Communications in Combinatorics and Optimization
Vol. xx, No. x (xxxx), pp. 1-23
https://doi.org/10.22049/cco.2025.30167.2363

CCO
Commun. Comb. Optim.

Research Article

A neutrosophic approach to solving constrained optimization problems using Karush-Kuhn-Tucker conditions

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Received: 29 November 2024; Accepted: 2 July 2025
Published Online: 17 July 2025

Abstract: In this article, we investigate the solution of constrained optimization problems using the Karush Kuhn Tucker (KKT) condition with single-valued neutrosophic triangular number coefficients. Our approach introduces new neutrosophic arithmetic operations applied to the parametric representations of neutrosophic numbers, along with the neutrosophic ranking of the parametric forms of Triangular Neutrosophic Numbers. The primary objective of this study is to develop a robust framework for solving constrained Single-Valued Neutrosophic Nonlinear Programming Problems using the KKT condition, effectively managing uncertainty and imprecision in optimization. We present and prove an important theorem for the KKT condition under neutrosophic environments, contributing to the theoretical foundation of this method. Furthermore, a detailed numerical example illustrates the practical application of the proposed approach. The results are compared with those of existing methods, demonstrating the effectiveness and advantages of the neutrosophic-based solution.

Keywords: KKT condition, constrained optimization, single-valued neutrosophic number, neutrosophic set, neutrosophic programming, uncertainty management

AMS Subject classification: 03B52, 03E72, 90C70, 90C30

1. Introduction

Ahmad et al. proposed addressing multi-objective nonlinear programming problems within a neutrosophic hesitant fuzzy framework [1]. Al-Naemi introduced a novel

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conjugate gradient formula, β_k^{Gh}, derived from the memoryless self-scaled DFP quasi-Newton (QN) method [2]. Anuradha and Sobana developed a model for an intuitionistic fuzzy multi-objective nonlinear programming problem (IFMONLPP), incorporating intuitionistic fuzzy numbers (IFNs) to represent all coefficients and constraints [3]. Chakraborty et al. extended the concept of neutrosophic numbers, suggesting that nonlinear neutrosophic numbers are more suitable for decision-makers than their linear counterparts [4]. Emam and Youssef proposed a neutrosophic framework combined with a fuzzy-based technique to determine the optimal compromise solution for bi-level multi-objective quadratic programming problems. Their approach incorporates neutrosophic parameters in both the objective functions and constraints [5]. Ghadle and Pawar proposed an alternative approach to Wolfe's modified simplex method. This technique simplifies solving quadratic programming problems (QPP) related to NLPP [6]. Hanachi et al. implemented a new hybrid method in which the parameter β_k is calculated as a convex combination of three parameters $\beta_k^{FR}, \beta_k^{PRP} and \beta_k^{DY}$. The method was shown to ensure sufficient descent and global convergence. Practical evaluations determined this approach is faster and more effective than previously utilized methods [8]. Iqbal et al. scrutinized a multi-objective nonlinear programming problem within a linear Diophantine fuzzy framework, addressing mixed conflicting objectives. Additionally, they highlighted that the linear Diophantine fuzzy set forms the foundation of the interactive approach used to address nonlinear fractional programming problems. In this approach, when the decision-maker (DM) specifies the level set parameter α, the max-min problem is resolved using Zimmermann's min operator technique [9, 10]. Irene and Sudha examined a new method to address Neutrosophic Fuzzy Quadratic Programming Problems (FNQPP) by employing a Taylor Series expansion [11]. Khalifa et al. addressed a challenging nonlinear programming problem in which the coefficients of the objective function are expressed as neutrosophic numbers, and the constraints are governed by fuzzy inequalities [15]. Lachhwani conducted extensive research on nonlinear neutrosophic numbers (NLNNs) and nonlinear neutrosophic linear programming problems (NLN-LPPs) [16]. A comparison of the convergence performance of widely used solvers for both constrained and unconstrained cases was conducted by Lavezzi et al. and provided recommendations for choosing the solvers for nonlinear programming problems [17]. Maissam Jdid illustrated how binary integers can be utilized to transform specific nonlinear models into linear forms, ensuring integer solutions that align with the inherent characteristics of the problems under consideration. Also, their work explored significant methods for solving nonlinear models focusing on the Lagrangian multiplier approach for models with equality constraints, which were subsequently reformulated using neutrosophic science principles [12, 13]. A new fuzzy arithmetic designed for fuzzy calculus and to solve fuzzy linear equations was developed by Ming Ma et al. [18]. Mishra et al. investigated a nonlinear fuzzy fractional signomial programming problem in which all coefficients in the constraints and objective functions are modeled as fuzzy numbers. They proposed two solution approaches: one leveraging nearest interval approximation through parametric interval-valued functions and the other employing a fuzzy α-cut combined with a min-max optimization technique [19]. Othman and Abdulrazzaq determined a control system for a bearingless brushless DC (BBLDC) motor designed for use in an artificial heart pump. Their work included simulations of the system and an analysis of advancements in medical devices, developments in IoT, detector designs, remote monitoring technologies and fuzzy logic-based healthcare solutions [20]. A fuzzy mathematical model incorporating Beale's condition to address such nonlinear programming problems (NLPPs) was discussed by Palanivel and Muralikrishna. This model illustrated how quadratic programming problems could be solved using membership functions (MFs) [14]. Purnima Raj and Ranjana presented a method for solving FNLPPs with inequality constraints often encountered in practical cases and involves transforming the FNLPP into sharp equality constraints by applying the KKT conditions [21]. Raju provided an introduction to optimization methods, covering their historical background, key developments, and foundational concepts [22]. Reig-Mullor and Salas-Molina explored an alternative definition of nonlinear neutrosophic numbers (NLNN) to address certain limitations highlighted in existing research. They also examined the fundamental properties of NLNN, including α, β, γ-cuts, variance, standard deviation and possibilistic mean [23]. A new matrix game where payoffs are expressed as single-valued neutrosophic numbers (SVNNs) was formulated by Seikh et al. To determine the optimal strategies and values for the players, they developed two nonlinear multi-objective programming models [25]. Sharma et al. demonstrated an optimization approach to tackle nonlinear separable programming problems, where the coefficients and variables are modeled as generalized trapezoidal intuitionistic fuzzy numbers and classified as fully intuitionistic fuzzy nonlinear separable programming problems [26]. Sudha and Hepzibah explored the substitution of fuzzy coefficients in the objective functions and constraints with single-valued triangular neutrosophic numbers. They reformulated a quadratic programming problem with triangular neutrosophic number coefficients into a Fuzzy Neutrosophic Linear Complementarity Problem (FNLCP) and developed an algorithm to solve the resulting model effectively [27]. Uma Maheswari and Ganesan devised a fuzzy adaptation of the Kuhn-Tucker conditions to solve fully fuzzy nonlinear programming problems, facilitating the determination of optimal fuzzy solutions. They utilized the Gradient method to transform the original problem into an unconstrained multivariable fuzzy optimization problem for further analysis [28]. Vanaja and Ganesan proposed an innovative interior fuzzy penalty function approach for addressing Fuzzy Nonlinear Programming Problems (FNLPPs). Their methodology incorporated advanced fuzzy arithmetic and ranking techniques based on the parametric representation of triangular fuzzy numbers. Additionally, they developed an exterior penalty method utilizing fuzzy-valued functions, offering an effective framework for solving these FNLPPs [7, 29].

Nomenclature

$\tilde{\chi}$	Variables	$\tilde{f}^{\mathcal{N}}$	Neutrosophic Objective Function
$\tilde{h}_i^{\mathcal{N}}$	Equality Neutrosophic Constraint	$\tilde{g}_i^{\mathcal{N}}$	Inequality Neutrosophic Constraint
$\approx$	Equivalent in fuzzy sense	$\succ$	Greater than in fuzzy sense
$\prec$	Less than in fuzzy sense	$\succeq$	Greater than or equal to in fuzzy sense
$\preceq$	Less than or equal to in fuzzy sense	λ	Fuzzy lagrange multiplier
$\tilde{M}^{\mathcal{N}}, \tilde{N}^{\mathcal{N}}$	Neutrosophic numbers	m_*	Left fuzziness index of $\tilde{M}$
m^*	Right fuzziness index of $\tilde{M}$	m_0	Location index of $\tilde{M}$
$\tilde{M} = (\langle m_{T_0}, m_{T_*}, m_{T^*}\rangle;$ $\langle m_{I_0}, m_{I_*}, m_{I^*}\rangle;$ $\langle m_{F_0}, m_{F_*}, m_{F^*}\rangle)$	Parametric form of $\tilde{M}^{\mathcal{N}}$	β	β- cut of fuzzy number

2. Preliminaries

Definition 1. [24] A *Single-Valued Neutrosophic Set* (SVNS), denoted as $\tilde{M}^{\mathcal{N}}$, is defined over a universe of discourse X and is characterized by three membership functions: a truth-membership function $T_{\tilde{M}^{\mathcal{N}}}$, an indeterminacy-membership function $I_{\tilde{M}^{\mathcal{N}}}$, and a falsity-membership function $F_{\tilde{M}^{\mathcal{N}}}$. Each of these functions maps elements of X to the interval [0,1], i.e., $T_{\tilde{M}^{\mathcal{N}}}; I_{\tilde{M}^{\mathcal{N}}}; F_{\tilde{M}^{\mathcal{N}}} : X \to [0,1]$. The SVNS is expressed as: $\tilde{M}^{\mathcal{N}} = \{<\chi, (T_{\tilde{M}^{\mathcal{N}}}(\chi), I_{\tilde{M}^{\mathcal{N}}}(\chi), F_{\tilde{M}^{\mathcal{N}}}(\chi) > /\chi \in X\}$, where the sum of the truth, indeterminacy, and falsity membership values satisfies the condition: $0 \leq T_{\tilde{M}^{\mathcal{N}}}(\chi) + I_{\tilde{M}^{\mathcal{N}}}(\chi) + F_{\tilde{M}^{\mathcal{N}}}(\chi) \leq 3$ for all $\chi \in X$.

