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Research Article

Relation between ABS index with some other topological indices

Artes Rosalio Jr[1,2], R.U. Gobithaasan[2,*], Roslan Hasni[3], N. Jafari Rad[4]

[1]Department of Mathematics, College of Arts and Sciences,
Mindanao State University - Tawi-Tawi College of Technology and Oceanography,
7500 Bongao, Tawi-Tawi, Philippines
rosalioartes@msutawi-tawi.edu.ph

[2]School of Mathematical Sciences, Universiti Sains Malaysia, 11800 USM Penang, Malaysia
*gobithaasan@usm.my

[3]Special Interest Group on Modelling and Data Analytics (SIGMDA),
Faculty of Computer Science and Mathematics, Universiti Malaysia Terengganu,
21030 UMT Kuala Nerus, Terengganu, Malaysia
hroslan@umt.edu.my

[4]Department of Mathematics, Shahed University, Tehran, Iran
n.jafarirad@shahed.ac.ir

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Abstract: The study of atom-bond sum-connectivity index emerged recently as a variant of atom-bond connectivity index by replacing the product in the denominator in each of the fractions corresponding to every edge by the sum of the degrees. In this paper, we established relationships between the ABS index with some other existing degree-based topological indices in terms of minimum degree and maximum degree of the graph.

Keywords: quantitative structure-activity relationship (QSAR), quantitative structure-property relationship (QSPR), ABS index, Zagreb index Sombor index.

AMS Subject classification: Provide the AMS code.

1. Introduction

In chemical sciences, topological indices (TIs) have been found to be useful in chemical documentation, isomer discrimination, structure-property relationships, and structure-activity relationships [10, 16]. There has been considerable interest in the

* *Corresponding Author*

general problem of determining topological indices as presented by Kulli [22] in 2019. TIs are best applied to recognize the physical properties, chemical reactions and biological activities. The numeric values of TIs predict various physical and chemical properties of the involved organic compounds in the molecular graphs such as volume, density, pressure, weight, boiling point, freezing point, vaporization point, heat of formation, and heat of evaporation [25, 27]. Moreover, they are used to study the quantitative structure-activity relationship (QSAR) and quantitative structure-property relationship (QSPR) and medical behaviors of different drugs in the subject of cheminformatics and pharmaceutical industries [10, 15, 30–32]. Aarthi et at. [1] investigated the maximum value of atom-bond sum-connectivity among the class of bicyclic graphs on n vertices. They also demonstrated the role of atom-bond sum-connectivity in explaining structure–property relationship.

In topological indices, the atom-bond connectivity index is one of the celebrated indices. This topological index was first introduced by Estrada et al. [13] in 1998. It is a useful topological index employed in studying the stability of alkanes and the strain energy of cycloalkanes [21]. The *atom-bond connectivity index* of a nontrivial graph G, denoted by $ABC(G)$, is defined as

$$ABC(G) = \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u d_v}},$$

where d_u is the degree of vertex u in G. In [13], Estrada et al. used ABC index for the purpose of modeling thermodynamic properties of organic chemical compounds. In 2008, Estrada published another paper [12], in which ABC index is used as a tool to explain the stability of branched alkanes. This captured the attention of several mathematicians working on topological indices, resulting in a remarkable number of research papers on the mathematical properties of this topological index [8, 9].

Hasni et al. [19] studied links between the difference of Randić and ABC indices with certain well-studied topological indices. They derived some bounds for the difference of Randić index and atom-bond connectivity index with minimum degree δ, maximum degree Δ, and size of G. The *minimum degree* δ of G and the *maximum degree* Δ of G are, respectively, defined as follows:

$$\delta = \min_{v \in V(G)} \{\deg_G(v)\}, \qquad \Delta = \max_{v \in V(G)} \{\deg_G(v)\}.$$

In 2021, Phanjoubami and Mawiong [26] established some new results relating the Sombor index and some well-studied topological indices: Zagreb indices, forgotten index, harmonic index, (general) sum-connectivity index and symmetric division deg index. Moreover, Du, Jahanbani and Sheikholeslami [11] investigated the relationships between Randic index and several topological indices. Cruz et al. [7] obtained results on extremal values of vertex-degree based topological indices, such as the generalized Geometric-Arithmetic indices and the generalized Atom-Bond-Connectivity indices.

