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Research Article

Generalized Beck's Zero-Divisor Graph: A Graph Associated with a ring induced by a module-submodule pair

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Abstract: Given a commutative ring R, a left R-module M, and an R-submodule $N \subseteq M$, the graph $G(R; M, N)$, induced on R by the pair (M, N), is a simple graph with vertex set $R^* = R\backslash\{0\}$. Distinct vertices r and s are adjacent if $rsN = 0$. This graph generalizes Beck's zero-divisor graph $G(R)$. We analyze connectivity, completeness, bipartiteness, cycles, diameter, girth, independence/clique/chromatic numbers, and domination numbers, often under specific algebraic constraints on R or N. Applications to $\mathbb{Z}_n$-modules illustrate these results. By linking $G(R; M, N)$ to $G(R)$, we derive graph invariants for $G(R)$ efficiently and vice versa, deepening insights into algebraic structures and their graph-theoretic analogs.

Keywords: graphs, commutative rings, semiprime ideals, graph invariants.

AMS Subject classification: 05C07, 05C25, 05C38, 05C40, 05C69

1. Introduction

Algebra and graph theory are two key areas of mathematics, each with its own focus and methods, but they come together in interesting and useful ways. This connection happens when algebra helps us understand graphs, or when graphs help us solve problems in algebra. This crossover has led to the creation of "algebraic graph theory," where algebraic ideas are used to study graphs, and graph-based methods are applied to explore algebraic structures. By combining the two, we gain new ideas and ways to better understand both fields and how they can be applied.

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One of the most important co-study between graphs and algebraic structures is the zero-divisor graphs of commutative rings. The concept of zero-divisor graph was first introduced by I. Beck in 1988 [4]. Beck's interest was in the computation of the chromatic number of the graph by using the algebraic properties of the commutative ring. Following Beck's introduction, the concept of zero-divisor graph was further refined, expanded, and generalized to other algebraic structures as one can see for example in [1, 2, 5, 7–9, 11].

Building on the extensive work around generalized zero-divisor graphs, we introduce a new version of Beck's zero-divisor graph for commutative rings. What makes this graph unique is that it's defined by a pair consisting of a module and its submodule rather than the usual generalizations that go from rings to modules. Given a commutative ring R, a left R-module M, and an R-submodule N of M, our graph $G(R; M, N)$ is a simple graph with vertex set $R^* = R - 0$, and two distinct vertices r and s are adjacent if $rsN = 0$. This graph expands on Beck's zero-divisor graph $G(R) - 0$ (with the zero-vertex removed), as $G(R) - 0 \subseteq G(R; M, N)$. In fact, $G(R) - 0 = G(R; M, N)$ when $Ann_R(N) = 0$, where $Ann_R(N)$ refers to the annihilator of N in R. Throughout the paper, we highlight the strong connections between $G(R; M, N)$ and $G(R)$, seen through the interaction between the algebraic properties of R and the graph properties of $G(R; M, N)$. We also provide formulas for graph characteristics like the clique number, independence number, chromatic number, girth, and more. For example, we show in Corollary 3 that when $Ann_R(N)$ is a semiprime ideal, $\omega(G(R; M, N)) = |Ann_R(N)| + \omega\left(G\left(\frac{R}{Ann_R(N)}\right)\right) - 2$, and if $Ann_R(N)$ is a prime ideal, $\alpha(G(R; M, N)) = \alpha\left(G\left(\frac{R}{Ann_R(N)}\right)\right) \cdot |Ann_R(N)|$. These formulas, along with others, not only help calculate the graph invariants of $G(R; M, N)$ in terms of their counterparts in $G(R)$, but they also provide methods to compute the graph invariants of $G(R)$ in terms of their counterparts of $G(R; M, N)$. As a result, the findings in this paper will be helpful in solving coloring and optimization problems.

In Section 2, we cover the basics needed for this paper. In Section 3, we explore and analyze $G(R; M, N)$, looking at aspects like connectivity, completeness, bipartiteness, and cycles. We also determine specific values or provide rules for calculating the diameter, girth, independence number, clique number, chromatic number, domination number, and vertex degree, sometimes with restrictions on the ring, module, or submodule. In Section 4, we apply the results from Section 3 to the ring $\mathbb{Z}_n$ and provide examples to illustrate their applications.

Numerous additional properties of this new graph remain to be investigated. These aspects are left for future research and interested scholars.

2. Background

This section provides a review of the fundamental concepts related to rings and graphs. All the results presented here are drawn from [3, 6]. Throughout the paper, we assume that R is a non-zero commutative ring with unity $1 \neq 0$. The set of units in R is denoted by $U(R)$. We begin by outlining some preliminaries from Ring Theory.

Definition 1. A proper ideal I of a ring R is said to be maximal if I is not contained in another proper ideal of R.

Definition 2. A proper ideal I of R is semiprime if whenever $x^2 \in I$, then $x \in I$.

Definition 3. A proper ideal I of R is prime if whenever $xy \in I$, then $x \in I$ or $y \in I$.

It's obvious that every prime ideal is semiprime.

Definition 4. The radical $Rad(I)$ or $\sqrt{I}$ of a proper ideal I of R is the set $\{x \in R : x^n \in I \text{ for some } n \in \mathbb{N}\}$.

It is easy to see that $\sqrt{I}$ is an ideal of R. Moreover, an ideal I is semiprime if and only if $I = \sqrt{I}$.

Definition 5. Given a left R-module M and a left R-submodule N of M, the annihilator of N in R is defined to be the set $Ann_R(N) = \{r \in R : rN = 0\}$.

The set $Ann_R(N)$ is an ideal of R. Also, if N and L are left submodules of M such that $N \subseteq L$, then $Ann_R(N) \supseteq Ann_R(L)$.