Definition 2. [24] A *Single-Valued Neutrosophic Set (SVNS)*, represented as $\tilde{M}^{\mathcal{N}} = \{<\chi, (T_{\tilde{M}^{\mathcal{N}}}(\chi), I_{\tilde{M}^{\mathcal{N}}}(\chi), F_{\tilde{M}^{\mathcal{N}}}(\chi) > /\chi \in X\}$ is considered neutrosophic normal if there exist at least three distinct points $\chi_0, \chi_1, \chi_2 \in X$ such that the truth-membership function achieves its maximum value of 1 at χ_0, the indeterminacy-membership function reaches 1 at χ_1, and the falsity-membership function equals 1 at χ_2.

Definition 3. [24] A *Single-Valued Neutrosophic Set* (SVNS), expressed as $\tilde{M}^{\mathcal{N}} = \{<\chi, (T_{\tilde{M}^{\mathcal{N}}}(\chi), I_{\tilde{M}^{\mathcal{N}}}(\chi), F_{\tilde{M}^{\mathcal{N}}}(\chi) > /\chi \in X\}$ is described as neutrosophic convex if it satisfies the following conditions for any $\chi_1, \chi_2 \in X$ and $\lambda \in [0,1]$ the following conditions are satisfied.

1. $T_{\tilde{M}^{\mathcal{N}}}(\lambda\chi_1 + (1-\lambda)\chi_2) \geq \min\{T_{\tilde{M}^{\mathcal{N}}}(\chi_1), T_{\tilde{M}^{\mathcal{N}}}(\chi_2)\}$
2. $I_{\tilde{M}^{\mathcal{N}}}(\lambda\chi_1 + (1-\lambda)\chi_2) \leq \max\{I_{\tilde{M}^{\mathcal{N}}}(\chi_1), I_{\tilde{M}^{\mathcal{N}}}(\chi_2)\}$
3. $F_{\tilde{M}^{\mathcal{N}}}(\lambda\chi_1 + (1-\lambda)\chi_2) \leq \max\{F_{\tilde{M}^{\mathcal{N}}}(\chi_1), F_{\tilde{M}^{\mathcal{N}}}(\chi_2)\}$

i.e.,$\tilde{M}^{\mathcal{N}}$ is neutrosophic convex, if its truth membership function is fuzzy convex, indeterminacy membership function and falsity membership function is fuzzy concave.

Definition 4. [24] A *Single-Valued Neutrosophic Number* (SVNN)
$\tilde{M}^{\mathcal{N}} = \{<\chi, (T_{\tilde{M}^{\mathcal{N}}}(\chi), I_{\tilde{M}^{\mathcal{N}}}(\chi), F_{\tilde{M}^{\mathcal{N}}}(\chi)) > |\chi \in X\}$, subset of a real line, is called generalised neutrosophic number if

1. $\tilde{M}^{\mathcal{N}}$ is neutrosophic normal, i.e. there exists atleast three points $\chi_0, \chi_1, \chi_2 \in X$ such that $T_{\tilde{M}^{\mathcal{N}}}(\chi_0) = 1; I_{\tilde{M}^{\mathcal{N}}}(\chi_1) = 1; F_{\tilde{M}^{\mathcal{N}}}(\chi_2) = 1$.

2. $\tilde{M}^{\mathcal{N}}$ is neutrosophic convex.

3. $T_{\tilde{M}^{\mathcal{N}}}(\chi)$ is upper semi-continuous, $I_{\tilde{M}^{\mathcal{N}}}(\chi)$ is lower semi continuous and $F_{\tilde{M}^{\mathcal{N}}}(\chi)$ is lower semi continuous.

4. $\tilde{M}^{\mathcal{N}}$ is support, i.e. $S(\tilde{M}^{\mathcal{N}}) = \chi \in \chi : T_{\tilde{M}^{\mathcal{N}}} > 0, I_{\tilde{M}^{\mathcal{N}}} < 1, F_{\tilde{M}^{\mathcal{N}}} < 1$ is bounded.

Definition 5. [24] A single valued neutrosophic number $\tilde{M}^{\mathcal{N}}$ is Triangular Neutrosophic Number (TNN) and is denoted by
$\tilde{M}^{\mathcal{N}} = (T_{\tilde{M}^{\mathcal{N}}}; I_{\tilde{M}^{\mathcal{N}}}; F_{\tilde{M}^{\mathcal{N}}}) = (\langle m_{T_1}, m_{T_2}, m_{T_3}\rangle; \langle m_{I_1}, m_{I_2}, m_{I_3}\rangle; \langle m_{F_1}, m_{F_2}, m_{F_3}\rangle)$ having the membership function, indeterminacy function and non-membership function as follows.M

$$\mu_T(\chi) = \begin{cases} \dfrac{\chi - m_{T_1}}{m_{T_2} - m_{T_1}}, & m_{T_1} \le \chi \le m_{T_2} \\ \dfrac{m_{T_3} - \chi}{m_{T_3} - m_{T_2}}, & m_{T_2} \le \chi \le m_{T_3} \\ 0, & \text{elsewhere} \end{cases}$$

$$\mu_I(\chi) = \begin{cases} \dfrac{\chi - m_{I_1}}{m_{I_2} - m_{I_1}}, & m_{I_1} \le \chi \le m_{I_2} \\ \dfrac{m_{I_3} - \chi}{m_{I_3} - m_{I_2}}, & m_{I_2} \le \chi \le m_{I_3} \\ 0, & \text{elsewhere} \end{cases}$$

$$\mu_F(\chi) = \begin{cases} \dfrac{\chi - m_{F_1}}{m_{F_2} - m_{F_1}}, & m_{F_1} \le \chi \le m_{F_2} \\ \dfrac{m_{F_3} - \chi}{m_{F_3} - m_{F_2}}, & m_{F_2} \le \chi \le m_{F_3} \\ 0, & \text{elsewhere} \end{cases}$$

We use $F(R)$ to represent the set of all triangular neutrosophic numbers defined on R.

Definition 6. [24] The (α, β, γ)-cut of neutrosophic set is denoted by $F(\alpha, \beta, \gamma)$, where $(\alpha, \beta, \gamma) \in [0, 1]$ and are fixed numbers, such that $\alpha + \beta + \gamma \le 3$ and is defined as $F(\alpha, \beta, \gamma) = (\mu_T(\chi), \mu_I(\chi), \mu_F(\chi))$, where $\chi \in X, \mu_T(\chi) \ge \alpha, \mu_I(\chi) \le \beta, \mu_F(\chi) \le \gamma$.

Definition 7. A triangular neutrosophic number $\tilde{M}^{\mathcal{N}}$ can also be represented as a pair $\tilde{M}_T^{\mathcal{N}} = (\underline{m}_T; \overline{m}_T), \tilde{M}_I^{\mathcal{N}} = (\underline{m}_I; \overline{m}_I), \tilde{M}_F^{\mathcal{N}} = (\underline{m}_F; \overline{m}_F)$ of functions $\underline{m}_T(\beta), \overline{m}_T(\beta), \underline{m}_I(\beta), \overline{m}_I(\beta), \underline{m}_F(\beta), \overline{m}_F(\beta),\ 0 \le \beta \le 1$ which satisfy the following requirements:

- $\underline{m}_T(\beta)$ is lower bound of the membership function, which is a monotonic increasing and left-continuous function .

- $\overline{m}_T(\beta)$ is upper bound of the membership function, which is a monotonic decreasing and left-continuous function.

- $\underline{m}_I(\beta)$ is lower bound of the indeterminacy function, characterized by being a monotonic increasing and left-continuous function.

- $\overline{m}_I(\beta)$ is upper bound of the indeterminacy function, which is monotonic decreasing and left-continuous.
- $\underline{m}_F(\beta)$ is lower bound of the non-membership function, defined as a monotonic increasing and left-continuous function.
- $\overline{m}_F(\beta)$ is upper bound of the non-membership function, characterized by being monotonic decreasing and left-continuous.
- $\underline{m}_T(\beta) \leq \overline{m}_T(\beta), \underline{m}_I(\beta) \leq \overline{m}_I(\beta), \underline{m}_F(\beta) \leq \overline{m}_F(\beta), 0 \leq \beta \leq 1.$

Definition 8. **(Parametric Form)**
Let $\tilde{M}^{\mathcal{N}} = (m_T, m_I, m_F)$ be a triangular neutrosophic number and

$$\overline{m}_T(\beta) = m_{T_3} - (m_{T_3} - m_{T_2})\beta,\ \underline{m}_T(\beta) = m_{T_1} + (m_{T_2} - m_{T_1})\beta$$

$$\overline{m}_I(\beta) = m_{I_3} - (m_{I_3} - m_{I_2})\beta,\ \underline{m}_I(\beta) = m_{I_1} + (m_{I_2} - m_{I_1})\beta$$

$$\overline{m}_F(\beta) = m_{F_3} - (m_{F_3} - m_{F_2})\beta,\ \underline{m}_F(\beta) = m_{F_1} + (m_{F_2} - m_{F_1})\beta,\ \beta \in [0,1]$$

The parametric form of the TNN is defined as

$$\tilde{M}^{\mathcal{N}} = (T_{\tilde{M}^{\mathcal{N}}}; I_{\tilde{M}^{\mathcal{N}}}; F_{\tilde{M}^{\mathcal{N}}}) = (\langle m_{T_0}, m_{T_*}, m_{T^*}\rangle; \langle m_{I_0}, m_{I_*}, m_{I^*}\rangle; \langle m_{F_0}, m_{F_*}, m_{F^*}\rangle)$$

where $m_{T_*} = m_{T_0} - \underline{m}_T$ and $m_{T^*} = \overline{m}_T - m_{T_0}$, $m_{I_*} = m_{I_0} - \underline{m}_I$ and $m_{I^*} = \overline{m}_I - m_{I_0}$, $m_{F_*} = m_{F_0} - \underline{m}_F$ and $m_{F^*} = \overline{m}_F - m_{F_0}$ are the left and right fuzziness index functions respectively.
The number $m_{T_0} = \left(\dfrac{\underline{m}_T(1) + \overline{m}_T(1)}{2}\right), m_{I_0} = \left(\dfrac{\underline{m}_I(1) + \overline{m}_I(1)}{2}\right)$,
$m_{F_0} = \left(\dfrac{\underline{m}_F(1) + \overline{m}_F(1)}{2}\right)$ is called the location index number. When $\beta = 1$, we get $m_{T_0} = m_{T_2}, m_{I_0} = m_{I_2}, m_{F_0} = m_{F_2}$.