Ali et al. [2] introduced the ABS index by amalgamating the core idea of the SC and ABC indices, a new molecular descriptor was put forward—the atom-bond sum-connectivity (ABS) index. They determined the graphs attaining the extreme values of the ABS index over the classes of (molecular) trees and general graphs of a fixed order. In their paper, they have shown that ABS index increases when a non-isolated edge is inserted between any two non-adjacent vertices.

For a (molecular) graph G, the *atom-bond sum-connectivity (ABS) index* is defined as

$$ABS(G) = \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u + d_v}}.$$

This molecular descriptor replaces the denominator $d_u d_v$ in the ABC index by $d_u + d_v$ for $uv \in E(G)$.

Hu and Wang [20] presented the extremal trees with the maximum ABS index among all trees of a given order with matching number or diameter, respectively. Moreover, they also determined trees with a perfect matching having the maximum ABS index.

The *harmonic index* of G is given by

$$H(G) = \sum_{uv \in E(G)} \frac{2}{d_u + d_v}.$$

Some results on harmonic index can be found in [33] where Zhong established the minimum and maximum values of the harmonic index for simple connected graphs and trees, and characterized the corresponding extremal graphs.

Ali et al. [2] obtained the following sharp upper bound for ABS index in terms of harmonic index and the size of the graph.

Proposition 1. [2] *Let $H(G)$ be the harmonic index of the graph G. If G is a graph with m edges, then*

$$ABS(G) \leq \sqrt{m(m - H(G))},$$

with equality if and only if either $m = 0$ or there is a fixed number k such that $d_u + d_v = k$ holds for every edge $uv \in E(G)$.

The above result was established using the well-known Cauchy-Schwartz Inequality.

Lemma 1 (Cauchy-Schwartz Inequality). *For all sequences of real numbers $\langle a_i \rangle_{i=1}^n$ and $\langle b_i \rangle_{i=1}^n$, we have*

$$\left(\sum_{i=1}^n a_i^2 \right) \left(\sum_{i=1}^n b_i^2 \right) \geq \left(\sum_{i=1}^n a_i b_i \right)^2,$$

equality holds if and only if $a_i = k b_i$ for a nonzero constant $k \in \mathbb{R}$.

In 2023, Lin [24] established results on relations between atom-bond sum-connectivity index and other connectivity indices particularly the Randić and harmonic indices. Below are some of the bounds established by Lin [24].
The *Randić index* of a graph G is given by

$$R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u d_v}}.$$

Theorem 1. [24] *Let G be a connected graph with the maximum degree Δ and minimum degree δ. Then*

$$\sqrt{\delta(\delta-1)}R(G) \leq ABS(G) \leq \sqrt{\Delta(\Delta-1)}R(G),$$

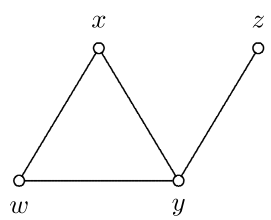
with equality if and only if G is regular.

Theorem 2. [24] *Let G be a connected graph with the maximum degree Δ and minimum degree δ. Then*

$$\sqrt{\delta(\delta-1)}H(G) \leq ABS(G) \leq \sqrt{\Delta(\Delta-1)}H(G),$$

with equality if and only if G is regular.

Recently, Ali et al. [3] made a survey of the mathematical properties of the ABS index. They collected known bounds and extremal results regarding the ABS index. Moreover, they proposed a number of open problems and conjectures, arising from the reported results.
In 2024, Swathi et al. [29] established relations between the atom-bond sum connectivity index with harmonic index, the first, hyper and augmented Zagreb indices, the general sum-connectivity index, the Randić index, and the sum-connectivity F-index. Also, Li, Ye, and Lu [23] established sharp upper bounds for the ABS indices of graphs on the basis of their fixed parameters such as chromatic number, clique number, connectivity and matching number.
Our focus in this paper is on extremal results and bounds of ABS index with other degree-based topological indices different from those established in [29].
Consider the graph below:



Let us examine the values of summands of some distance-based topological indices with respect to the graph above. Here, $d_w = 2$, $d_x = 2$, $d_y = 3$, and $d_z = 1$.