Next, we turn to some preliminaries from graph theory concerning undirected graphs. In this section, G denotes an undirected graph. The number of vertices in G is referred to as the order of the graph. The set of vertices in G is denoted by $V(G)$. If two vertices u and v are adjacent, we write this as $u \leftrightarrow v$.

Definition 6. Let v be a vertex in G. The open neighborhood $N(v)$ of v is the set of all vertices adjacent to v.

If G is a simple undirected graph, then $v \notin N(v)$. If $N(v) = \emptyset$, then v is said to be an isolated vertex.

Definition 7. The degree of a vertex v of G is the number of edges incident to v. The degree of v is denoted by $deg_G(v)$ (or $deg(v)$ if there is no confusion with the underlined graph).

The minimum of the degrees of the vertices is denoted by $\delta(G)$, while the maximum of the degrees of the vertices is denoted by $\Delta(G)$. When G is a simple graph, then $deg(v) = |N(v)|$, where $|N(v)|$ means the cardinality of $N(v)$. Hence, v is isolated if and only if $deg(v) = 0$.

Definition 8. Let v and u be two vertices of G. The distance $d(u, v)$ between v and u is the length of a shortest path between them. The diameter of G, denoted by $diam(G)$, is defined to be the maximum of the set $\{d(u, v) : u, v \in V(G)\}$.

Definition 9. A graph G is connected if there is a path between any two distinct vertices of G.

Definition 10. By the girth of G, we mean the length of a shortest cycle in G. The girth of G is denoted by $g(G)$. If G has no cycles, then we write $g(G) = \infty$.

Definition 11. A graph is said to be complete if it is a simple graph and every two distinct vertices are adjacent. The complete graph on n vertices is denoted by K_n.

Definition 12. A subgraph of G which is a complete graph is called a clique of G. The order of a clique with the largest number of vertices is called the clique number of G and it is denoted by $\omega(G)$.

Definition 13. A dominating set D of G is a nonempty subset of $V(G)$ such that each vertex of G is either in D or adjacent to a vertex in D. The minimum of the set $\{|D| : D \text{ is a dominating set of } G\}$ is called the domination number of G and is denoted by $\gamma(G)$.

Definition 14. A simple graph G is called bipartite if we can partition $V(G)$ into two disjoint nonempty subsets (each subset is called a part) such that the vertices belonging to the same subset are not adjacent to each other.

Definition 15. A subset S of vertices of a graph G is called independent if no two vertices in S are adjacent. The cardinality of largest independent set is called the independence number of G and is denoted by $\alpha(G)$.

3. The graph of rings induced by a pair of module and submodule

In this section, we define the graph $G(R; M, N)$ associated with a commutative ring R, defined by a pair (M, N) where M is a left R-module and N is an R-submodule of M. We then examine various properties of $G(R; M, N)$, including connectivity, completeness, bipartiteness, and cycles. Additionally, we calculate or provide methods to determine key values such as diameter, girth, independence number, clique number, chromatic number, domination number, and vertex degree, potentially with some restrictions on the ring, its ideals, or the submodule N.

3.1. Basic Properties

Definition 16. Let R be a commutative ring, M a left R-module, and N be a left R-submodule of M. The graph of R induced by the pair (M, N), denoted by $G(R; M, N)$, is the simple graph whose vertices are the elements of $R^* = R - 0$ and two distinct vertices $r, s \in R^*$ are adjacent if $rsN = 0$ (or equivalently $rs \in Ann_R(N)$).

The following lemma has a key role in this paper.

Lemma 1. *Let R be a commutative ring, M a left R-module, and N a left R-submodule of M. Then*

1. *For every $r \in R^*$ and $x \in Ann_R(N)\backslash\{0\}$ such that $r \neq x$, we have $r \leftrightarrow x$.*
2. *If $r \leftrightarrow s$, then $\alpha r \leftrightarrow \beta s$, for every $\alpha, \beta \in R^*$ provided that $\alpha r, \beta s \in R^*$ and $\alpha r \neq \beta s$.*
3. *Given a unit u and a non-unit element $r \neq 0$, we have $u \leftrightarrow r$ if and only if $r \in Ann_R(N)$. Therefore, $1 \leftrightarrow r$ if and only if $u \leftrightarrow r$.*
4. *Given distinct units u_1 and u_2 of R, we have $u_1 \leftrightarrow u_2$ if and only if $N = 0$.*

Proof. 1, 2, and 3 follow from the fact that $Ann_R(N)$ is an ideal of R. For Part 4, the proof is straightforward. □

Theorem 1. *Let R be a commutative ring, M a left R-module, N a left R-submodule of M, and $r \in R^*$. Then*

$$deg(r) = \begin{cases} |R| - 2 & if\ r \in Ann_R(N) - 0 \\ |Ann_R(N)| - 1 & if\ r \in U(R) \\ \geq |Ann_R(N)| - 1 & if\ r \notin U(R) \cup Ann_R(N). \end{cases}$$

Proof. Apply Lemma 1. □

The graph $G(\mathbb{Z}_p, M, N)$ of $\mathbb{Z}_p$, where p is a prime number, defined by a pair (M, N) of module and submodule, respectively, is utterly specified by the following proposition whose proof follows from Lemma 1.

Proposition 1. *Let p be a prime number, M a $\mathbb{Z}_p-$module, and N a $\mathbb{Z}_p-$ submodule of M. Then*

$$G(\mathbb{Z}_p; M, N) = \begin{cases} K_{p-1} & if\ N = 0 \\ \overline{K}_{p-1} & if\ N \neq 0, \end{cases}$$

where $\overline{K}_{p-1}$ is the complement graph of K_{p-1} which consists of $p-1$ independent vertices.

Assume B is a set of vertices of a graph G, then $G - B$ denotes the subgraph of G obtained by deleting all vertices of B along with all edges incident to these vertices. If $B = \{v\}$, we write $G - B$ as $G - v$. On the other hand, if $R \ncong \mathbb{Z}_p$, where p is a prime number, we can assume without loss of generality that $|R| \geq 4$.

