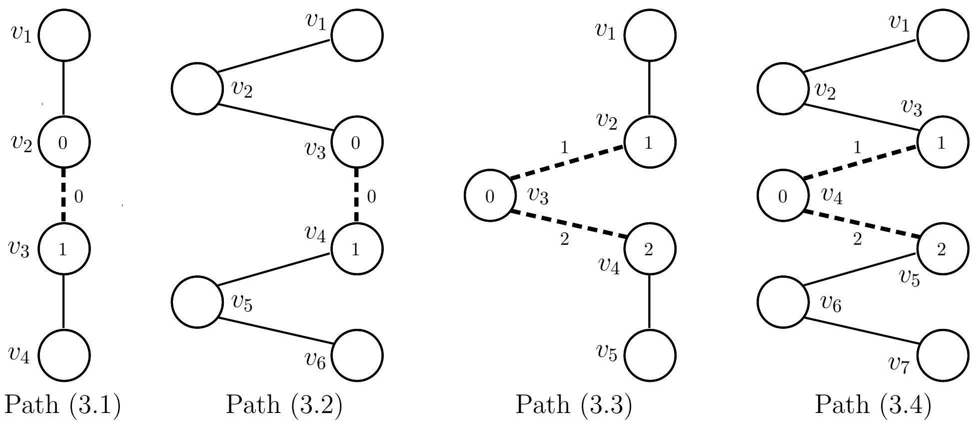

Figure 3. **The signed paths in 4 which are not additively graceful.**

Proof. As a consequence of Theorem 11, it suffices to prove the theorem for paths (3.1) and (3.2) only.

Suppose f is an additively graceful labeling of the signed path Theorem (3.1) and f^* is the induced edge labeling. Without loss of generality assume $f(v_2) = 0$ and $f(v_3) = 1$. If $f(v_1) = 2$, then $f^*(v_1, v_2) = f^*(v_3, v_4)$, giving a contradiction. If $f(v_1) = 3$, then $f^*(v_1, v_2) = 3$, which is again a contradiction.
Now suppose f is an additively graceful labeling of the signed path (3.2) and f^* is the induced edge labeling. Without loss of generality , assume that $f(v_3) = 0$ and $f(v_4) = 1$. Now the edge label 4 can only be obtained as $|4 - 0|$ or $|5 - 1|$.

Case 1. If $f(v_2) = 4$ then $f(v_5) \neq 5$. Also $f(v_1)$ cannot be 5 because this would cause $f^*(v_1, v_2) = f^*(v_5, v_6) = 1$. Hence vertex label 5 can only be assigned to v_6. Now no matter how vertex labels 2 and 3 are assigned, we get $f^*(v_1, v_2) = f^*(v_4, v_5)$, thus giving a contradiction.

Case 2. If $f(v_5) = 5$, then $f(v_2) \neq 4$. Also $f(v_6)$ cannot be 4 because this would cause $f^*(v_1, v_2) = f^*(v_5, v_6) = 1$. Hence vertex label 4 can only be assigned to v_1. Now no matter how vertex labels 2 and 3 are assigned, we get $f^*(v_2, v_3) = f^*(v_5, v_6)$, thus giving a contradiction. □

Lemma 5. *For $r \geq 3$, the signed path $S = u_2, u_1, v_1, v_2, \ldots, v_r$ with one negative edge (u_1, v_1) is an additively graceful signed graph.*

Proof. By Theorem 12, $P_r = v_1, v_2, \ldots, v_r$ has a labeling $g : V(P_r) \to R_r^{r-1}$ such that the induced edge labels are $1, 2, \ldots, r-1$ and $g(v_1) = 0$. Define $f : V(S) \to \{0, 1, \ldots, r+1\}$ by $f(u_1) = 0$, $f(u_2) = r$ and $f(v_j) = g(v_j) + 1$ for $j = 1, 2, \ldots, r$. It can be easily verified that f is an additively graceful labeling. □

Observe that the case $r = 1$ in Lemma 5 has been proved in Lemma 3 while the case $r = 2$ has been shown to not admit an additively graceful labeling in Lemma 4.

Lemma 6. *For $r \geq 4$, the signed path $S = u_3, u_2, u_1, v_1, v_2, \ldots, v_r$ with one negative edge (u_1, v_1) is an additively graceful signed graph.*

Proof. **Case 1.** $r = 4$.
The labeling $f(S) = (4, 5, 0, 1, 3, 6, 2)$ is an additively graceful labeling of S.