Topological Indices	Formula	wx	xy	wy	yz	Value
ABS Index	$\sum_{uv\in E(G)} \sqrt{\frac{d_u+d_v-2}{d_u+d_v}}$	$\sqrt{\frac{2}{4}}$	$\sqrt{\frac{3}{5}}$	$\sqrt{\frac{3}{5}}$	$\sqrt{\frac{2}{4}}$	2.96340
Harary Index	$\sum_{uv\in E(G)} \frac{2}{d_u+d_v}$	$\frac{2}{4}$	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{4}$	1.8
ABC Index	$\sum_{uv\in E(G)} \sqrt{\frac{d_u+d_v-2}{d_ud_v}}$	$\sqrt{\frac{2}{4}}$	$\sqrt{\frac{3}{6}}$	$\sqrt{\frac{3}{6}}$	$\sqrt{\frac{2}{3}}$	2.93781
Randić Index	$\sum_{uv\in E(G)} \frac{1}{\sqrt{d_ud_v}}$	$\frac{1}{2}$	$\frac{1}{\sqrt{6}}$	$\frac{1}{\sqrt{6}}$	$\frac{1}{\sqrt{3}}$	1.89384
First Zagreb Index	$\sum_{uv\in E(G)} d_u+d_v$	4	5	5	4	18
Second Zagreb Index	$\sum_{uv\in E(G)} d_ud_v$	4	6	6	3	19
Sombor Index	$\sum_{uv\in E(G)} \sqrt{d_u^2+d_v^2}$	$2\sqrt{2}$	$\sqrt{13}$	$\sqrt{13}$	$\sqrt{10}$	13.20180
Modified Sombor	$\sum_{uv\in E(G)} \frac{1}{\sqrt{d_u^2+d_v^2}}$	$\frac{1}{2\sqrt{2}}$	$\frac{1}{\sqrt{13}}$	$\frac{1}{\sqrt{13}}$	$\frac{1}{\sqrt{10}}$	1.22448

In the ABS index, the edges wx and yz have equal values although the end-vertices of the edges have different degrees. But the same edges differs in values for the ABC index. This is due to the replacement of the product in the denominator for ABC index by the sum of the degrees for ABS index. We will establish their relationships in Section 2. With the same graph, the value for modified Sombor index is much smaller than the ABS index. The relationships between these two topological indices will be established in the second part of Section 4. In the above table, the second Zagreb index has the highest value. The relationship of this TI with the ABS index is established in Section 3.

We will now examine the relationships between ABS index with other degree-based topological indices given the minimum degree and maximum degree of the graph.

2. Relation Between ABS Index with ABC Index

Note that if G has no pendant vertex, then $d_ud_v \geq d_u + d_v$ for every pair of vertices u and v in G. Consequently, $ABC(G) \leq ABS(G)$. Now, we will establish bounds of ABS index of general graphs in terms of ABC index with minimum degree δ and maximum degree Δ.

Theorem 3. *Let G be a connected graph of order at least 3 with minimum degree δ and maximum degree Δ. Then*

$$\frac{\delta}{\sqrt{2\Delta}}ABC(G) \leq ABS(G) \leq \frac{\Delta}{\sqrt{2\delta}}ABC(G).$$

Equality holds for both if and only if G is regular.

Proof. Let $u, v \in V(G)$ such that $uv \in E(G)$. Then

$$\begin{aligned} ABC(G) &= \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u d_v}} \\ &= \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u d_v}} \cdot \frac{\sqrt{d_u + d_v}}{\sqrt{d_u + d_v}} \\ &\leq \frac{\sqrt{2\Delta}}{\delta} \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u + d_v}} \\ &= \frac{\sqrt{2\Delta}}{\delta} ABS(G). \end{aligned}$$