2.1. Arithmetic Operations on Neutrosophic Numbers

A novel fuzzy arithmetic framework is introduced based on the parametric representation of triangular neutrosophic numbers. This approach represents these numbers through location index functions and fuzziness index functions associated with membership, indeterminacy, and non-membership degrees. A new neutrosophic arithmetic operation is proposed, where the lattice rule defined by the least upper bound and greatest lower bound in the lattice L governs the fuzziness index functions, while the location index adheres to conventional arithmetic operations.
That is for $m, n \in L$, $m \vee n = \max\{m, n\}$ and $m \wedge n = \min\{m, n\}$. For any two neutrosophic numbers $\tilde{M}^{\mathcal{N}}$ and $\tilde{N}^{\mathcal{N}}$ the arithmetic operations are defined as

$$\begin{aligned}\tilde{M}^{\mathcal{N}} * \tilde{N}^{\mathcal{N}} =& (\langle m_{T_0}, m_{T_*}, m_{T^*}\rangle; \langle m_{I_0}, m_{I_*}, m_{I^*}\rangle; \langle m_{F_0}, m_{F_*}, m_{F^*}\rangle) \\ & * (\langle n_{T_0}, n_{T_*}, n_{T^*}\rangle; \langle n_{I_0}, n_{I_*}, n_{I^*}\rangle; \langle n_{F_0}, n_{F_*}, n_{F^*}\rangle) \\ =& (\langle m_{T_0} * n_{T_0}, m_{T_*} \vee n_{T_*}, m_{T^*} \vee n_{T^*}\rangle, \langle m_{I_0} * n_{I_0}, m_{I_*} \vee n_{I_*}, m_{I^*} \vee n_{I^*}\rangle, \\ & \langle m_{F_0} * n_{F_0}, m_{F_*} \vee n_{F_*}, m_{F^*} \vee n_{F^*}\rangle)\end{aligned}$$

In particular for $\tilde{M}^{\mathscr{N}} = (\langle m_{T_0}, m_{T_*}, m_{T^*}\rangle; \langle m_{I_0}, m_{I_*}, m_{I^*}\rangle; \langle m_{F_0}, m_{F_*}, m_{F^*}\rangle)$,
$\tilde{N}^{\mathscr{N}} = (\langle n_{T_0}, n_{T_*}, n_{T^*}\rangle; \langle n_{I_0}, n_{I_*}, n_{I^*}\rangle; \langle n_{F_0}, n_{F_*}, n_{F^*}\rangle) \in F(R)$, we have
Addition:

$$\begin{aligned}\tilde{M}^{\mathscr{N}} * \tilde{N}^{\mathscr{N}} =& (\langle m_{T_0}, m_{T_*}, m_{T^*}\rangle; \langle m_{I_0}, m_{I_*}, m_{I^*}\rangle; \langle m_{F_0}, m_{F_*}, m_{F^*}\rangle) \\ &+ (\langle n_{T_0}, n_{T_*}, n_{T^*}\rangle; \langle n_{I_0}, n_{I_*}, n_{I^*}\rangle; \langle n_{F_0}, n_{F_*}, n_{F^*}\rangle) \\ =& (\langle m_{T_0} + n_{T_0}, m_{T_*} \vee n_{T_*}, m_{T^*} \vee n_{T^*}\rangle, \langle m_{I_0} + n_{I_0}, m_{I_*} \vee n_{I_*}, m_{I^*} \vee n_{I^*}\rangle, \\ & \langle m_{F_0} + n_{F_0}, m_{F_*} \vee n_{F_*}, m_{F^*} \vee n_{F^*}\rangle)\end{aligned}$$

Subtraction:

$$\begin{aligned}\tilde{M}^{\mathscr{N}} * \tilde{N}^{\mathscr{N}} =& (\langle m_{T_0}, m_{T_*}, m_{T^*}\rangle; \langle m_{I_0}, m_{I_*}, m_{I^*}\rangle; \langle m_{F_0}, m_{F_*}, m_{F^*}\rangle) \\ &- (\langle n_{T_0}, n_{T_*}, n_{T^*}\rangle; \langle n_{I_0}, n_{I_*}, n_{I^*}\rangle; \langle n_{F_0}, n_{F_*}, n_{F^*}\rangle) \\ =& (\langle m_{T_0} - n_{T_0}, m_{T_*} \vee n_{T_*}, m_{T^*} \vee n_{T^*}\rangle, \langle m_{I_0} - n_{I_0}, m_{I_*} \vee n_{I_*}, m_{I^*} \vee n_{I^*}\rangle, \\ & \langle m_{F_0} - n_{F_0}, m_{F_*} \vee n_{F_*}, m_{F^*} \vee n_{F^*}\rangle)\end{aligned}$$

Multiplication:

$$\begin{aligned}\tilde{M}^{\mathscr{N}} * \tilde{N}^{\mathscr{N}} =& (\langle m_{T_0}, m_{T_*}, m_{T^*}\rangle; \langle m_{I_0}, m_{I_*}, m_{I^*}\rangle; \langle m_{F_0}, m_{F_*}, m_{F^*}\rangle) \\ &\times (\langle n_{T_0}, n_{T_*}, n_{T^*}\rangle; \langle n_{I_0}, n_{I_*}, n_{I^*}\rangle; \langle n_{F_0}, n_{F_*}, n_{F^*}\rangle) \\ =& (\langle m_{T_0} \times n_{T_0}, m_{T_*} \vee n_{T_*}, m_{T^*} \vee n_{T^*}\rangle, \langle m_{I_0} \times n_{I_0}, m_{I_*} \vee n_{I_*}, m_{I^*} \vee n_{I^*}\rangle, \\ & \langle m_{F_0} \times n_{F_0}, m_{F_*} \vee n_{F_*}, m_{F^*} \vee n_{F^*}\rangle)\end{aligned}$$

Division:

$$\begin{aligned}\tilde{M}^{\mathscr{N}} * \tilde{N}^{\mathscr{N}} =& (\langle m_{T_0}, m_{T_*}, m_{T^*}\rangle; \langle m_{I_0}, m_{I_*}, m_{I^*}\rangle; \langle m_{F_0}, m_{F_*}, m_{F^*}\rangle) \\ &\div (\langle n_{T_0}, n_{T_*}, n_{T^*}\rangle; \langle n_{I_0}, n_{I_*}, n_{I^*}\rangle; \langle n_{F_0}, n_{F_*}, n_{F^*}\rangle) \\ =& (\langle m_{T_0} \div n_{T_0}, m_{T_*} \vee n_{T_*}, m_{T^*} \vee n_{T^*}\rangle, \langle m_{I_0} \div n_{I_0}, m_{I_*} \vee n_{I_*}, m_{I^*} \vee n_{I^*}\rangle, \\ & \langle m_{F_0} \div n_{F_0}, m_{F_*} \vee n_{F_*}, m_{F^*} \vee n_{F^*}\rangle)\end{aligned}$$

provided $n_{T_0}, n_{I_0}, n_{F_0} \neq 0$.

2.2. Ranking of neutrosophic Numbers

The ranking of neutrosophic numbers plays an essential role in the decision-making process within a fuzzy environment. Various authors in the literature have proposed different types of ranking methods. This article uses a highly effective ranking technique based on the graded mean.
For $\tilde{M}^{\mathscr{N}} = (\langle m_{T_0}, m_{T_*}, m_{T^*}\rangle; \langle m_{I_0}, m_{I_*}, m_{I^*}\rangle; \langle m_{F_0}, m_{F_*}, m_{F^*}\rangle) \in F(R)$, define $R : F(R) \to R$ by

$$R(T_{\tilde{M}^{\mathscr{N}}}) = \left(\frac{m_{T_*} + 4m_{T_0} + m_{T^*}}{6}\right).$$

$$R(I_{\tilde{M}^{\mathcal{N}}}) = \left(\frac{m_{I_*} + 4m_{I_0} + m_{I^*}}{6}\right).$$

$$R(F_{\tilde{M}^{\mathcal{N}}}) = \left(\frac{m_{F_*} + 4m_{F_0} + m_{F^*}}{6}\right).$$

For any two triangular neutrosophic numbers

$$\tilde{M}^{\mathcal{N}} = (T_{\tilde{M}^{\mathcal{N}}}; I_{\tilde{M}^{\mathcal{N}}}; F_{\tilde{M}^{\mathcal{N}}}) = (\langle m_{T_0}, m_{T_*}, m_{T^*}\rangle; \langle m_{I_0}, m_{I_*}, m_{I^*}\rangle; \langle m_{F_0}, m_{F_*}, m_{F^*}\rangle) and$$
$$\tilde{N}^{\mathcal{N}} = (T_{\tilde{N}^{\mathcal{N}}}; I_{\tilde{N}^{\mathcal{N}}}; F_{\tilde{N}^{\mathcal{N}}}) = (\langle n_{T_0}, n_{T_*}, n_{T^*}\rangle; \langle n_{I_0}, n_{I_*}, n_{I^*}\rangle; \langle n_{F_0}, n_{F_*}, n_{F^*}\rangle) \in F(R)$$

we have the following comparison:

- If $R(T_{\tilde{M}^{\mathcal{N}}}) < R(T_{\tilde{N}^{\mathcal{N}}}); R(I_{\tilde{M}^{\mathcal{N}}}) < R(I_{\tilde{N}^{\mathcal{N}}}), R(F_{\tilde{M}^{\mathcal{N}}}) < R(F_{\tilde{N}^{\mathcal{N}}})$, then $\tilde{M}^{\mathcal{N}} \prec \tilde{N}^{\mathcal{N}}$
- If $R(T_{\tilde{M}^{\mathcal{N}}}) > R(T_{\tilde{N}^{\mathcal{N}}}); R(I_{\tilde{M}^{\mathcal{N}}}) > R(I_{\tilde{N}^{\mathcal{N}}}), R(F_{\tilde{M}^{\mathcal{N}}}) > R(F_{\tilde{N}^{\mathcal{N}}})$, then $\tilde{M}^{\mathcal{N}} \succ \tilde{N}^{\mathcal{N}}$
- If $R(T_{\tilde{M}^{\mathcal{N}}}) = R(T_{\tilde{N}^{\mathcal{N}}}); R(I_{\tilde{M}^{\mathcal{N}}}) = R(I_{\tilde{N}^{\mathcal{N}}}), R(F_{\tilde{M}^{\mathcal{N}}}) = R(F_{\tilde{N}^{\mathcal{N}}})$, then $\tilde{M}^{\mathcal{N}} \approx \tilde{N}^{\mathcal{N}}$.