Theorem 2. *Let R be a commutative ring, M a left R-module, and N a left R-submodule of M. Then the following statements are equivalent:*

1. *$G(R; M, N)$ is connected.*
2. *$Ann_R(N) \neq 0$.*
3. *$G(R; M, N) \neq G(R) - 0$.*

Moreover, for connected $G(R; M, N)$, we have $diam(G(R; M, N)) \leq 2$, and $diam(G(R; M, N)) = 2$ if and only if $N \neq 0$.

Proof. If $R \cong \mathbb{Z}_p$, where p is a prime number, then the result follows by Proposition 1 and by noticing that $N = 0$ if and only if $Ann_{\mathbb{Z}_p}(N) \neq 0$. Therefore assume $R \ncong \mathbb{Z}_p$.
$1 \Rightarrow 2$: Suppose $G(R; M, N)$ is connected, and assume, for contrary, that $Ann_R(N) = 0$. Let $t \in R\backslash\{0, 1\}$. Then by Lemma 1, $1 \not\leftrightarrow t$. Thus, 1 is an isolated vertex and hence $G(R; M, N)$ is disconnected which is a contradiction.
$2 \Rightarrow 3$: Assume $Ann_R(N) \neq 0$. Let $0 \neq r \in Ann_R(N)$. Then $r \leftrightarrow 1$ in $G(R; M, N)$ but $r \not\leftrightarrow 1$ in $G(R) - 0$. So, we get $G(R; M, N) \neq G(R) - 0$.
$3 \Rightarrow 1$: Assume $G(R; M, N) \neq G(R) - 0$. Since both graphs have the same vertex set and $G(R) - 0 \subseteq G(R; M, N)$, there exist $x, y \in R^*$ such that $x \leftrightarrow y$ in $G(R; M, N)$ but $x \not\leftrightarrow y$ in $G(R) - 0$. That is, $xy \in Ann_R(N)$ and $xy \neq 0$. Now let $r, s \in R^*$. If $r \leftrightarrow s$, then $d(r, s) = 1$. If $r \not\leftrightarrow s$, then both r and s are different from xy (otherwise we obtain a contradiction with item (1) of Lemma 1). So, we have the path $r \leftrightarrow xy \leftrightarrow s$ of length 2. Thus, $d(r, s) = 2$. Consequently, $G(R; M, N)$ is connected. The fact that $diam(G(R; M, N)) = 2$ if and only if $N \neq 0$ will be a direct conclusion of Theorem 3. □

3.2. Completeness and Girth

Theorem 3. *Let R be a commutative ring, M a left R-module, and N a left R-submodule of M. Then $G(R; M, N)$ is a complete graph if and only if $N = 0$.*

Proof. If $R \cong \mathbb{Z}_p$, where p is a prime number, then the result follows by Proposition 1. Therefore we assume $R \ncong \mathbb{Z}_p$.
$(\Rightarrow)$: Suppose $G(R; M, N)$ is complete. Then we have $1 \leftrightarrow r$, for each $r \in R\backslash\{0, 1\}$. Thus, $R\backslash\{0, 1\} \subseteq Ann_R(N)$. Since $R \ncong \mathbb{Z}_p$, there exists $r \in R\backslash\{0, 1\}$ such that $r + 1 \notin \{0, 1\}$. Hence $r + 1 \in Ann_R(N)$. Since $Ann_R(N)$ is an ideal of R, we get $1 = r + 1 - r \in Ann_R(N)$. We conclude that $RN = 0$ which implies $N = 0$.
$(\Leftarrow)$: Assume $N = 0$. Then $Ann_R(N) = R$ which yields $G(R; M, N)$ is complete. □

Corollary 1. *Let R be a commutative ring, M a left R-module, and N a left R-submodule of M. If $|U(R)| \geq 2$, then the following statements are equivalent:*

1. *$G(R; M, N)$ is a complete graph.*
2. *$N = 0$.*
3. *$U(R)$ is not an independent set.*

Proof. Based on Theorem 3, we focus on proving the equivalence between items (1) and (3). Suppose $G(R; M, N)$ is a complete graph. Let $1 \neq u \in U(R)$. Then $1 \leftrightarrow u$ and hence $U(R)$ is not independent. Conversely, assume $U(R)$ is not independent. Then there exist distinct elements $u, v \in U(R)$ such that $u \leftrightarrow v$. Thus $uvN = 0$ which implies $N = 0$. So, $rsN = 0$ or equivalently $r \leftrightarrow s$ for every distinct elements $r, s \in R^*$. Consequently, we obtain that $G(R; M, N)$ is complete. □

One must pay attention to the condition $|U(R)| \geq 2$ in Corollary 1. Actually, when $|U(R)| = 1$, then $U(R)$ is independent vacuously. However the independence of $U(R)$ alone is not enough to guarantee the completeness of the graph $G(R; M, N)$. The next example illustrates this matter.

Example 1. Consider the Boolean ring $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2 = \{0, 1, 1^-, 1^+\}$. Here $0 = (0,0), 1^- = (1,0), 1^+ = (0,1)$ and $1 = (1,1)$. It's easy to see that $U(R) = \{1\}$ and $1^- \cdot 1^+ = 0$. Let $M = R[x]$ and $N =< 1^+ \cdot x >$. We have $N \neq 0$ and $Ann_R(N) = \{0, 1^-\}$. Also, $G(R; M, N)$ is not complete since $1 \not\leftrightarrow 1^+$. The graph $G(R; M, N)$ is represented below:

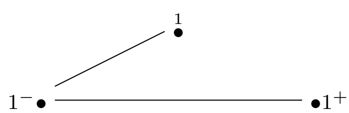

Figure 1. $G(\mathbb{Z}_2 \oplus \mathbb{Z}_2; (\mathbb{Z}_2 \oplus \mathbb{Z}_2)[x], < 1^+ \cdot x >)$

Next, we explore the cycles and girth of $G(R; M, N)$. We start by introducing the definition of demiprime ideals, which generalize the concept of prime ideals.