Thus,

$$ABS(G) \geq \frac{\delta}{\sqrt{2\Delta}} ABC(G). \tag{2.1}$$

Suppose equality holds in 2.1. From the above relations, this means that $\frac{\sqrt{d_u + d_v}}{\sqrt{d_u d_v}} = \frac{\sqrt{2\Delta}}{\delta}$, which further implies that $\frac{d_u + d_v}{d_u d_v} = \frac{2\Delta}{\delta^2}$. Thus, there exists $k \in \mathbb{R}$ such that $d_u + d_v = 2k\Delta$ and $d_u d_v = k\delta^2$. Now, $2k\Delta = d_u + d_v \leq 2\Delta$. Consequently, $k = 1$ and $d_u = d_v = \Delta$. Hence, $\delta^2 = k\delta^2 = d_u d_v = \Delta^2$ for every $uv \in E(G)$. Thus, $d_u = \Delta = \delta$ for every $u \in V(G)$ and hence G is regular. Conversely, if G is regular then it can be easily seen that equality holds.
Next we show the second inequality. Now,

$$\begin{aligned} ABC(G) &= \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u d_v}} \\ &= \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u d_v}} \cdot \frac{\sqrt{d_u + d_v}}{\sqrt{d_u + d_v}} \\ &\geq \frac{\sqrt{2\delta}}{\Delta} \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u + d_v}} \\ &= \frac{\sqrt{2\delta}}{\Delta} ABS(G). \end{aligned}$$

Hence,

$$ABS(G) \leq \frac{\Delta}{\sqrt{2\delta}} ABC(G). \tag{2.2}$$

Suppose equality holds in 2.2. Then $\frac{\sqrt{d_u + d_v}}{\sqrt{d_u d_v}} = \frac{\sqrt{2\delta}}{\Delta}$. Similar arguements with the above gives us $d_u = \delta = \Delta$ for every $u \in V(G)$ and conclude that G is regular. Conversely, if G is regular then it can be easily seen that equality holds. □

3. Relation Between ABS Index with Zagreb Indices

One of the oldest graph invariants is the well-known Zagreb index first introduced in [18], where Gutman and Trinajstić examined the dependence of total π-electron energy on molecular structure. For a (molecular) graph G, the *first Zagreb index* $M_1(G)$ and the *second Zagreb index* $M_2(G)$ are, respectively, defined as follows:

$$M_1(G) = \sum_{uv\in E(G)} d_u + d_v, \qquad M_2(G) = \sum_{uv\in E(G)} d_u d_v.$$

In 2021, Filipovski [14] obtained various bounds of the First Zagreb index in terms of the degree sequence of a graph.
Now we present an upper bound for the ABS index in terms of the first Zagreb index $M_1(G)$ and harmonic index.

Theorem 4. *Let G be a graph of size m. Then*

$$ABS(G) \leq \frac{\sqrt{2}}{2}\sqrt{(M_1(G)-2m)H(G)}.$$

Proof. Let G denotes a graph with m edges and the first Zagreb index $M_1(G)$. By the ABS definition and the Cauchy-Schwartz inequality, we have

$$\begin{aligned} ABS(G) &= \sum_{uv\in E(G)} \sqrt{\frac{d_u+d_v-2}{d_u+d_v}} \\ &\leq \sqrt{\sum_{uv\in E(G)} \left(\sqrt{d_u+d_v-2}\right)^2 \sum_{uv\in E(G)} \left(\frac{1}{\sqrt{d_u+d_v}}\right)^2} \\ &= \sqrt{\sum_{uv\in E(G)} (d_u+d_v-2) \sum_{uv\in E(G)} \frac{1}{d_u+d_v}} \\ &= \frac{\sqrt{2}}{2}\sqrt{(M_1(G)-2m)H(G)} \end{aligned}$$

The proof is complete. □

Theorem 4 was established in the paper of Ali et al. [4] (THEOREM 3.4) by using a slightly different way.
Next, we present bounds for ABS index in terms of the first Zagreb index $M_1(G)$ with minimum degree δ and maximum degree Δ.
We will use the following straightforward result.

Lemma 2. *Let G be a connected graph of order at least 3. Then for any two adjacent vertices $u, v \in V(G)$, $d_u + d_v \geq 3$.*

Theorem 5. *Let G be a connected graph of order at least 3 with minimum degree δ and maximum degree Δ. Then*

$$\frac{1}{2\Delta\sqrt{2\Delta}}M_1(G) \leq ABS(G) < \frac{1}{2\delta}M_1(G).$$

Proof. Let $u, v \in V(G)$ such that $uv \in E(G)$. Then by Lemma 2, $d_u + d_v - 2 \geq 1$. Now,

$$\begin{aligned} M_1(G) &= \sum_{uv\in E(G)} d_u + d_v \\ &\leq 2\Delta \sum_{uv\in E(G)} \sqrt{\frac{d_u+d_v}{d_u+d_v}} \qquad (3.1) \\ &\leq 2\Delta\sqrt{2\Delta} \sum_{uv\in E(G)} \sqrt{\frac{1}{d_u+d_v}} \qquad (3.2) \\ &\leq 2\Delta\sqrt{2\Delta} \sum_{uv\in E(G)} \sqrt{\frac{d_u+d_v-2}{d_u+d_v}} \qquad (3.3) \\ &= 2\Delta\sqrt{2\Delta}ABS(G). \end{aligned}$$