3. Neutrosophic Non Linear Programming Problems (NNLPP)

Let general NNLPP

$$\begin{aligned} &\min \tilde{f}^{\mathcal{N}}(\tilde{\chi}) \\ &\text{subject to } \tilde{h}_i^{\mathcal{N}}(\tilde{\chi}) \approx \tilde{0} \text{ for } i = 1, 2, \cdots, l \\ &\qquad \tilde{g}_j^{\mathcal{N}}(\tilde{\chi}) \preceq \tilde{0} \text{ for } j = 1, 2, \cdots, m, \\ &\qquad \tilde{\chi} \succeq \tilde{0} \end{aligned} \tag{3.1}$$

where $\tilde{f}^{\mathcal{N}}, \tilde{h}_1^{\mathcal{N}}, \cdots, \tilde{h}_l^{\mathcal{N}}, \tilde{g}_1^{\mathcal{N}}, \cdots, \tilde{g}_m^{\mathcal{N}}$ are continuous neutrosophic valued functions defined on R^n.

Definition 9. A vector $\tilde{\chi} = (\tilde{\chi}_1, \tilde{\chi}_2, \tilde{\chi}_3, \cdots, \tilde{\chi}_n)$ is said to be a feasible solution to the NNLPP (3.1) if it meets all the constraints and adheres to the non-negativity condition of the NNLPP (3.1).

The collection of all such feasible solutions is known as the feasible region, which is defined by these criteria.

$$F = \{\tilde{\chi} \in F(R)^n / \tilde{h}_i^{\mathcal{N}}(\tilde{\chi}) \approx \tilde{0}, \text{ for } i = 1, 2, 3, \cdots, l,\ \tilde{g}_j^{\mathcal{N}}(\tilde{\chi}) \preceq \tilde{0}, \text{ for } j = 1, 2, 3, \cdots, m\}.$$

Definition 10. A feasible solution $\tilde{\chi}^0$ to the NNLPP (3.1) is said to be an optimal solution to the NNLPP (3.1) if $\tilde{f}^{\mathcal{N}}(\tilde{\chi}^0) \preceq \tilde{f}^{\mathcal{N}}(\tilde{\chi})$, for all $\tilde{\chi} \in F$.

3.1. Karush Kuhn Tucker optimality conditions

The conditions derived by mathematicians Kuhn and Tucker are necessary for $\tilde{f}^{\mathcal{N}}(\tilde{\chi})$ to be extreme in the case of multivariable problems with inequality constraints.
The Khun-Tucker conditions are the necessary conditions to be satisfied at relative maximum of $\tilde{f}^{\mathcal{N}}(\tilde{\chi})$ with inequality constraints $\tilde{g}_j^{\mathcal{N}}(\tilde{\chi}) \preceq \tilde{0}$, expressed as

1. **Stationarity:** $\frac{\partial \tilde{f}^{\mathcal{N}}}{\partial \tilde{\chi}_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial \tilde{g}_j^{\mathcal{N}}}{\partial \tilde{\chi}_i} \approx \tilde{0} \qquad i = 1, 2, \cdots, n, \quad \lambda_j \preceq \tilde{0}$

   where λ_j are the Lagrange multipliers.
   If the set of active constraints is not known, then the Khun-Tucker conditions can be stated as follows for the case of maximize $\tilde{f}^{\mathcal{N}}(\tilde{\chi})$, subject to, $\tilde{g}_j^{\mathcal{N}}(\tilde{\chi}) \preceq 0$
2. **Primal feasibility:** $\tilde{g}_j^{\mathcal{N}} \preceq \tilde{0}, \quad j = 1, 2, \cdots, m$
3. **Complementary slackness:** $\lambda_j \tilde{g}_j^{\mathcal{N}} \approx \tilde{0}, \quad j = 1, 2, \cdots, m$
4. **Dual feasibility:** $\lambda_j \geq 0, \quad j = 1, 2, \cdots, m$

Table 1. **Summary of values of Lagrange multipliers**

	$\min \tilde{f}^{\mathcal{N}}(\tilde{\chi})$	$\max \tilde{f}^{\mathcal{N}}(\tilde{\chi})$
$\tilde{g}_j^{\mathcal{N}}(\tilde{\chi}) \preceq 0$	$\lambda_j \geq 0$	$\lambda_j \leq 0$
$\tilde{g}_j^{\mathcal{N}}(\tilde{\chi}) \succeq 0$	$\lambda_j \leq 0$	$\lambda_j \geq 0$

Theorem 1. *Let $\tilde{f}^{\mathcal{N}} : X \to F(R)$ be a neutrosophic-valued function that is convex and continuously differentiable. Here, $\tilde{f}^{\mathcal{N}}$ represents the neutrosophic objective function, and $\tilde{g}_j^{\mathcal{N}}$ denotes the set of all associated neutrosophic constraints. Assume there exists a point $\tilde{\chi}^0$ such that $\tilde{g}_j^{\mathcal{N}}(\tilde{\chi}^0) - \tilde{b}_i \preceq \tilde{0}$, for all $i = 1, 2 \cdots, n$, then $\tilde{\chi}^0$ is considered an optimal solution for the given Neutrosophic Nonlinear Programming Problem (NNLPP) as defined in (3.1) over the feasible region, if and only if there exist multipliers $\lambda_i \geq 0$ satisfying the Karush Kuhn Tucker (KKT) first-order conditions:*
(i). $\frac{\partial \tilde{f}^{\mathcal{N}}}{\partial \tilde{\chi}_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial \tilde{g}_j^{\mathcal{N}}}{\partial \tilde{\chi}_i} \approx \tilde{0}$ *and*
(ii). $\lambda_i(\tilde{g}_j^{\mathcal{N}}(\tilde{\chi}^0) - \tilde{b}_i) \approx \tilde{0}$*, for all* $j = 1, 2 \cdots, m$.

Proof. Assume that $\tilde{f}^{\mathcal{N}}$ is a convex, continuously differentiable function, and let $\tilde{\chi}^0$ represent the optimal solution for the NNLPP defined in (3.1). This implies that there exists some ($\tilde{\chi} \not\approx \tilde{\chi}^0$) such that $\tilde{f}^{\mathcal{N}}(\tilde{\chi}^0) \preceq \tilde{f}^{\mathcal{N}}(\tilde{\chi})$. Given that $\tilde{g}_j^{\mathcal{N}}$ is also a convex and continuously differentiable function, the feasible set F is defined as: $F = \{\tilde{\chi} \in F(R)^n : \tilde{g}_j^{\mathcal{N}}(\tilde{\chi}) - \tilde{b}_i \preceq \tilde{0}, i = 1, 2, \cdots, n\}$. Therefore $F = \{\tilde{\chi}^0 \in F(R)^n : \tilde{g}_j^{\mathcal{N}}(\tilde{\chi}^0) - \tilde{b}_i \preceq \tilde{0}, i = 1, 2, \cdots, n\}$. As a result, for $\tilde{\chi}^0 \in F$, we have $\tilde{g}_j^{\mathcal{N}}(\tilde{\chi}^0) - \tilde{b}_i \preceq \tilde{0}$, satisfying all constraints. Consequently, the problem becomes an optimization problem with a neutrosophic objective function $\tilde{f}^{\mathcal{N}}(\tilde{\chi})$ subject to the neutrosophic constraints. By the Karush Kuhn Tucker (KKT) theorem, there exist non-negative multipliers $\lambda_i \geq 0$,

such that the following conditions hold:
(i). $\dfrac{\partial \tilde{f}^{\mathcal{N}}}{\partial \tilde{\chi}_i} + \sum_{j=1}^{m} \lambda_j \dfrac{\partial \tilde{g}_j^{\mathcal{N}}}{\partial \tilde{\chi}_i} \approx \tilde{0}$ and
(ii). $\lambda_i(\tilde{g}_j^{\mathcal{N}}(\tilde{\chi}^0) - \tilde{b}_i) \approx \tilde{0}$, for all $i = 1, 2 \cdots, n$ hold.
Conversely: Let the KKT first-order conditions hold for the NNLPP defined in (3.1). We now need to show that $\tilde{\chi}^0$ is indeed an optimal solution for this problem. Assume, for contradiction, that $\tilde{\chi}^0$ is not an optimal solution. Then, there must exist some $\tilde{\chi} \not\approx \tilde{\chi}^0$ such that $\tilde{f}^{\mathcal{N}}(\chi) \preceq \tilde{f}^{\mathcal{N}}(\tilde{\chi}^0)$, since $\tilde{f}^{\mathcal{N}}$ is a convex and continuously differentiable function, this contradicts the KKT conditions, which state that $\tilde{g}_j^{\mathcal{N}}(\tilde{\chi}^0 - \tilde{b}_i) \approx \tilde{0}$. Thus, $\tilde{\chi}^0$ must indeed be the optimal solution for the NNLPP defined in (3.1). □

4. Algorithm

This section provides the step-by-step procedure for the Neutrosophic Khun-Tucker conditions. This algorithm is developed to iteratively improve the solution by considering uncertainties and inconsistencies during the process. They represent the principles of Neutrosophy and provide valuable insights into managing uncertainty during optimization.

Algorithm 1 KKT-Based Solution for SVNNLPP

Input: SVNNLPP: objective function $\tilde{f}^{\mathcal{N}}(\tilde{\chi})$, constraints $\tilde{g}_i^{\mathcal{N}}(\tilde{\chi}) \preceq \tilde{0}$
    Neutrosophic parameters (truth, indeterminacy, falsity)
Begin
Convert SVNNLPP into a crisp optimization problem
    Apply score or accuracy function to neutrosophic parameters
Form the Lagrangian function
    $L(\tilde{\chi}, \lambda) = \tilde{f}^{\mathcal{N}}(\tilde{\chi}) + \sum \lambda_i \tilde{g}_i^{\mathcal{N}}(\tilde{\chi})$
Derive the KKT conditions
    Stationarity: $\nabla \tilde{f}^{\mathcal{N}}(\tilde{\chi}) + \sum \lambda_i \nabla \tilde{g}_i^{\mathcal{N}}(\tilde{\chi}) \approx \tilde{0}$
    Primal feasibility: $\tilde{g}_i^{\mathcal{N}}(\tilde{\chi}) \preceq \tilde{0}$
    Dual feasibility: $\lambda_i \succeq \tilde{0}$
    Complementary slackness: $\lambda_i \tilde{g}_i^{\mathcal{N}}(\tilde{\chi}) \approx \tilde{0}$
Identify all possible combinations of active/inactive constraints
For each combination:
    Solve the system of KKT equations
    Check if the solution is feasible
    Verify complementary slackness
From all feasible solutions, choose the one with the best objective value
Output: The optimal solution and corresponding values
End

Note:
During the iterative process, the feasibility of neutrosophic constraints is checked using neutrosophic arithmetic, the graded mean ranking method and the definition of feasibility for neutrosophic numbers. These tools ensure that the constraints are properly evaluated at each step of the algorithm.