Case 2. $r \geq 5$. By Theorem 12, $P_{r-2} = v_3, v_4, \ldots, v_r$ has a labeling $g : V(P_{r-2}) \to R_{r-2}^{r-3}$ such that the induced edge labels are $1, 2, \ldots, r-3$ and $g(v_3) = 0$. Define $f : V(S) \to \{0, 1, \ldots, r+2\}$ by $f(u_3, u_2, u_1, v_1, v_2) = (2, r, 0, 1, r+2)$ and $f(v_j) = g(v_j) + 3$ for $j = 3, 4, \ldots, r$. It can easily be verified that f is an additively graceful labeling. □

Lemma 7. *For $r \geq 6$, the signed path $S = u_6, u_5, \ldots, u_1, v_1, v_2, \ldots, v_r$ with one negative edge (u_1, v_1) is an additively graceful signed graph.*

Proof. By Theorem 8, $P_{r-3} = v_4, v_5, \ldots, v_r$ has a graceful labeling $g : V(P_{r-3}) \to \{0, 1, \ldots, r-4\}$ such that $g(v_4) = r-6$. Define $f : V(S) \to \{0, 1, \ldots, r+5\}$ by $f(u_6, u_5, \ldots, u_1)$ is $(r+4, 3, r+3, 4, r+2, 0)$, $f(v_1, v_2, v_3)$ is $(1, r+5, 2)$ and $f(v_j) = g(v_j) + 5$ for $j = 4, 5, \ldots, r$. It can be easily verified that f is an additively graceful labeling. □

Lemma 8. *Let $l \geq 4$, $l \neq 6$ and let S_l denote the signed path $v_2, v_1, u_1, u_2, \ldots, u_l$ with one negative edge (v_1, u_1). Let $p > l + \lceil \frac{l}{2} \rceil$ be a positive integer. There exists a vertex labeling $f_l^p : V(S_l) \to L_l^p$, where $L_l^p = \{0, 1, \ldots, \lceil \frac{l}{2} \rceil\} \cup \{p-1, p-2, \ldots, p - \lfloor \frac{l}{2} \rfloor\} \cup \{p-l\}$, satisfying,*

1. *The induced edge labels on the positive edges are $p-2, p-3, \ldots, p-(l+1)$.*
2. *$f_l^p(u_1) = 0$, $f_l^p(v_1) = 1$, $f_l^p(v_2) = p - l$.*
3. *$f_l^p(u_l) = \begin{cases} p - (\frac{l}{2} - 1) & \text{if } l \text{ is even.} \\ \frac{l-1}{2} & \text{if } l \text{ is odd.} \end{cases}$*

Proof. We prove the result by induction on l. For $l = 4, 5, 9$, consider the labelings $f(S_l)$ equal to $(p-4, 1, 0, p-2, 2, p-1)$, $(p-5, 1, 0, p-2, 3, p-1, 2)$ and $(p-9, 1, 0, p-2, 3, p-1, 2, p-4, 5, p-3, 4)$ respectively. It can be easily verified that these labelings have range L_l^p and satisfy conditions (1), (2), and (3), whenever $p > l + \lceil \frac{l}{2} \rceil$ (see Figure 4).