Definition 17. Let R be a commutative ring and I a proper ideal of R. The ideal I is said to be a demiprime ideal if whenever distinct elements $a, b \in R$ such that $ab \in I$, then $a \in I$ or $b \in I$.

Remark 1. It is clear that an ideal is prime if and only if it is both semiprime and demiprime. On the other hand, it is easy to see that if I is not demiprime, then $|R| \geq 4$.

Example 2. In $\mathbb{Z}_4$, the ideal $\{0\}$ is not prime since $2^2 \equiv 0 \ (mod\ 4)$ but it is demiprime because by direct calculations we have if $x \not\equiv y \ (mod\ 4)$, then $x.y \not\equiv 0 \ (mod\ 4)$.

In the previous example the demiprime ideal which is not prime is the zero ideal. So, the question that arises is whether there is a nonzero demiprime ideal which is not prime. The answer is found in Corollary 2 which states that demiprime ideals which are not prime ideals must be zero ideals.

Proposition 2. *Let R be a commutative ring and I a nonzero ideal of R such that I is not semiprime. Then I is not demiprime.*

Proof. Let $a \notin I$ such that $a^2 \in I$, and $i \in I - \{0\}$. Then a and $a + i$ are distinct elements of R that lie outside I such that $a(a + i) = a^2 + ai \in I$. This means that I is not demiprime. □

Corollary 2. *Let R be a commutative ring and I a demiprime ideal of R that is not a prime ideal. Then $I = \{0\}$.*

Proof. The proof follows by noticing that the statement of the corollary is the contrapositive statement of Proposition 2 and that an ideal is prime if and only if it is both demiprime and semiprime. □

In the next theorem, we compute the girth of $G(R; M, N)$. Since $G(R; M, N) = G(R) - 0$ when $Ann_R(N) = 0$ and this case has been studied in the literature (see for example [10]), we shall assume $Ann_R(N) \neq 0$.

Theorem 4. *Let R be a commutative ring such that $|R| \geq 4$, M a left R-module, and N a left R-submodule of M such that $Ann_R(N) \neq 0$. Then $G(R; M, N)$ has a cycle only in the following cases:*

1. *$N = 0$.*
2. *$N \neq 0$ and $|Ann_R(N)| > 2$.*
3. *$N \neq 0$, $|Ann_R(N)| = 2$, and $Ann_R(N)$ is not a demiprime ideal.*

Besides, when a cycle exists, the girth of $G(R; M, N)$ is 3.

Proof. 1. If $N = 0$, then $G(R; M, N)$ is complete by Theorem 3. Since $|R| \geq 4$, we have a triangle inside $G(R; M, N)$.
2. Assume $N \neq 0$ and $|Ann_R(N)| > 2$. Since $N \neq 0$, we have $1 \notin Ann_R(N)$. On the other hand, since $|Ann_R(N)| > 2$, we can find two distinct nonzero elements $r, s \in Ann_R(N)$. So, by Lemma 1, we obtain the triangle $1 \leftrightarrow r \leftrightarrow s \leftrightarrow 1$.
3. Assume $N \neq 0$, $|Ann_R(N)| = 2$, and $Ann_R(N)$ is not a demiprime ideal. Let $Ann_R(N) = \{0, x\}$. Since $Ann_R(N)$ is not demiprime, there exist two distinct elements $r, s \notin Ann_R(N)$ such that $rs \in Ann_R(N)$. Thus, we have the triangle $r \leftrightarrow x \leftrightarrow s \leftrightarrow r$.
We conclude from the above argument that we always have a triangle. Therefore $g(G(R; M, N)) = 3$.

In the final part of the proof, we confirm that in the remaining case $G(R;M,N)$ does not possess cycles. Assume $N \neq 0$, $|Ann_R(N)| = 2$, and $Ann_R(N)$ is a demiprime ideal. Then for every distinct elements $r, s \notin Ann_R(N)$, we have $rs \notin Ann_R(N)$, which means that $r \not\leftrightarrow s$. Thus $G(R;M,N)$ is a star graph with center x and hence no cycle is included in the graph. □

Theorem 5. *Let R be a commutative ring such that $|R| \geq 4$, M a left R-module, and N a left R-submodule of M such that $Ann_R(N) \neq 0$. Then the following statements are equivalent:*

1. *$G(R;M,N)$ is a tree.*
2. *$G(R;M,N)$ is a star graph.*
3. *$N \neq 0$, $|Ann_R(N)| = 2$, and $Ann_R(N)$ is a demiprime ideal.*
4. *$G(R;M,N)$ is bipartite.*

Proof. $1 \Rightarrow 2$: is obvious.
$2 \Rightarrow 3$: Assume $G(R;M,N)$ is a star graph with center $x \neq 0$. Since $G(R;M,N)$ is not complete, we have $N \neq 0$ by Corollary 1 and $1 \notin Ann_R(N)$. So, $x \neq 1$ and hence $x \leftrightarrow 1$. By Lemma 1, $x \in Ann_R(N)$. Since every pair of distinct nonzero elements not equal to x are not adjacent, we obtain $Ann_R(N) = \{0, x\}$ and $Ann_R(N)$ is demiprime.
$3 \Rightarrow 4$: Suppose $N \neq 0$, $|Ann_R(N)| = 2$, and $Ann_R(N)$ is a demiprime ideal. Let $Ann_R(N) = \{0, x\}$. Thus every pair of distinct nonzero elements not equal to x are not adjacent. So, $G(R;M,N)$ is a star graph and hence bipartite.
$4 \Rightarrow 1$: Assume $G(R;M,N)$ is bipartite. If $G(R;M,N)$ has a cycle, then by Theorem 4, $g(G(R;M,N)) = 3$. That is, $G(R;M,N)$ has a triangle which implies $G(R;M,N)$ cannot be bipartite and this is a contradiction. Therefore $G(R;M,N)$ does not have cycles and hence it is a tree. □