5. Numerical Example

Example 1. Consider a NLPP discussed by Ghadle and Pawar [6]

$$\begin{aligned} \max f(\chi_1,\chi_2) =& 2\chi_1 + 3\chi_2 - 2\chi_1^2 \\ \text{sub. to } g_1(\chi) =& \chi_1 + 4\chi_2 \leq 4 \\ g_2(\chi) =& \chi_1 + \chi_2 \leq 2 \\ & \chi_1, \chi_2 \geq 0. \end{aligned} \tag{5.1}$$

Solution: Suppose that the coefficients in the objective function and in the constraints are uncertain and are modeled as neutrosophic triangular numbers, then the NLPP (5.1) becomes a SVNNLPP as

$$\begin{aligned} \max \tilde{f}^{\mathcal{N}}(\tilde{\chi}_1,\tilde{\chi}_2) =& \tilde{2}\tilde{\chi}_1 + \tilde{3}\tilde{\chi}_2 - \tilde{2}\tilde{\chi}_1^2 \\ \text{sub. to } \tilde{g}_1^{\mathcal{N}}(\tilde{\chi}) =& \tilde{1}\tilde{\chi}_1 + \tilde{4}\tilde{\chi}_2 \preceq \tilde{4} \\ \tilde{g}_2^{\mathcal{N}}(\tilde{\chi}) =& \tilde{1}\tilde{\chi}_1 + \tilde{1}\tilde{\chi}_2 \preceq \tilde{2} \\ & \tilde{\chi}_1, \tilde{\chi}_2 \succeq \tilde{0}. \end{aligned} \tag{5.2}$$

$$\begin{aligned} \text{Assume that } \tilde{2} = \langle(1,2,3);(0.6,0.4,0.1)\rangle, \tilde{3} = \langle(2,3,4);(0.2,0.3,0.5)\rangle, \\ \tilde{1} = \langle(0,1,2);(0.2,0.3,0.5)\rangle, \tilde{4} = \langle(3,4,5);(0.4,0.3,0.2)\rangle \end{aligned}$$

Then the SVNNLPP (5.2) becomes

$$\begin{aligned} \max \tilde{f}^{\mathcal{N}}(\tilde{\chi}_1,\tilde{\chi}_2) =& \langle(1,2,3);(0.6,0.4,0.1)\rangle\tilde{\chi}_1 + \langle(2,3,4);(0.2,0.3,0.5)\rangle\tilde{\chi}_2 \\ & - \langle(1,2,3);(0.6,0.4,0.1)\rangle\tilde{\chi}_1^2 \\ \text{sub. to } \tilde{g}_1^{\mathcal{N}}(\tilde{\chi}) =& \langle(0,1,2);(0.2,0.3,0.5)\rangle\tilde{\chi}_1 + \langle(3,4,5);(0.4,0.3,0.2)\rangle\tilde{\chi}_2 \\ & \preceq \langle(3,4,5);(0.4,0.3,0.2)\rangle \\ \tilde{g}_2^{\mathcal{N}}(\tilde{\chi}) =& \langle(0,1,2);(0.2,0.3,0.5)\rangle\tilde{\chi}_1 + \langle(0,1,2);(0.2,0.3,0.5)\rangle\tilde{\chi}_2 \\ & \preceq \langle(1,2,3);(0.6,0.4,0.1)\rangle \\ & \tilde{\chi}_1, \tilde{\chi}_2 \succeq \tilde{0}. \end{aligned} \tag{5.3}$$

Express the SVNNLPP (5.3) in its parametric form as

$$\begin{aligned} \max \tilde{f}^{\mathcal{N}}(\tilde{\chi}_1,\tilde{\chi}_2) =& \langle(2,1-\beta,1-\beta);(0.6,0.4,0.1)\rangle\tilde{\chi}_1 \\ & + \langle(3,1-\beta,1-\beta);(0.2,0.3,0.5)\rangle\tilde{\chi}_2 \\ & - \langle(2,1-\beta,1-\beta);(0.6,0.4,0.1)\rangle\tilde{\chi}_1^2 \\ \text{subject to } \tilde{g}_1^{\mathcal{N}}(\tilde{\chi}) =& \langle(1,1-\beta,1-\beta);(0.2,0.3,0.5)\rangle\tilde{\chi}_1 \\ & + \langle(4,1-\beta,1-\beta);(0.4,0.3,0.2)\rangle\tilde{\chi}_2 \\ & \preceq \langle(4,1-\beta,1-\beta);(0.4,0.3,0.2)\rangle \\ \tilde{g}_2^{\mathcal{N}}(\tilde{\chi}) =& \langle(1,1-\beta,1-\beta);(0.2,0.3,0.5)\rangle\tilde{\chi}_1 \\ & + \langle(1,1-\beta,1-\beta);(0.2,0.3,0.5)\rangle\tilde{\chi}_2 \\ & \preceq (2,1-\beta,1-\beta);(0.6,0.4,0.1) \\ & \text{and } \tilde{\chi}_1, \tilde{\chi}_2 \succeq \tilde{0}. \end{aligned} \tag{5.4}$$

Stationarity: $\dfrac{\partial \tilde{f}_i^{\mathcal{N}}}{\partial \tilde{\chi}_i} + \sum_{i=1}^{m} \lambda_i \dfrac{\partial \tilde{g}_i^{\mathcal{N}}}{\partial \tilde{\chi}_i} \approx \tilde{0}$

$$\begin{aligned}
&\langle(2, 1-\beta, 1-\beta); (0.6, 0.4, 0.1)\rangle - \langle(4, 1-\beta, 1-\beta); (0.6, 0.4, 0.1)\rangle\tilde{\chi}_1\\
&+ \lambda_1[\langle(1, 1-\beta, 1-\beta); (0.2, 0.3, 0.5)\rangle]\\
&+ \lambda_2[\langle(1, 1-\beta, 1-\beta); (0.2, 0.3, 0.5)\rangle] \approx \tilde{0}\\
&\langle(3, 1-\beta, 1-\beta); (0.2, 0.3, 0.5)\rangle + \lambda_1[\langle(4, 1-\beta, 1-\beta); (0.4, 0.3, 0.2)\rangle]\\
&+ \lambda_2[\langle(1, 1-\beta, 1-\beta); (0.2, 0.3, 0.5)\rangle] \approx \tilde{0}
\end{aligned} \tag{5.5}$$

Primal feasibility: $\tilde{g}_i^{\mathcal{N}}(\tilde{\chi}_i) \preceq \tilde{0}$

$$\begin{aligned}
&\langle(1, 1-\beta, 1-\beta); (0.2, 0.3, 0.5)\rangle\tilde{\chi}_1 + \langle(4, 1-\beta, 1-\beta); (0.4, 0.3, 0.2)\rangle\tilde{\chi}_2\\
&- \langle(4, 1-\beta, 1-\beta); (0.4, 0.3, 0.2)\rangle \preceq \tilde{0}\\
&\langle(1, 1-\beta, 1-\beta); (0.2, 0.3, 0.5)\rangle\tilde{\chi}_1 + \langle(1, 1-\beta, 1-\beta); (0.2, 0.3, 0.5)\rangle\tilde{\chi}_2\\
&- \langle(2, 1-\beta, 1-\beta); (0.6, 0.4, 0.1)\rangle \preceq \tilde{0}
\end{aligned} \tag{5.6}$$

Complementary slackness: $\lambda_i[\tilde{g}_i^{\mathcal{N}}(\tilde{\chi}_i)] \approx \tilde{0}$

$$\begin{aligned}
&\lambda_1[\langle(1, 1-\beta, 1-\beta); (0.2, 0.3, 0.5)\rangle\tilde{\chi}_1\\
&+ \langle(4, 1-\beta, 1-\beta); (0.4, 0.3, 0.2)\rangle\tilde{\chi}_2 - \langle(4, 1-\beta, 1-\beta); (0.4, 0.3, 0.2)\rangle] \approx \tilde{0}\\
&\lambda_2[\langle(1, 1-\beta, 1-\beta); (0.2, 0.3, 0.5)\rangle\tilde{\chi}_1 + \langle(1, 1-\beta, 1-\beta); (0.2, 0.3, 0.5)\rangle\tilde{\chi}_2\\
&- \langle(2, 1-\beta, 1-\beta); (0.6, 0.4, 0.1)\rangle] \approx \tilde{0}
\end{aligned} \tag{5.7}$$

Dual feasibility: $\lambda_1, \lambda_2 \preceq \tilde{0}$
Here we have two Langrange's multiplier λ_1, λ_2 which can take zero or non zero positive values. Thus four solutions corresponding to the following four combinations of $\lambda_i (i = 1, 2)$ values can be obtained.
Case(i): When $\lambda_1 = 0, \lambda_2 = 0$, from equations (5.5), (5.6) and (5.7), we have

$$\tilde{\chi}_1 = \frac{\langle(2, 1-\beta, 1-\beta); (0.6, 0.4, 0.1)\rangle}{\langle(4, 1-\beta, 1-\beta); (0.6, 0.4, 0.1)\rangle}, \quad \tilde{\chi}_2 = 0$$

$\therefore$ The solution is feasible. This solution also satisfy all the conditions.