Hence,

$$ABS(G) \geq \frac{1}{2\Delta\sqrt{2\Delta}}M_1(G). \qquad (3.4)$$

Moreover,

$$\begin{aligned} M_1(G) &= \sum_{uv\in E(G)} d_u + d_v \\ &\geq 2\delta \sum_{uv\in E(G)} \sqrt{\frac{d_u+d_v}{d_u+d_v}} \\ &> 2\delta \sum_{uv\in E(G)} \sqrt{\frac{d_u+d_v-2}{d_u+d_v}} \\ &= 2\delta ABS(G). \end{aligned}$$

Hence,

$$ABS(G) < \frac{1}{2\delta}M_1(G). \qquad (3.5)$$

Combining Inequalities 3.4 and 3.5 gives the desired result. □

In the above result, sharpness of bounds cannot be attained. This is due to the fact that if we have equality, then line 3.1 implies that $d_u + d_v = 2\Delta$ which means that $\Delta = d_u$ for every $u \in V(G)$ and hence G is regular. But in this case, $1 < d_u + d_v - 2$ from lines 3.2 and 3.3. For the upper bound, we use the fact that $d_u+d_v > d_u+d_v-2$. We establish sharp bounds of ABS index in terms of first Zagreb index for classes of graphs without pendant vertices.

Theorem 6. *Let G be a connected graph with minimum degree $\delta \geq 2$ and maximum degree Δ. Then*

$$\frac{\sqrt{\delta-1}}{2\Delta\sqrt{\Delta}}M_1(G) \leq ABS(G) \leq \frac{\sqrt{\Delta-1}}{2\delta\sqrt{\delta}}M_1(G).$$

Equality holds for both if and only if G is regular.

Proof. Let G be a graph with minimum degree $\delta \geq 2$. Then $\sqrt{2\delta-2} > 0$. Let $u, v \in V(G)$ such that $uv \in E(G)$. Now, Lemma 2 asserts that $d_u + d_v - 2 > 0$. Hence,

$$\begin{aligned}
M_1(G) &= \sum_{uv\in E(G)} d_u + d_v \\
&\leq 2\Delta \sum_{uv\in E(G)} \sqrt{\frac{d_u+d_v-2}{d_u+d_v}} \cdot \sqrt{\frac{d_u+d_v}{d_u+d_v-2}} \\
&\leq \frac{2\Delta\sqrt{2\Delta}}{\sqrt{2\delta-2}} \sum_{uv\in E(G)} \sqrt{\frac{d_u+d_v-2}{d_u+d_v}} \\
&= \frac{2\Delta\sqrt{\Delta}}{\sqrt{\delta-1}} ABS(G).
\end{aligned}$$

Hence,

$$ABS(G) \geq \frac{\sqrt{\delta-1}}{2\Delta\sqrt{\Delta}}M_1(G). \tag{3.6}$$

Suppose equality holds in 3.6. Then $2\Delta = d_u + d_v$. This equality can only be attained when $d_u = d_v = \Delta$ for every $uv \in E(G)$. Consequently, $d_u = \delta = \Delta$ for every $u \in V(G)$ and hence G is regular. Conversely, if G is regular then it can be easily seen that equality holds.
Next, we show the second inequality. Now,

$$\begin{aligned}
M_1(G) &= \sum_{uv\in E(G)} d_u + d_v \\
&\geq 2\delta \sum_{uv\in E(G)} \sqrt{\frac{d_u+d_v-2}{d_u+d_v}} \cdot \sqrt{\frac{d_u+d_v}{d_u+d_v-2}} \\
&\geq \frac{2\delta\sqrt{2\delta}}{\sqrt{2\Delta-2}} \sum_{uv\in E(G)} \sqrt{\frac{d_u+d_v-2}{d_u+d_v}} \\
&= \frac{2\delta\sqrt{\delta}}{\sqrt{\Delta-1}} ABS(G).
\end{aligned}$$