3.3. Domination Number, Cliques, Chromatic Number, and Independence Number

Theorem 6. *Let R be a commutative ring, M a left R-module, and N a left R-submodule of M. Then*

$$\gamma(G(R;M,N)) = \begin{cases} \gamma(G(R) - 0) & \text{if } Ann_R(N) = 0 \\ 1 & \text{if } Ann_R(N) \neq 0. \end{cases}$$

Proof. By Theorem 2, if $Ann_R(N) = 0$, then $G(R;M,N) = G(R) - 0$. However, if $Ann_R(N) \neq 0$, then there exists $x \in Ann_R(N) - 0$. The set $\{x\}$ is a dominating set by Lemma 1. □

In the remaining part of this subsection, a maximal clique means a clique that is, not included in a larger clique. The reader should pay attention to the fact that the number of vertices in a maximal clique is not necessarily equal to the clique number of $G(R;M,N)$. However, the clique number of $G(R;M,N)$ equals the order of a maximal clique with the largest number of vertices compared to the other maximal cliques.

Theorem 7. *Let R be a commutative ring, M a left R-module, and N a left R-submodule of M. Then*

$$\omega(G(R;M,N)) = \begin{cases} |R|-1 & if\ N=0 \\ |Ann_R(N)| & if\ N\neq 0\ and \\ & Ann_R(N)\ is\ demiprime \\ |Ann_R(N)|+\omega(G(R;M,N)-Ann_R(N))-1 & if\ N\neq 0\ and \\ & Ann_R(N)\ is\ not\ demiprime. \end{cases}$$

Moreover, in the last case $\omega(G(R;M,N)-Ann_R(N))\geq 2$.

Proof. • If $N=0$, then $G(R;M,N)$ is a complete graph by Theorem 3. Therefore $\omega(G(R;M,N))=|R|-1$.
• If $N\neq 0$ and $Ann_R(N)$ is demiprime, then the product of any two distinct vertices outside $Ann_R(N)$ remains outside $Ann_R(N)$. Thus the set $R-Ann_R(N)$ is independent and therefore, by Lemma 1, $(Ann_R(N)-0)\cup\{r\}$, where $r\notin Ann_R(N)$, are the only maximal cliques in $G(R;M,N)$. Consequently, $\omega(G(R;M,N))=|Ann_R(N)|$.
• If $N\neq 0$ and $Ann_R(N)$ is not demiprime, then there exists two distinct non-unit elements $r,s\in R^*-Ann_R(N)$ such that $rs\in Ann_R(N)$ (if r is unit, then $s\in Ann_R(N)-0$ which is a contradiction). Thus, $r\leftrightarrow s$ and hence $\omega(G(R;M,N)-Ann_R(N))\geq 2$. Also, $(Ann_R(N)-0)\cup\{s,r\}$ is a clique which yields $\omega(G(R;M,N))\geq |Ann_R(N)|+1$. Now, since every element in $Ann_R(N)-0$, if any, is adjacent to every element outside $Ann_R(N)$, we obtain that whenever Q is a maximal clique in the subgraph $G(R;M,N)-Ann_R(N)$ with the largest number of vertices (i.e., $|Q|=\omega(G(R;M,N)-Ann_R(N))))$, then $(Ann_R(N)-0)\cup Q$ is a maximal clique of $G(R;M,N)$ with the largest number of vertices. So, $\omega(G(R;M,N))=|Ann_R(N)-0|+|Q|=|Ann_R(N)|-1+\omega(G(R;M,N)-Ann_R(N))$. □

In the upcoming work, we shall show that

$$\omega(G(R;M,N)-Ann_R(N))=\omega\left(G\left(\frac{R}{Ann_R(N)}\right)\right)-1,$$

when $Ann_R(N)$ is a semiprime ideal of R. We start from the following lemma that investigates, in Parts 1 and 2, the adjacency between different cosets of $Ann_R(N)$ in R, and, in Part 3, the adjacency between vertices within a coset of $Ann_R(N)$ in R.

Lemma 2. *Let R be a commutative ring, M a left R-module, and N a left R-submodule of M. Then*

1. *If $r\not\equiv s\ (mod\ Ann_R(N))$, then $r\leftrightarrow s$ in $G(R;M,N)$ if and only if $r+Ann_R(N)\leftrightarrow s+Ann_R(N)$ in $G\left(\frac{R}{Ann_R(N)}\right)$.*

2. *If $r \not\equiv s \pmod{\sqrt{Ann_R(N)}}$, then $r \leftrightarrow s$ in $G(R;M,N)$ implies $r + \sqrt{Ann_R(N)} \leftrightarrow s + \sqrt{Ann_R(N)}$ in $G\left(\frac{R}{\sqrt{Ann_R(N)}}\right)$.*

3. *If $x \in R^*$ and $r, s \in Ann_R(N)\backslash\{0\}$ are two distinct elements such that $r + x \leftrightarrow s + x$ in $G(R;M,N)$, then $x \in \sqrt{Ann_R(N)}$. Further, the converse holds if $Ann_R(N)$ is semiprime.*

Proof. 1. Let $r, s \in R^*$ such that $r \not\equiv s \pmod{Ann_R(N)}$. This implies $r \neq s$. We have

$$\begin{aligned} r \leftrightarrow s \text{ in } G(R;M,N) &\Leftrightarrow rs \in Ann_R(N) \\ &\Leftrightarrow (r + Ann_R(N))(s + Ann_R(N)) = Ann_R(N) \\ &\Leftrightarrow r + Ann_R(N) \leftrightarrow s + Ann_R(N) \text{ in } G\left(\frac{R}{Ann_R(N)}\right). \end{aligned}$$