Case(ii): When $\lambda_1 \neq 0, \lambda_2 = 0$, from equations (5.5), (5.6) and (5.7), we have

$$\begin{aligned}
&\tilde{\chi}_1 = \frac{\langle(5, 1-\beta, 1-\beta); (0.6, 0.4, 0.5)\rangle}{\langle(16, 1-\beta, 1-\beta); (0.6, 0.4, 0.2)\rangle}, \tilde{\chi}_2 = \frac{\langle(59, 1-\beta, 1-\beta); (0.6, 0.4, 0.5)\rangle}{\langle(64, 1-\beta, 1-\beta); (0.6, 0.4, 0.2)\rangle}\\
&\lambda_1 = \frac{-\langle(3, 1-\beta, 1-\beta); (0.2, 0.3, 0.5)\rangle}{\langle(4, 1-\beta, 1-\beta); (0.4, 0.3, 0.2)\rangle}
\end{aligned}$$

$\therefore$ This solution is also feasible and satisfy all the conditions.
$\max \tilde{f}^{\mathcal{N}}(\tilde{\chi}_1, \tilde{\chi}_2) = \dfrac{\langle(459, 1-\beta, 1-\beta); (0.6, 0.4, 0.2)\rangle}{\langle(128, 1-\beta, 1-\beta); (0.6, 0.4, 0.2)\rangle}$
Case(iii): When $\lambda_1 = 0, \lambda_2 \neq 0$, from equations (5.5), (5.6) and (5.7), we have

$$\tilde{\chi}_1 = \frac{-\langle(1, 1-\beta, 1-\beta); (0.6, 0.4, 0.1)\rangle}{\langle(4, 1-\beta, 1-\beta); (0.6, 0.4, 0.1)\rangle}, \quad \tilde{\chi}_2 = 0$$
$$\lambda_2 = -\langle(3, 1-\beta, 1-\beta); (0.2, 0.3, 0.5)\rangle$$

$\therefore$ This solution is infeasible.
Case(iv): When $\lambda_1 \neq 0, \lambda_2 \neq 0$, from equations (5.5), (5.6) and (5.7), we have

$$\tilde{\chi}_1 = \frac{\langle(4, 1-\beta, 1-\beta); (0.6, 0.4, 0.5)\rangle}{\langle(3, 1-\beta, 1-\beta); (0.4, 0.3, 0.5)\rangle}, \tilde{\chi}_2 = \frac{\langle(2, 1-\beta, 1-\beta); (0.6, 0.4, 0.2)\rangle}{\langle(3, 1-\beta, 1-\beta); (0.4, 0.3, 0.5)\rangle}$$
$$\lambda_1 = \frac{-\langle(19, 1-\beta, 1-\beta); (0.6, 0.4, 0.5)\rangle}{\langle(9, 1-\beta, 1-\beta); (0.4, 0.3, 0.5)\rangle}, \lambda_2 = \frac{\langle(103, 1-\beta, 1-\beta); (0.6, 0.4, 0.5)\rangle}{\langle(9, 1-\beta, 1-\beta); (0.4, 0.3, 0.5)\rangle}$$

$\therefore$ This solution is also feasible but not satisfy the dual feasibility condition.
Since the maximum value of $\tilde{f}^{\mathcal{N}}(\tilde{\chi}_1, \tilde{\chi}_2)$ is obtained for Case(ii), where $\lambda_1 \neq 0, \lambda_2 = 0$, The optimal solution is

$$\tilde{\chi}_1 = \frac{\langle(5, 1-\beta, 1-\beta); (0.6, 0.4, 0.5)\rangle}{\langle(16, 1-\beta, 1-\beta); (0.6, 0.4, 0.2)\rangle}, \tilde{\chi}_2 = \frac{\langle(59, 1-\beta, 1-\beta); (0.6, 0.4, 0.5)\rangle}{\langle(64, 1-\beta, 1-\beta); (0.6, 0.4, 0.2)\rangle}$$
$$\lambda_1 = \frac{-\langle(3, 1-\beta, 1-\beta); (0.2, 0.3, 0.5)\rangle}{\langle(4, 1-\beta, 1-\beta); (0.4, 0.3, 0.2)\rangle}$$

That is the optimal solution of the SVNNLPP (5.1) is

$$\tilde{\chi}_1 = \left\langle \left(\frac{-11000000000000}{16} + \beta, \frac{5}{16}, \frac{21}{16} - \beta \right); (0.6, 0.4, 0.5) \right\rangle,$$
$$\tilde{\chi}_2 = \left\langle \left(\frac{-5}{64} + \beta, \frac{59}{64}, \frac{123}{64} - \beta \right); (0.6, 0.4, 0.5) \right\rangle$$
$$\text{with } \max \tilde{f}^{\mathcal{N}}(\tilde{\chi}_1, \tilde{\chi}_2) = \left\langle \left(\frac{562}{256} + \beta, \frac{818}{256}, \frac{1074}{256} - \beta \right); (0.6, 0.4, 0.5) \right\rangle.$$

6. An application in Home Appliances

[22] Himaja Home Appliances agreed to supply Srija and Company around 50 fans at the end of the first month and arond 50 each at the end of the second and third months. The cost of manufacturing fans in any month is given by χ^2 rupees, where χ is the number of fans manufactured in that month. The company can manufacture

more number of fans in a month than 50 and carry forward the surplus to the next month. However an inventory carrying charge around rupees 20 per fan is to maintain the total cost to the minimum. (Assume no initial inventory and no surplus stock at the end of the month).

Solution: As the number of fans to be manufactured in every month is uncertain and the carrying charge is also uncertain, we model these uncertain parameters as single valued neutrosophic triangular numbers. Then the mathematical formulation of the given SVNNLPP becomes

Given that the production cost of $\tilde{\chi}$ units manufactured in any month $= \tilde{\chi}^2$. Since $\tilde{\chi}_1, \tilde{\chi}_2$ and $\tilde{\chi}_3$ are the number of fans to be manufactured in the first, second and third month respectively, we have $\tilde{\chi}_1^2, \tilde{\chi}_2^2$ and $\tilde{\chi}_3^2$. And there is no initial inventory, hence no holding charge in the first month. But, if $\tilde{\chi}_1$ exceeds 50, the holding cost in the second month $= 20(\tilde{\chi}_1 - 50)$. Also, if $\tilde{\chi}_1 + \tilde{\chi}_2$ exceeds 100 the holding cost in the third month $= 20(\tilde{\chi}_1 + \tilde{\chi}_2 - 100)$,. Hence, the total cost(Objective function is to minimize Z)$= \tilde{\chi}_1^2 + \tilde{\chi}_2^2 + \tilde{\chi}_3^2 + 20(\tilde{\chi}_1 - 50) + 20(\tilde{\chi}_1 + \tilde{\chi}_2 - 100)$.

$$\begin{aligned} \min \tilde{f}^{\mathcal{N}}(\tilde{\chi}) =& \tilde{\chi}_1^2 + \tilde{\chi}_2^2 + \tilde{\chi}_3^2 + \tilde{40}\tilde{\chi}_1 + \tilde{20}\tilde{\chi}_2 - 3\tilde{000} \\ \text{sub. to } \tilde{g}_1^{\mathcal{N}}(\tilde{\chi}) =& \tilde{\chi}_1 \succeq \tilde{50} \\ \tilde{g}_2^{\mathcal{N}}(\tilde{\chi}) =& \tilde{\chi}_1 + \tilde{\chi}_2 \succeq 1\tilde{00} \\ \tilde{g}_3^{\mathcal{N}}(\tilde{\chi}) =& \tilde{\chi}_1 + \tilde{\chi}_2 + \tilde{\chi}_3 \approx 1\tilde{50} \\ & \tilde{\chi}_1, \tilde{\chi}_2, \tilde{\chi}_3 \succeq \tilde{0}. \end{aligned} \tag{6.1}$$

We can convert this problem into a two variable problem by using the direct substitution method. Since we have one exact constraint.

$$\tilde{g}_3^{\mathcal{N}}(\tilde{\chi}) \Rightarrow \tilde{\chi}_1 + \tilde{\chi}_2 + \tilde{\chi}_3 = 1\tilde{50} \Rightarrow \tilde{\chi}_3 = 1\tilde{50} - \tilde{\chi}_1 - \tilde{\chi}_2$$

On substituting this $\tilde{\chi}_3$, the above problem becomes

$$\begin{aligned} \min \tilde{f}^{\mathcal{N}}(\tilde{\chi}) =& 2\tilde{\chi}_1^2 + 2\tilde{\chi}_2^2 + \tilde{40}\tilde{\chi}_1 + \tilde{20}\tilde{\chi}_2 + 19\tilde{500} \\ \text{sub. to } \tilde{g}_1^{\mathcal{N}}(\tilde{\chi}) =& \tilde{\chi}_1 \succeq \tilde{50} \\ \tilde{g}_2^{\mathcal{N}}(\tilde{\chi}) =& \tilde{\chi}_1 + \tilde{\chi}_2 \succeq 1\tilde{00} \\ & \tilde{\chi}_1, \tilde{\chi}_2, \tilde{\chi}_3 \succeq \tilde{0}. \end{aligned} \tag{6.2}$$

$$\begin{aligned} \text{Assume that } & \tilde{2} = \langle(1,2,3);(0.8,0.6,0.4)\rangle, \tilde{1} = \langle(0,1,2);(0.9,0.7,0.5)\rangle, \\ & \tilde{40} = \langle(39,40,41);(0.6,0.3,0.2)\rangle, \tilde{20} = \langle(19,20,21);(0.8,0.2,0.6)\rangle, \\ & 19\tilde{500} = \langle(19499,19500,19501);(0.6,0.3,0.6)\rangle, \\ & \tilde{50} = \langle(49,50,51);(0.7,0.7,0.5)\rangle, 1\tilde{00} = \langle(99,100,101);(0.4,0.2,0.1)\rangle \end{aligned}$$

Now the problem (6.2) becomes

$$\begin{aligned}\min \tilde{f}^{\mathscr{N}}(\tilde{\chi}) &= \langle(1,2,3);(0.8,0.6,0.4)\rangle\tilde{\chi}_1^2 + \langle(1,2,3);\\ &\quad(0.8,0.6,0.4)\rangle\tilde{\chi}_2^2 + \langle(39,40,41);(0.6,0.3,0.2)\rangle\tilde{\chi}_1\\ &\quad+\langle(19,20,21);(0.8,0.2,0.6)\rangle\tilde{\chi}_2\\ &\quad+\langle(19499,19500,19501);(0.6,0.3,0.6)\rangle\\ \text{sub. to } \tilde{g}_1^{\mathscr{N}}(\tilde{\chi}) &= \langle(0,1,2);(0.9,0.7,0.5)\rangle\tilde{\chi}_1\\ &\quad\succeq\langle(49,50,51);(0.7,0.7,0.5)\rangle\\ \tilde{g}_2^{\mathscr{N}}(\tilde{\chi}) &= \langle(0,1,2);(0.9,0.7,0.5)\rangle\tilde{\chi}_1+\langle(0,1,2);\\ &\quad(0.9,0.7,0.5)\rangle\tilde{\chi}_2\succeq\langle(99,100,101);(0.4,0.2,0.1)\rangle\\ &\quad\tilde{\chi}_1,\tilde{\chi}_2\succeq\tilde{0}.\end{aligned}\tag{6.3}$$