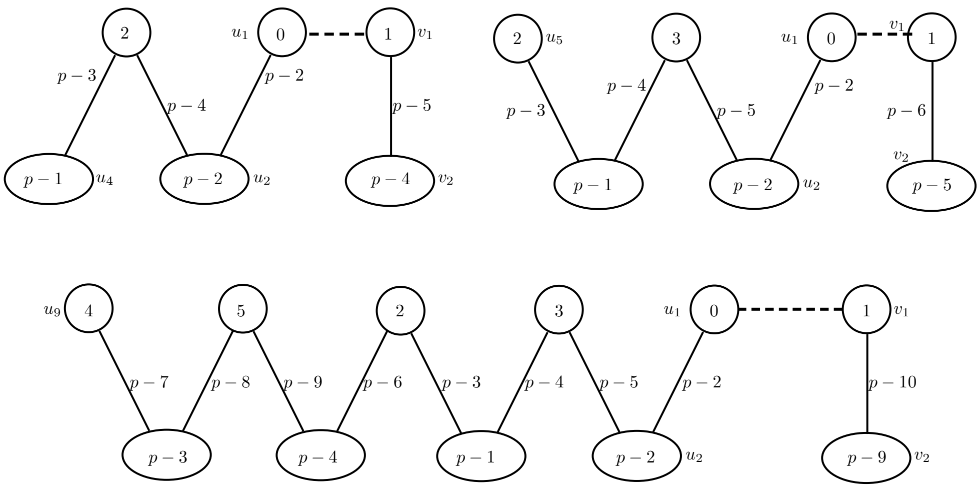

Figure 4. **Labelings for S_4, S_5 and S_9**

Assume that for some $l \in \mathbb{N}$, there exists a labeling $f_l^{p'} : V(S_l) \to L_l^{p'}$ satisfying conditions (1), (2) and (3) for every $p' > l + \lceil \frac{l}{2} \rceil$. Now consider S_{l+3} and fix a positive integer p, such that $p > l + 3 + \lceil \frac{l+3}{2} \rceil$. It follows that $p > l + \lceil \frac{l}{2} \rceil$ and hence by the induction hypothesis, there exists a vertex labeling $f_l^p : V(S_l) \to L_l^p$ for the subsigned path S_l of S_{l+3} satisfying (1), (2) and (3).

Now define $f^p_{l+3} : V(S_{l+3}) \to L^p_{l+3}$ as follows; $f^p_{l+3}(v_1) = 1$, $f^p_{l+3}(v_2) = p-(l+3)$ and,

If l is even,
$$f^p_{l+3}(u_i) = \begin{cases} f^p_l(u_i) & \text{if } i = 1, 2, \dots, l. \\ \frac{l}{2} + 2 & \text{if } i = l+1. \\ p - (\frac{l}{2} + 1) & \text{if } i = l+2. \\ \frac{l}{2} + 1 & \text{if } i = l+3. \end{cases}$$

If l is odd,
$$f^p_{l+3}(u_i) = \begin{cases} f^p_l(u_i) & \text{if } i = 1, 2, \dots, l. \\ p - (\frac{l-1}{2}) - 2 & \text{if } i = l+1. \\ \frac{l-1}{2} + 2 & \text{if } i = l+2. \\ p - (\frac{l-1}{2}) - 1 & \text{if } i = l+3. \end{cases}$$

It can be verified that f^p_{l+3} indeed has range L^p_{l+3} and satisfies conditions (1), (2) and (3). Illustrations of this labeling are provided in Figure 5. □

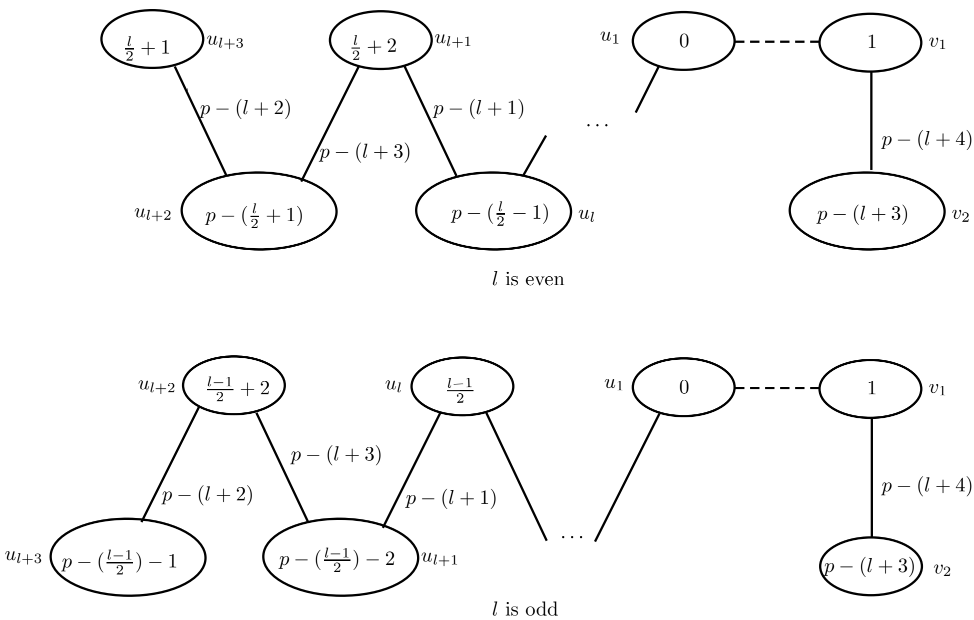

Figure 5.