2. Let $r, s \in R^*$ such that $r \not\equiv s \pmod{\sqrt{Ann_R(N)}}$. This implies $r \neq s$. We have

$$\begin{aligned} r \leftrightarrow s \text{ in } G(R;M,N) &\Leftrightarrow rs \in Ann_R(N) \subseteq \sqrt{Ann_R(N)} \\ &\Rightarrow (r + \sqrt{Ann_R(N)})(s + \sqrt{Ann_R(N)}) = \sqrt{Ann_R(N)} \\ &\Rightarrow r + \sqrt{Ann_R(N)} \leftrightarrow s + \sqrt{Ann_R(N)} \text{ in } G\left(\frac{R}{\sqrt{Ann_R(N)}}\right). \end{aligned}$$

3. Let $x \in R^*$ and $r, s \in Ann_R(N)\backslash\{0\}$ such that $r \neq s$. Then

$$\begin{aligned} r + x \leftrightarrow s + x \text{ in } G(R;M,N) &\Leftrightarrow (r + x)(s + x) \in Ann_R(N) \\ &\Leftrightarrow rs + rx + sx + x^2 \in Ann_R(N) \\ &\Leftrightarrow x^2 \in Ann_R(N) \\ &\Rightarrow x \in \sqrt{Ann_R(N)}. \end{aligned}$$

The rest is straightforwardly proved. □

Remark 2. In Lemma 2, the converse of (3) is not necessarily true. To see this Let $R = \mathbb{Z}$, $M = \frac{\mathbb{Z}}{16\mathbb{Z}}$, and $N = \frac{2\mathbb{Z}}{16\mathbb{Z}}$. Note that $Ann_R(N) = 8\mathbb{Z}$ and $\sqrt{Ann_R(N)} = 2\mathbb{Z}$. Take $2, 10 \in \sqrt{Ann_R(N)}$. We have $2 \neq 10$ and $10 \equiv 2 \pmod{Ann_R(N)}$ but $2 \times 10 = 20 \notin Ann_R(N)$ which means $2 \not\leftrightarrow 10$ in $G(R;M,N)$. The reason behind the failure of the converse of (3) is that $Ann_R(N)$ is not a semiprime ideal (note that $4 \notin Ann_R(N)$ but $4^2 \in Ann_R(N)$). We have seen in Lemma 2 (3) that if $Ann_R(N)$ is semiprime, then the converse of (3) is satisfied.

Lemma 3. *Let R be a commutative ring, M a left R-module, and N a left R-submodule of M. Then any maximal clique in $G(R;M,N)$ must contain $Ann_R(N)\backslash\{0\}$.*

Proof. Let Λ be a maximal clique in $G(R; M, N)$. Assume, for contrary, that $Ann_R(N)\backslash\{0\} \nsubseteq \Lambda$. By Lemma 1, $\Lambda \cup Ann_R(N)\backslash\{0\}$ is a clique such that $\Lambda \subsetneq \Lambda \cup Ann_R(N)\backslash\{0\}$. This contradicts that Λ is a maximal clique. $\square$

The next theorem gives an estimation of the clique number of $G(R; M, N)$ in terms of $Ann_R(N)$ and the quotient of R by $\sqrt{Ann_R(N)}$ and the $\sqrt{Ann_R(N)}$. To make things easy, we introduce the notation $G(B \subseteq R; M, N)$ which denotes the subgraph of $G(R; M, N)$ whose vertex set is the subset $B \subseteq R^*$.

Theorem 8. *Let R be a commutative ring, M a left R-module, and N a left R-submodule of M. Then $\varphi \leq \omega(G(R; M, N)) \leq \psi$ where*
$\varphi = \omega(G(\sqrt{Ann_R(N)} \subseteq R; M, N))$ and $\psi = \varphi + \omega\left(G\left(\frac{R}{\sqrt{Ann_R(N)}}\right)\right) - 1$.

Proof. The inequality $\omega(G(\sqrt{Ann_R(N)} \subseteq R; M, N)) \leq \omega(G(R; M, N))$ follows directly. We prove the other inequality. By Lemma 2, for every $x \notin \sqrt{Ann_R(N)}$, the coset $x + Ann_R(N)$ forms an independent set in $G(R; M, N)$. So any clique in $G(R; M, N) - \sqrt{Ann_R(N)}$ includes vertices from different cosets of $Ann_R(N)$. Thus, once again by Lemma 2, each clique of $G(R; M, N) - \sqrt{Ann_R(N)}$ induces a clique in $G\left(\frac{R}{\sqrt{Ann_R(N)}}\right)$ with the same cardinality consisting of the corresponding cosets of $Ann_R(N)$. Now, let Λ be a maximal clique whose cardinality is $\omega(G(R; M, N))$. Noticing that $\Lambda \cap \sqrt{Ann_R(N)}$ is a clique in $G(\sqrt{Ann_R(N)} \subseteq R; M, N)$ and $\Lambda \cap (R - \sqrt{Ann_R(N)})$ is a clique in $G(R; M, N) - \sqrt{Ann_R(N)}$, we obtain

$$
\begin{aligned}
|\Lambda| &= |\Lambda \cap \sqrt{Ann_R(N)}| + |\Lambda \cap (R - \sqrt{Ann_R(N)})| \\
&\leq \omega(G(\sqrt{Ann_R(N)} \subseteq R; M, N)) + \omega\left(G\left(\frac{R}{\sqrt{Ann_R(N)}}\right)\right) - 1.
\end{aligned}
$$