The parametric form of (6.3) becomes

$$\begin{aligned}\min \tilde{f}^{\mathscr{N}}(\tilde{\chi})) =&\langle(2,1-\beta,1-\beta);(0.8,0.6,0.4)\rangle\tilde{\chi}_1^2\\ &+\langle(2,1-\beta,1-\beta);(0.8,0.6,0.4)\rangle\tilde{\chi}_2^2\\ &+\langle(40,1-\beta,1-\beta);(0.6,0.3,0.2)\rangle\tilde{\chi}_1\\ &+\langle(20,1-\beta,1-\beta);(0.8,0.2,0.6)\rangle\tilde{\chi}_2\\ &+\langle(19500,1-\beta,1-\beta);(0.6,0.3,0.6)\rangle\\ \text{sub. to } \tilde{g}_1^{\mathscr{N}}(\tilde{\chi}) =&\langle(1,1-\beta,1-\beta);(0.9,0.7,0.5)\rangle\tilde{\chi}_1\\ &\succeq\langle(50,1-\beta,1-\beta);(0.7,0.7,0.5)\rangle\\ \tilde{g}_2^{\mathscr{N}}(\tilde{\chi}) =&\langle(1,1-\beta,1-\beta);(0.9,0.7,0.5)\rangle\tilde{\chi}_1\\ &+\langle(1,1-\beta,1-\beta);(0.9,0.7,0.5)\rangle\tilde{\chi}_2\\ &\succeq\langle(100,1-\beta,1-\beta);(0.4,0.2,0.1)\rangle\\ &\tilde{\chi}_1,\tilde{\chi}_2\succeq\tilde{0}.\end{aligned}\tag{6.4}$$

Applying the KKT conditions, we have

Stationarity: $\dfrac{\partial\tilde{f}_i^{\mathscr{N}}}{\partial\tilde{\chi}_i}+\sum_{i=1}^m\lambda_i\dfrac{\partial\tilde{g}_i^{\mathscr{N}}}{\partial\tilde{\chi}_i}\approx\tilde{0}$

$$\begin{aligned}\langle(4,1-\beta,1-\beta);&(0.8,0.6,0.4)\rangle\tilde{\chi}_1+\langle(40,1-\beta,1-\beta);(0.6,0.3,0.2)\rangle\\ &+\lambda_1[\langle(1,1-\beta,1-\beta);(0.9,0.7,0.5)\rangle]\\ &+\lambda_2[\langle(1,1-\beta,1-\beta);(0.9,0.7,0.5)\rangle]\approx\tilde{0}\\ \langle(4,1-\beta,1-\beta);&(0.8,0.6,0.4)\rangle\tilde{\chi}_2+\langle(20,1-\beta,1-\beta);(0.8,0.2,0.6)\rangle\\ &+\lambda_2[\langle(1,1-\beta,1-\beta);(0.9,0.7,0.5)\rangle]\approx\tilde{0}\end{aligned}\tag{6.5}$$

Primal feasibility: $\tilde{g}_i^{\mathcal{N}}(\tilde{\chi}_i) \preceq \tilde{0}$

$$\begin{aligned}
&\langle(1,1-\beta,1-\beta);(0.9,0.7,0.5)\rangle\tilde{\chi}_1 \\
&\qquad - \langle(50,1-\beta,1-\beta);(0.7,0.7,0.5)\rangle \preceq \tilde{0} \\
&\langle(1,1-\beta,1-\beta);(0.9,0.7,0.5)\rangle\tilde{\chi}_1 \\
&\qquad + \langle(1,1-\beta,1-\beta);(0.9,0.7,0.5)\rangle\tilde{\chi}_2 \\
&\qquad - \langle(100,1-\beta,1-\beta);(0.4,0.2,0.1)\rangle \preceq \tilde{0}
\end{aligned} \tag{6.6}$$

Complementary slackness: $\lambda_i[\tilde{g}_i^{\mathcal{N}}(\tilde{\chi}_i)] \approx \tilde{0}$

$$\begin{aligned}
&\lambda_1[\langle(1,1-\beta,1-\beta);(0.9,0.7,0.5)\rangle\tilde{\chi}_1 \\
&\qquad - \langle(50,1-\beta,1-\beta);(0.7,0.7,0.5)\rangle] \approx \tilde{0} \\
&\lambda_2[\langle(1,1-\beta,1-\beta);(0.9,0.7,0.5)\rangle\tilde{\chi}_1 \\
&\qquad + \langle(1,1-\beta,1-\beta);(0.9,0.7,0.5)\rangle\tilde{\chi}_2 \\
&\qquad - \langle(100,1-\beta,1-\beta);(0.4,0.2,0.1)\rangle] \approx \tilde{0}
\end{aligned} \tag{6.7}$$

Dual feasibility: $\lambda_1, \lambda_2 \preceq \tilde{0}$
Here we have two Langrange's multiplier λ_1, λ_2 which can take zero or non zero positive values. Thus four solutions corresponding to the following four combinations of $\lambda_i (i = 1, 2)$ values can be obtained.
Case(i): When $\lambda_1 = 0, \lambda_2 = 0$, from equations (6.5), (6.6) and (6.7), we have

$$\tilde{\chi}_1 = \frac{-\langle(40,1-\beta,1-\beta);(0.6,0.3,0.2)\rangle}{\langle(4,1-\beta,1-\beta);(0.8,0.6,0.4)\rangle}, \tilde{\chi}_2 = \frac{-\langle(20,1-\beta,1-\beta);(0.8,0.2,0.6)\rangle}{\langle(4,1-\beta,1-\beta);(0.8,0.6,0.4)\rangle}$$

$\therefore$ The solution is infeasible.
Case(ii): When $\lambda_1 \neq 0, \lambda_2 = 0$, from equations (6.5), (6.6) and (6.7), we have

$$\begin{aligned}
&\tilde{\chi}_1 = \langle(50,1-\beta,1-\beta);(0.9,0.7,0.5)\rangle, \tilde{\chi}_2 = -\langle(5,1-\beta,1-\beta);(0.8,0.6,0.6)\rangle \\
&\lambda_1 = -\langle(240,1-\beta,1-\beta);(0.9,0.7,0.5)\rangle
\end{aligned}$$

$\therefore$ This solution also infeasible.
Case(iii): When $\lambda_1 = 0, \lambda_2 \neq 0$, from equations (6.5), (6.6) and (6.7), we have

$$\begin{aligned}
&\tilde{\chi}_1 = \langle(47.5,1-\beta,1-\beta);(0.9,0.7,0.5)\rangle, \tilde{\chi}_2 = \langle(52.5,1-\beta,1-\beta);(0.9,0.7,0.6)\rangle \\
&\tilde{\chi}_3 = \langle(50,1-\beta,1-\beta);(0.9,0.7,0.6)\rangle, \lambda_2 = -\langle(230,1-\beta,1-\beta);(0.9,0.7,0.5)\rangle
\end{aligned}$$

$\therefore$ The solution is feasible but not satisfy condition (6.6).

Case(iv): When$\lambda_1 \neq 0, \lambda_2 \neq 0$, from equations (6.5), (6.6) and (6.7), we have

$$\tilde{\chi}_1 = \langle(50, 1-\beta, 1-\beta); (0.9, 0.7, 0.6)\rangle, \tilde{\chi}_2 = \langle(50, 1-\beta, 1-\beta); (0.9, 0.7, 0.6)\rangle$$
$$\tilde{\chi}_3 = \langle(50, 1-\beta, 1-\beta); (0.9, 0.7, 0.5)\rangle, \lambda_1 = -\langle(20, 1-\beta, 1-\beta); (0.9, 0.7, 0.5)\rangle$$
$$\lambda_2 = -\langle(220, 1-\beta, 1-\beta); (0.9, 0.7, 0.5)\rangle$$

$\therefore$ The solution is feasible and satisfy all condition.
Hence the optimal solution of the given SVNNLPP (6.1) is
$\tilde{\chi}_1 = \langle(50, 1-\beta, 1-\beta); (0.9, 0.7, 0.5)\rangle, \tilde{\chi}_2 = \langle(50, 1-\beta, 1-\beta); (0.9, 0.7, 0.5)\rangle$,
$\tilde{\chi}_3 = \langle(50, 1-\beta, 1-\beta); (0.9, 0.7, 0.5)\rangle$ with
$\min \tilde{f}^{\mathcal{N}}(\tilde{\chi}) = \langle(7500, 1-\beta, 1-\beta); (0.9, 0.7, 0.6)\rangle$.
That is Himaja Home Appliances has to supply $\tilde{\chi}_1 = \langle(49+\beta, 50, 51-\beta); (0.9, 0.7, 0.5)\rangle$ fans at the end of first month, $\tilde{\chi}_2 = \langle(49+\beta, 50, 51-\beta); (0.9, 0.7, 0.5)\rangle$ fans at the end of second month and $\tilde{\chi}_3 = \langle(49+\beta, 50, 51-\beta); (0.9, 0.7, 0.5)\rangle$ fans at the end of third month with minimum cost $\min \tilde{f}^{\mathcal{N}}(\tilde{\chi}) = \langle(7499+\beta, 7500, 7501-\beta); (0.9, 0.7, 0.6)\rangle$.

7. Result and Discussion

The Neutrosophic optimal solution of the SVNNLPP (5.1) for different values of β.