Hence,

$$ABS(G) \leq \frac{\sqrt{\Delta-1}}{2\delta\sqrt{\delta}} M_1(G). \tag{3.7}$$

If equality holds in 3.7, then $2\delta = d_u + d_v$, which implies that $d_u = d_v = \delta$ for every $uv \in E(G)$. Consequently, $d_u = \delta = \Delta$ for every $u \in V(G)$ and hence G is regular. Conversely, if G is regular then it can be easily seen that equality holds. □

Corollary 1. *Let G be a connected graph with minimum degree $\delta \geq 2$ and maximum degree Δ. If G is not regular, then*

$$ABS(G) < \frac{\sqrt{\Delta-1}}{2\delta\sqrt{\delta}} M_1(G).$$

Comparing with Theorem 5, we have

$$\frac{\sqrt{\Delta-1}}{2\delta\sqrt{\delta}} M_1(G) \leq \frac{1}{2\delta} M_1(G),$$

which implies that $\sqrt{\Delta-1} \leq \sqrt{\delta}$. Thus, if G is a graph with $\Delta = \delta+1$, then Corollary 1 is better than Theorem 5.

The next result establishes the bounds of ABS index of G in terms of the second Zagreb index $M_2(G)$ with minimum degree δ and maximum degree Δ.

Theorem 7. *Let G be a connected graph of order at least 3 with minimum degree δ and maximum degree Δ. Then*

$$\frac{1}{\Delta^2\sqrt{2\Delta}} M_2(G) \leq ABS(G) < \frac{1}{\delta^2} M_2(G).$$

Proof. Let $u, v \in V(G)$ such that $uv \in E(G)$. Then by Lemma 2, $d_u + d_v - 2 \geq 1$. Now,

$$\begin{aligned}
M_2(G) &= \sum_{uv\in E(G)} d_u d_v \\
&\leq \Delta^2 \sum_{uv\in E(G)} \sqrt{\frac{d_u+d_v}{d_u+d_v}} \\
&\leq \Delta^2\sqrt{2\Delta} \sum_{uv\in E(G)} \sqrt{\frac{1}{d_u+d_v}} \\
&\leq \Delta^2\sqrt{2\Delta} \sum_{uv\in E(G)} \sqrt{\frac{d_u+d_v-2}{d_u+d_v}} \\
&= \Delta^2\sqrt{2\Delta}\, ABS(G).
\end{aligned}$$

Hence,

$$ABS(G) \geq \frac{1}{\Delta^2\sqrt{2\Delta}} M_2(G). \tag{3.8}$$

Moreover,

$$\begin{aligned} M_2(G) &= \sum_{uv \in E(G)} d_u d_v \\ &\geq \delta^2 \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v}{d_u + d_v}} \\ &> \delta^2 \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u + d_v}} \\ &= \delta^2 ABS(G). \end{aligned}$$

Hence,

$$ABS(G) < \frac{1}{\delta^2} M_2(G). \tag{3.9}$$

Combining Inequalities 3.8 and 3.9 gives the desired result. □

In the above result, the lower bound is not sharp since equating $d_u d_v = \Delta^2$ will lead us to a regular graph in which case $d_u + d_v - 1$ cannot be equal to 1. Also, in the upper bound $\sqrt{d_u + d_v}$ cannot be equal to $\sqrt{d_u + d_v - 2}$.
In the next result, we establish sharp bounds of ABS index in terms of second Zagreb index for some classes of graphs.

Theorem 8. *Let G be a connected graph with minimum degree $\delta \geq 2$ and maximum degree Δ. Then*

$$\frac{\sqrt{\delta - 1}}{\Delta^2\sqrt{\Delta}} M_2(G) \leq ABS(G) \leq \frac{\sqrt{\Delta - 1}}{\delta^2\sqrt{\delta}} M_2(G).$$

Equality holds for both if and only if G is regular.