The proof ends when we let $\varphi = \omega(G(\sqrt{Ann_R(N)} \subseteq R; M, N))$ and $\psi = \omega(G(\sqrt{Ann_R(N)} \subseteq R; M, N)) + \omega\left(G\left(\frac{R}{\sqrt{Ann_R(N)}}\right)\right) - 1$. $\square$

Corollary 3. *Let R be a commutative ring, M a left R-module, and N a left R-submodule of M.*

1. *If $Ann_R(N)$ is semiprime, then*

$$\omega(G(R; M, N)) = |Ann_R(N)| + \omega\left(G\left(\frac{R}{Ann_R(N)}\right)\right) - 2.$$

2. *If $Ann_R(N)$ is prime, then $\omega(G(R; M, N)) = |Ann_R(N)|$, which agrees with Theorem 7. Moreover, $\chi(G(R; M, N)) = |Ann_R(N)|$ and $\alpha(G(R; M, N)) = \alpha\left(G\left(\frac{R}{Ann_R(N)}\right)\right) \cdot |Ann_R(N)|$.*

Proof. 1. Suppose $Ann_R(N)$ is semiprime. Then we have $\sqrt{Ann_R(N)} = Ann_R(N)$. By Theorem 8, we have

$$|Ann_R(N)| - 1 \leq \omega(G(R; M, N)) \leq |Ann_R(N)| - 2 + \omega\left(G\left(\frac{R}{Ann_R(N)}\right)\right).$$

However, by 1 and 3 of Lemma 2 and Lemma 3, carrying the same discussion as in the proof of Theorem 8, we get

$$\begin{aligned}
|\Lambda| &= |\Lambda \cap Ann_R(N)| + |\Lambda \cap (R - Ann_R(N))| \\
&= |Ann_R(N)\backslash\{0\}| + \omega\left(G\left(\frac{R}{\sqrt{Ann_R(N)}}\right)\right) - 1 \\
&= |Ann_R(N)| - 1 + \omega\left(G\left(\frac{R}{\sqrt{Ann_R(N)}}\right)\right) - 1 \\
&= |Ann_R(N)| + \omega\left(G\left(\frac{R}{Ann_R(N)}\right)\right) - 2.
\end{aligned}$$

2. Suppose $Ann_R(N)$ is prime. Then $\dfrac{R}{Ann_R(N)}$ is an integral domain and hence $G\left(\frac{R}{Ann_R(N)}\right)$ is a star graph with center zero. Thus $\omega\left(G\left(\frac{R}{Ann_R(N)}\right)\right) = 2$. Therefore, $\omega(G(R; M, N)) = |Ann_R(N)|$, which is the same result obtained in Theorem 7 by noticing that a prime ideal is both semiprime and demiprime. On the other hand, $(Ann_R(N) - 0) \cup \{x\}$ is a maximal clique for every $x \notin Ann_R(N)$ and $R - Ann_R(N)$ is the largest independent set in $G(R; M, N)$. We conclude that $\chi(G(R; M, N)) = |Ann_R(N)|$ and by Lemma 2, $\alpha(G(R; M, N)) = |R - Ann_R(N)| = \alpha(G\left(\frac{R}{Ann_R(N)}\right)) \cdot |Ann_R(N)|$. □

4. Applications and Examples

This section applies the results from Section 3 to the ring $\mathbb{Z}_n$. We derive some concise and practical formulas for the graph invariants and include examples to illustrate these applications.

Application 9. Let $R = \mathbb{Z}_{pq} = \mathbb{Z}_p \oplus \mathbb{Z}_q$ where $q > 2$ and p are distinct prime numbers. Let $M = R$ and $N = \mathbb{Z}_p$. Then $Ann_R(N) = \mathbb{Z}_q$ and $\dfrac{R}{Ann_R(N)} \cong \mathbb{Z}_p$ which is a field. So, $Ann_R(N)$ is a maximal ideal and hence prime. Therefore by Corollary 3, we have $\omega(G(R; M, N)) = |Ann_R(N)| = q$, $\chi(G(R; M, N)) = |Ann_R(N)| = q$, $\alpha(G(R; M, N)) = \alpha(G(\mathbb{Z}_p)) \cdot |Ann_R(N)| = (p - 1) \cdot q$. Moreover, by Theorem 6, $\gamma(G(R; M, N)) = 1$. Furthermore, by Theorem 4, $g(G(R; M, N)) = 3$.

Example 3. The values obtained in Application 9 can be confirmed by observing the graph $G(\mathbb{Z}_6; \mathbb{Z}_6, \mathbb{Z}_2)$, where we agree that $\mathbb{Z}_6 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3$.

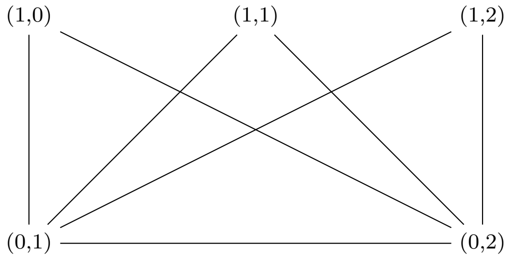

Figure 2. $G(\mathbb{Z}_6; \mathbb{Z}_6, \mathbb{Z}_2)$

We have $g(G(\mathbb{Z}_6; \mathbb{Z}_6, \mathbb{Z}_2)) = \omega(G(\mathbb{Z}_6; \mathbb{Z}_6, \mathbb{Z}_2)) = \alpha(G(\mathbb{Z}_6; \mathbb{Z}_6, \mathbb{Z}_2)) = \chi(G(\mathbb{Z}_6; \mathbb{Z}_6, \mathbb{Z}_2)) = 3$, and $\gamma(G(\mathbb{Z}_6; \mathbb{Z}_6, \mathbb{Z}_2)) = 1$. The graph $G(\mathbb{Z}_6; \mathbb{Z}_6, \mathbb{Z}_2)$ is illustrated in the figure above.