Table 2. **NOS for various values of $\beta \in [0, 1]$**

β	$\tilde{\chi}_1$	$\tilde{\chi}_2$
0	$\left\langle\left(\frac{-5}{64}, \frac{59}{64}, \frac{123}{64}\right); (0.6, 0.4, 0.5)\right\rangle$	$\left\langle\left(\frac{-11}{16}, \frac{5}{16}, \frac{21}{16}\right); (0.6, 0.4, 0.5)\right\rangle$
0.25	$\left\langle\left(\frac{11}{64}, \frac{59}{64}, \frac{107}{64}\right); (0.6, 0.4, 0.5)\right\rangle$	$\left\langle\left(\frac{-7}{16}, \frac{5}{16}, \frac{17}{16}\right); (0.6, 0.4, 0.5)\right\rangle$
0.5	$\left\langle\left(\frac{27}{64}, \frac{59}{64}, \frac{91}{64}\right); (0.6, 0.4, 0.5)\right\rangle$	$\left\langle\left(\frac{-3}{16}, \frac{5}{16}, \frac{13}{16}\right); (0.6, 0.4, 0.5)\right\rangle$
0.75	$\left\langle\left(\frac{43}{64}, \frac{75}{64}, \frac{123}{64}\right); (0.6, 0.4, 0.5)\right\rangle$	$\left\langle\left(\frac{1}{16}, \frac{5}{16}, \frac{9}{16}\right); (0.6, 0.4, 0.5)\right\rangle$
1	$\left\langle\left(\frac{59}{64}, \frac{59}{64}, \frac{59}{64}\right); (1, 1, 1)\right\rangle = \frac{59}{64}$	$\left\langle\left(\frac{5}{16}, \frac{5}{16}, \frac{5}{16}\right); (1, 1, 1)\right\rangle = \frac{5}{16}$

Table 3. **Continuation to Table 2**

β	$\tilde{f}^{\mathcal{N}}(\tilde{\chi})$
0	$\left\langle \left(\frac{562}{256}, \frac{818}{256}, \frac{1074}{256}\right); (0.6, 0.4, 0.5) \right\rangle$
0.25	$\left\langle \left(\frac{626}{256}, \frac{818}{256}, \frac{1010}{256}\right); (0.6, 0.4, 0.5) \right\rangle$
0.5	$\left\langle \left(\frac{690}{256}, \frac{818}{256}, \frac{946}{256}\right); (0.6, 0.4, 0.5) \right\rangle$
0.75	$\langle (\frac{754}{256}, \frac{818}{256}, \frac{882}{256}); (0.6, 0.4, 0.5) \rangle$
1	$\left\langle \left(\frac{818}{256}, \frac{818}{256}, \frac{818}{256}\right); (1, 1, 1) \right\rangle = \frac{818}{256}$

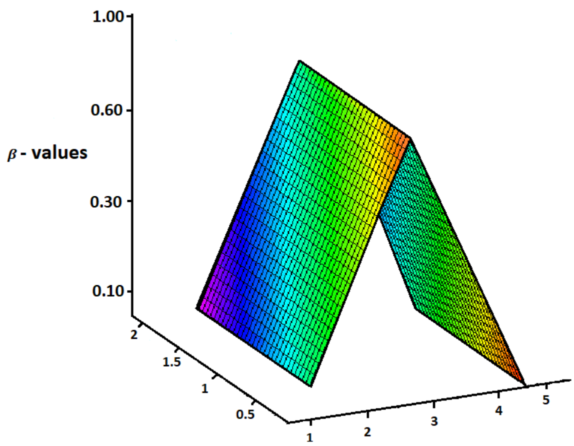

Figure 1. **Optimal solution for different values of** $\beta \in [0, 1]$

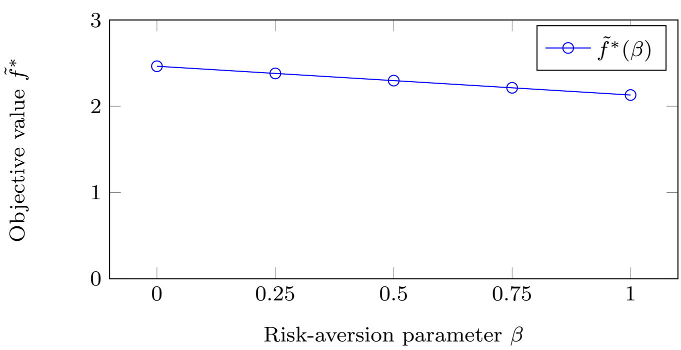

Figure 2. Numerical example 5.1: graded-mean optimal objective vs. risk-aversion β. Data copied from Table 3.

When $\beta = 1$, we see that $\tilde{\chi}_1 = \frac{59}{64}\tilde{\chi}_2 = \frac{5}{16}$ and $\max \tilde{f}^{\mathcal{N}}(\tilde{\chi}) = \frac{818}{256}$. This solution is same the crisp optimal solution $\chi^* = \left(\frac{59}{64}, \frac{5}{16}\right)$ with $\max f(\chi) = 3.195$ obtained by Ghadle and Pawar [6].
The Neutrosophic optimal solution of the SVNNLPP (6.1) for different values of β.

Table 4. NOS for various values of $\beta \in [0, 1]$

β	$\tilde{\chi}_1$	$\tilde{\chi}_2$
0	$\langle(49, 50, 51); (0.9, 0.7, 0.5)\rangle$	$\langle(49, 50, 51); (0.9, 0.7, 0.5)\rangle$
0.25	$\langle(49.25, 50, 50.75); (0.9, 0.7, 0.5)\rangle$	$\langle(49.25, 50, 50.75); (0.9, 0.7, 0.5)\rangle$
0.5	$\langle(49.5, 50, 50.5); (0.9, 0.7, 0.5)\rangle$	$\langle(49.5, 50, 50.5); (0.9, 0.7, 0.5)\rangle$
0.75	$\langle(49.75, 50, 50.25); (0.9, 0.7, 0.5)\rangle$	$\langle(49.75, 50, 50.25); (0.9, 0.7, 0.5)\rangle$
1	$\langle(50, 50, 50); (1, 1, 1)\rangle$ =50	$\langle(50, 50, 50); (0.9, 0.7, 0.5)\rangle$ =50

Table 5. Continuation to Table 4

β	$\tilde{\chi}_3$	$\tilde{f}^{\mathcal{N}}(\tilde{\chi})$
0	$\langle(49, 50, 51); (0.9, 0.7, 0.5)\rangle$	$\langle(7499, 7500, 7501); (0.9, 0.7, 0.6)\rangle$
0.25	$\langle(49.25, 50, 50.75); (0.9, 0.7, 0.5)\rangle$	$\langle(7499.25, 7500, 7500.75); (0.9, 0.7, 0.6)\rangle$
0.5	$\langle(49.5, 50, 50.5); (0.9, 0.7, 0.5)\rangle$	$\langle(7499.5, 7500, 7500.5); (0.9, 0.7, 0.6)\rangle$
0.75	$\langle(49.75, 50, 50.25); (0.9, 0.7, 0.5)\rangle$	$\langle(7499.75, 7500, 7500.25); (0.9, 0.7, 0.6)\rangle$
1	$\langle(50, 50, 50); (1, 1, 1)\rangle$ =(50)	$\langle(7500, 7500, 7500); (1, 1, 1)\rangle$ =7500

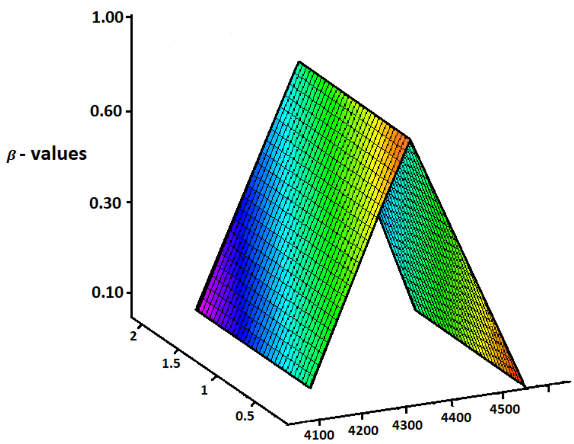

Figure 3. Optimal solution for different values of $\beta \in [0, 1]$

When $\beta = 1$, we see that $\tilde{\chi}_1 = 50, \tilde{\chi}_2 = 50, \tilde{\chi}_3 = 50$ and $\min \tilde{f}^{\mathcal{N}}(\tilde{\chi}) = 7500$.

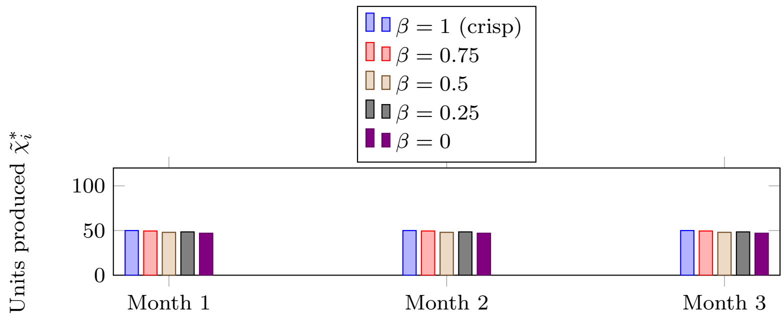

Figure 4. neutrosophic optimal production quantities under three levels of uncertainty tolerance β.

8. Conclusion

In this chapter, we discussed a solution concept for SVNNLPP involving neutrosophic triangular numbers. First, the given SVNNLPP is expressed in terms of its location index number , left and right fuzziness index functions. In the parametric forms of neutrosophic numbers, a new type of neutrosophic arithmetic and neutrosophic ranking are introduced and utilized. The neutrosophic versions of the Karush Kuhn Tucker (KKT) condition are used, and the neutrosophic optimal solution of the SVNNLPP is obtained without having to convert the given problem. The neutrosophic fuzzy optimal solution of the given SVNNLPP is tabulated for different values of $\beta \in [0, 1]$. It is important to note that by utilizing the suggested procedure and selecting an appropriate value for $\beta \in [0, 1]$, the decision maker has the flexibility to select his or her preferred optimal solution based on the situation. The numerical solutions have been presented and discussed.

Future Scope

The proposed Neutrosophic KKT-based method opens several avenues for future research. One promising direction is its extension to multi-objective optimization problems, where conflicting objectives can be modeled using neutrosophic sets to reflect trade-offs under uncertainty. Another potential area involves dynamic systems, where the method could be adapted to handle time-dependent parameters and constraints. Moreover, integrating this approach with machine learning techniques may enhance its ability to learn and update neutrosophic parameters from evolving data. Exploring hybrid models combining fuzzy, intuitionistic and neutrosophic frameworks also presents a valuable path for handling diverse forms of imprecision in real-world problems.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

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