Theorem 14 (Characterization of additively graceful signed paths). *A signed path S with n negative edges is additively graceful if and only if $n \le 2$ and S has at most one negative section, except that the four signed paths (3.1), (3.2), (3.3) and (3.4) are not additively graceful.*

Proof. By Lemma 1, if S is additively graceful then $n \leq 2$ and S has at most one negative section. To prove the converse, we first observe that by Lemma 2, a signed path with no negative edges is additively graceful.
Now consider the signed path $S = u_l, u_{l-1}, \dots, u_1, v_1, v_2, \dots, v_r$ with one negative edge (u_1, v_1). Without loss of generality assume $l \leq r$. By Lemmas 3, 5, 6 and 7, S is additively graceful for $l = 1, 2, 3$ and 6 respectively. Now suppose $l \geq 4$ and $l \neq 6$. Let $p = l + r$. It follows that $r \geq 4$ and $p > l + \lceil \frac{l}{2} \rceil$. By Theorem 8, the path $P_{r-1} = v_2, v_3, \dots, v_r$ admits a graceful labeling $g : V(P_{r-1}) \to \{0, 1, \dots, r-2\}$ such that $g(v_2) = r - (\lceil \frac{l}{2} \rceil + 1)$. For $S_l = u_l, u_{l-1}, \dots, u_1, v_1, v_2$, consider the labeling $f_l^p : V(S_l) \to L_l^p$ provided by Lemma 8. Define $f : V(S) \to \{0, 1, \dots, p-1\}$ by $f(w) = f_l^p(w)$ for $w \in S_l$ and $f(v_j) = g(v_j) + (\lceil \frac{l}{2} \rceil + 1)$ for $j = 2, 3, \dots, r$. One can readily verify that f is an additively graceful labeling of S.
Hence we have proved that apart from signed paths (3.1) and (3.2), all signed paths with one negative edge are additively graceful. Using Theorem 11, we conclude that apart from signed paths (3.3) and (3.4), all signed paths with two adjacent negative edges are additively graceful. □

4. Additively Graceful Signed Cycles

In this section we obtain a characterization of signed cycles which admit an additively graceful labeling. We shall refer to a (p, m, n) signed graph on the cycle C_p as a (p, m, n) *signed cycle*. By an *all negative* signed graph we shall mean a signed graph, all of whose edges are negative. Observation 15 follows directly from the labelings given in Figure 6.

Observation 15. Every $(3, m, n)$ signed cycle is additively graceful.

Lemma 9. *For $p \geq 4$, if a (p,m,n) signed cycle S is additively graceful then $n \leq 2$ and S has at most one negative section.*

Proof. For $p \geq 4$, let S be a (p, m, n) additively graceful signed cycle. If $n > 2$ then by Observation 10 the edges labeled 1 and 2 are incident with the vertex labeled 0. Now the negative edge labeled 3 must be incident with vertices labeled 0 and 3 or vertices labeled 1 and 2, which contradicts the fact that S is a cycle or that $p \geq 4$ respectively. Hence $n \leq 2$ and S has at most one negative section. □

Lemma 10. *The (p, m, n) signed cycle S with $n = 0$ is additively graceful if and only if $p \equiv 0$ or $3 \pmod 4$.*

Proof. Let S be a $(p, m, 0)$ signed cycle. If $p \equiv 0$ or $3 \pmod 4$ then by Theorem 4 and Observation 9 it follows that S is additively graceful. Conversely, if $p \equiv 1$ or 2

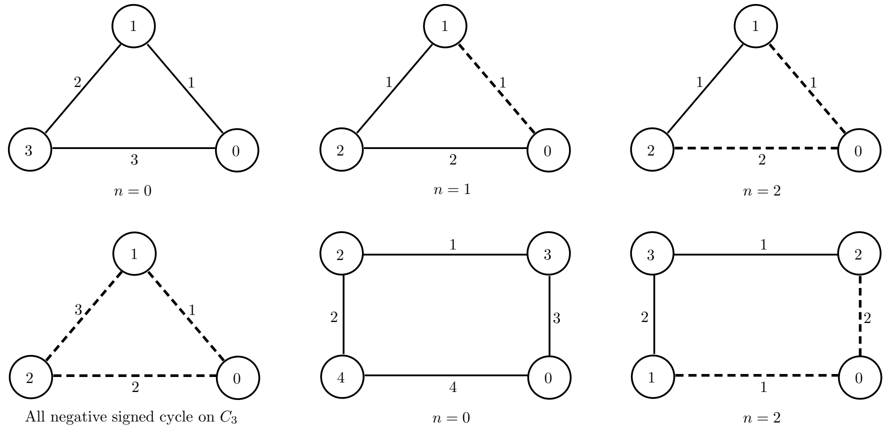

Figure 6. **All the additively graceful signed graphs on C_3 and C_4**

(mod 4) then $1+2+\cdots+p$, which is equal to $\frac{p}{2}(p+1)$ is odd. Hence using Theorem 2, we conclude that S is not additively graceful. $\square$