Application 10. Let $R = \mathbb{Z}_{2p} = \mathbb{Z}_2 \oplus \mathbb{Z}_p$ where $p > 2$ is a prime number. Let $M = R$ and $N = \mathbb{Z}_p$. Then $Ann_R(N) = \mathbb{Z}_2$ which is a prime ideal. By Theorem 5, $G(\mathbb{Z}_{2p}; \mathbb{Z}_{2p}, \mathbb{Z}_p)$ is a star graph. Hence, we have $g(G(R; M, N)) = \infty$, $\chi(G(R; M, N)) = \omega(G(R; M, N)) = 2$, $\alpha(G(R; M, N)) = 2p - 2$, $\gamma(G(R; M, N)) = 1$.

Example 4. The values obtained in Application 10 can be confirmed by observing the graph $G(\mathbb{Z}_6; \mathbb{Z}_6, \mathbb{Z}_3)$ shown in the figure below:

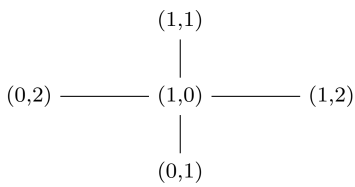

Figure 3. $G(\mathbb{Z}_6; \mathbb{Z}_6, \mathbb{Z}_3)$

where we agree that $\mathbb{Z}_6 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3$. We have $g(G(\mathbb{Z}_6; \mathbb{Z}_6, \mathbb{Z}_3)) = \infty$, $\omega(G(\mathbb{Z}_6; \mathbb{Z}_6, \mathbb{Z}_2)) = \chi(G(\mathbb{Z}_6; \mathbb{Z}_6, \mathbb{Z}_2)) = 2$, $\alpha(G(\mathbb{Z}_6; \mathbb{Z}_6, \mathbb{Z}_2)) = 4$, and $\gamma(G(\mathbb{Z}_6; \mathbb{Z}_6, \mathbb{Z}_2)) = 1$.

Application 11. Let $m = p_1 p_2 \ldots p_n$, where $p_1, \ldots, p_n$ are distinct prime numbers and $n \geq 4$, $R = \mathbb{Z}_m = \mathbb{Z}_{p_1} \oplus \ldots \oplus \mathbb{Z}_{p_n}$, $M = R$, and $N = \mathbb{Z}_{p_1 \ldots p_k}$ an R-submodule where $1 < k < n - 1$. Now, $Ann_R(N) = \mathbb{Z}_{p_{k+1} \ldots p_n} \ncong \mathbb{Z}_2$ and $\dfrac{R}{Ann_R(N)} = \mathbb{Z}_{p_1 \ldots p_k}$ which is not an integral domain. Thus $Ann_R(N)$ is not prime. Since it is semiprime, it is not demiprime. By Theorem 4, $g(G(R; M, N)) = 3$. Also $\gamma(G(R; M, N)) = 1$. On the other hand, by Corollary 3, $\omega(G(R; M, N)) = |Ann_R(N)| + \omega(G(\frac{R}{Ann_R(N)})) - 2 = p_{k+1} \cdot p_{k+2} \cdot \ldots \cdot p_n + k - 1$, where $\omega(G(Z_{p_1 \ldots p_k})) - 1 = \omega(G(Z_{p_1 \ldots p_k}) - 0) = k$ by Theorem 3 of [11]. More efficiently, if we let $q = p_1 \ldots p_k$, then

$$\omega(G(\mathbb{Z}_m; \mathbb{Z}_m, \mathbb{Z}_q)) = \frac{m}{q} - 1 + \text{ (the number of prime factors of } q)$$

.

Example 5. Let $m = 7735 = 5 \times 7 \times 13 \times 17$, $R = \mathbb{Z}_{7735}$, $M = R$, and $N = \mathbb{Z}_{91} = \mathbb{Z}_{7\times 13}$. Then by Application 11, $g(G(\mathbb{Z}_{7735}; \mathbb{Z}_{7735}, \mathbb{Z}_{91})) = 3$, $\gamma(G(\mathbb{Z}_{7735}; \mathbb{Z}_{7735}, \mathbb{Z}_{91})) = 1$, and $\omega((\mathbb{Z}_{7735}; \mathbb{Z}_{7735}, \mathbb{Z}_{91})) = \dfrac{7753}{91} - 1 + 2 = 85 + 1 = 86$.

Example 6. Let $R = \mathbb{Z}_n$ and $M = \mathbb{Z}_n[x]$.

1. If $N = \mathbb{Z}_n x$, then $Ann_R(\mathbb{Z}_n x) = 0$. By Theorem 2, $G(\mathbb{Z}_n; \mathbb{Z}_n[x], \mathbb{Z}_n x) = G(\mathbb{Z}_n) - 0$.

2. If $n = 2p$ where $p > 2$ is a prime number, and $N = \mathbb{Z}_n 2x$, then $Ann_R(N) = \{0, p\} \cong \mathbb{Z}_2$. Moreover, $\dfrac{\mathbb{Z}_n}{Ann_R(N)} \cong \mathbb{Z}_p$ which is a field. So, $Ann_R(N)$ is a maximal ideal and therefore a prime ideal. So, we obtain by Corollary 3 that $\chi(G(\mathbb{Z}_n; \mathbb{Z}_n[x], \mathbb{Z}_n 2x)) = \omega(G(\mathbb{Z}_n; \mathbb{Z}_n[x], \mathbb{Z}_n 2x)) = |Ann_R(N)| = 2$. In addition, $\gamma(G(\mathbb{Z}_n; \mathbb{Z}_n[x], \mathbb{Z}_n 2x)) = 1$ and again by Corollary 3, $\alpha(G(\mathbb{Z}_n; \mathbb{Z}_n[x], \mathbb{Z}_n 2x)) = 2(p-1)$.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

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