Lemma 11. *For $m \equiv 1$ or $2 \pmod 4$, the path P_{m+1} admits a labeling $f : V(P_{m+1}) \to \{0, 1, \ldots, m+1\}$ such that $f^*(E(P_{m+1})) = \{1, 2, \ldots, m\}$, where $f^*(uv) = |f(u)-f(v)|\ \forall\ uv \in E(P_{m+1})$. Moreover this labeling can be chosen such that the end vertices receive labels 0 and 1.*

Proof. **Case 1.** $m = 4k+1$ where k is a non-negative integer.
Let $P_{m+1} = a_0, b_0, a_1, b_1, \ldots, a_{2k}, b_{2k}$. If $k = 0$, take $f(P_2) = (0, 1)$, as the required vertex labeling of P_2. If $k \geq 1$, then define $f : V(P_{m+1}) \to \{0, 1, \ldots, m+1\}$ by,

$$f(a_i) = \begin{cases} 0 & \text{if } i = 0. \\ 2k+2+i & \text{if } 1 \leq i \leq 2k. \end{cases} \tag{4.1}$$

$$f(b_i) = \begin{cases} 2k+2-i & \text{if } 0 \leq i \leq k. \\ 2k+1-i & \text{if } k+1 \leq i \leq 2k. \end{cases} \tag{4.2}$$

Case 2. $m = 4k+2$ where k is a non-negative integer.
Consider $P_{m+1} = a_0, b_1, a_1, b_2, a_2, \ldots, b_{2k+1}, a_{2k+1}$ and define $f : V(P_{m+1}) \to \{0, 1, \ldots, m+1\}$ by,

$$f(a_i) = \begin{cases} 0 & \text{if } i = 0. \\ 2k+2-i & \text{if } 1 \leq i \leq 2k+1. \end{cases} \tag{4.3}$$

$$f(b_i) = \begin{cases} 2k+1+i & \text{if } 1 \leq i \leq k+1. \\ 2k+2+i & \text{if } k+2 \leq i \leq 2k+1. \end{cases} \tag{4.4}$$

It can be easily verified that the labelings in Equations (4.1), (4.2), (4.3) and (4.4), which are illustrated in Figure 7, satisfy the required conditions. □

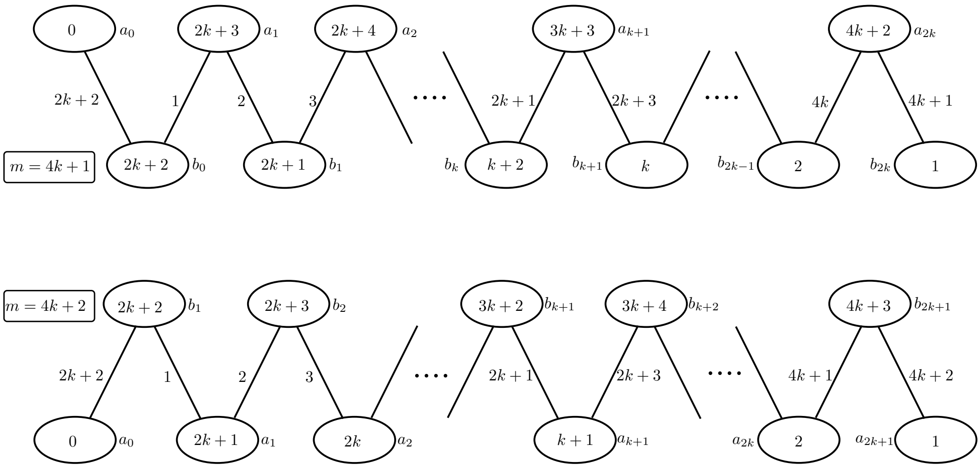

Figure 7.

Remark 1. It is worth noting that there exists equally elegant, alternate labelings to Equations (4.1), (4.2), (4.3) and (4.4), which work just as well to prove Lemma 11. These are given in Equations (4.5), (4.6), (4.7) and (4.8) respectively.
Case 1. $m = 4k + 1$ where $k > 0$.

$$f(a_i) = \begin{cases} 0 & \text{if } i = 0. \\ 2k+1+i & \text{if } 1 \le i \le k. \\ 2k+2+i & \text{if } k+1 \le i \le 2k. \end{cases} \tag{4.5}$$

$$f(b_i) = 2k + 1 - i \text{ where } 0 \le i \le 2k. \tag{4.6}$$

Case 2. $m = 4k + 2$ where $k \ge 0$.

$$f(a_i) = \begin{cases} 0 & \text{if } i = 0. \\ 2k+3-i & \text{if } 1 \le i \le k+1. \\ 2k+2-i & \text{if } k+2 \le i \le 2k+1. \end{cases} \tag{4.7}$$

$$f(b_i) = 2k + 2 + i \text{ where } 1 \le i \le 2k+1. \tag{4.8}$$

Lemma 12. *The (p,m,n) signed cycle S with $n = 1$ or 2, is additively graceful if and only if $m \equiv 1$ or 2 (mod 4) and S has a single negative section.*

Proof. For $n = 1$ or 2, let S be an additively graceful signed cycle. In either case, by Theorem 7, it follows that $(m^2 + m) \equiv 2 \pmod 4$ and hence $m \equiv 1$ or $2 \pmod 4$. Using Observation 10, we can conclude that, S has a single negative section.
Conversely, suppose S is a (p, m, n) signed graph with a single negative section with $n = 1$ or 2 and $m \equiv 1$ or $2 \pmod 4$. Consider the path P_{m+1} with the vertex labeling f provided by Lemma 11.

Case 1. If $n = 1$.
Connect the end vertices of P_{m+1} by a negative edge to obtain S. Now f itself is an additively graceful labeling of S.

Case 2. If $n = 2$.
Insert a vertex w and connect it to the end vertices of P_{m+1} using two negative edges, to obtain S. Now define $g : V(S) \to \{0, 1, \ldots, m+2\}$ by $g(v) = f(v) + 1$ for $v \in V(P_{m+1})$ and $g(w) = 0$. It can be easily verified that g is an additively graceful labeling of S. $\square$

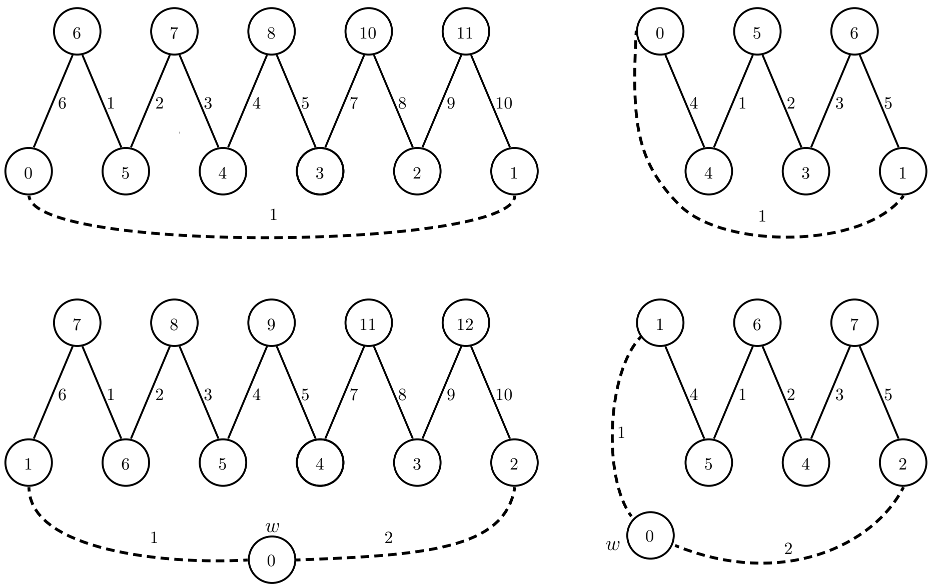

Figure 8. **Additively graceful labelings of signed cycles with $m = 5$ and 10**

Theorem 16 follows directly from Observation 15 and Lemmas 9, 10 and 12.

Theorem 16 (Characterization of additively graceful signed cycles). *A signed cycle S with m positive and n negative edges, is additively graceful if and only if one among the following 4 conditions is satisfied,*

1. $n = 0$ *and* $m \equiv 0$ *or* $3 \pmod 4$.

2. $n = 1$ *and* $m \equiv 1$ *or* $2 \pmod 4$.

3. $n = 2$, $m \equiv 1$ *or* $2 \pmod 4$ *and* S *contains a single negative section.*

4. S *is the all negative signed cycle on* C_3.

Acknowledgement: The authors are thankful to the reviewers for their valuable suggestions which helped improve the manuscript.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

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