Proof. Let G be a graph with minimum degree $\delta \geq 2$. Then $\sqrt{2\delta - 2} > 0$. Let $u, v \in V(G)$ such that $uv \in E(G)$. Now, Lemma 2 asserts that $d_u + d_v - 2 > 0$. Hence,

$$\begin{aligned} M_2(G) &= \sum_{uv \in E(G)} d_u d_v \\ &\leq \Delta^2 \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u + d_v}} \cdot \sqrt{\frac{d_u + d_v}{d_u + d_v - 2}} \\ &\leq \frac{\Delta^2\sqrt{2\Delta}}{\sqrt{2\delta - 2}} \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u + d_v}} \\ &= \frac{\Delta^2\sqrt{\Delta}}{\sqrt{\delta - 1}} ABS(G). \end{aligned}$$

Hence,

$$ABS(G) \geq \frac{\sqrt{\delta-1}}{\Delta^2\sqrt{\Delta}} M_2(G). \tag{3.10}$$

Suppose that equality holds in 3.10. Then $d_u d_v = \Delta^2$, which forces $d_u = d_v = \Delta$ for every $uv \in E(G)$. Consequently, $d_u = \Delta = \delta$ for every $u \in V(G)$ and hence G is regular. Conversely, if G is regular then it can be easily seen that equality holds.
To show the second inequality, we have

$$\begin{aligned}
M_2(G) &= \sum_{uv\in E(G)} d_u d_v \\
&\geq \delta^2 \sum_{uv\in E(G)} \sqrt{\frac{d_u+d_v-2}{d_u+d_v}} \cdot \sqrt{\frac{d_u+d_v}{d_u+d_v-2}} \\
&\geq \frac{\delta^2\sqrt{2\delta}}{\sqrt{2\Delta-2}} \sum_{uv\in E(G)} \sqrt{\frac{d_u+d_v-2}{d_u+d_v}} \\
&= \frac{\delta^2\sqrt{\delta}}{\sqrt{\Delta-1}} ABS(G).
\end{aligned}$$

Hence,

$$ABS(G) \leq \frac{\sqrt{\Delta-1}}{\delta^2\sqrt{\delta}} M_2(G). \tag{3.11}$$

If equality holds in 3.11, then $d_u d_v = \delta^2$ and $d_u = d_v = \delta$ for every $uv \in E(G)$. By connectedness of G, we must have $d_u = \delta = \Delta$ for every $u \in V(G)$ and thus G is regular. Conversely, if G is regular then it can be easily seen that equality holds. $\square$

Corollary 2. *Let G be a connected graph with minimum degree $\delta \geq 2$ and maximum degree Δ. If G is not regular, then*

$$ABS(G) < \frac{\sqrt{\Delta-1}}{\delta^2\sqrt{\delta}} M_2(G).$$

Comparing with Theorem 7, we have

$$\frac{\sqrt{\Delta-1}}{\delta^2\sqrt{\delta}} M_2(G) \leq \frac{1}{\delta^2} M_2(G),$$

which implies that $\sqrt{\Delta-1} \leq \sqrt{\delta}$. Thus, if G is a graph with $\Delta = \delta+1$, then Corollary 2 is better than Theorem 7.

4. Relation Between ABS index with Sombor Index and modified Sombor index

The *Sombor index* and *modified Sombor index* of a graph G are defined respectively as follows:

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2}, \qquad SO^m(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u^2 + d_v^2}}.$$

Both topological indices were introduced in the 2020s, and have already found a variety of chemical, physicochemical, and network-theoretical applications [17]. Some results on Sombor index can be found in [5, 6]. In 2023, Saha [28] established relations between Sombor index and modified Sombor index with other degree-based topological indices.

The following result establishes the bounds for ABS index in terms of the Sombor index of G with minimum degree δ and maximum degree Δ.

Theorem 9. *Let G be a connected graph of order at least 3 with minimum degree δ and maximum degree Δ. Then*

$$\frac{1}{2\Delta\sqrt{\Delta}} SO(G) \leq ABS(G) < \frac{1}{\sqrt{2}\delta} SO(G).$$

Proof. Let $u, v \in V(G)$ such that $uv \in E(G)$. Then by Lemma 2, $d_u + d_v - 2 \geq 1$. Hence,

$$\begin{aligned} SO(G) &= \sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2} \\ &= \sum_{uv \in E(G)} \sqrt{\frac{(d_u^2 + d_v^2)(d_u + d_v)}{d_u + d_v}} \\ &\leq 2\Delta\sqrt{\Delta} \sum_{uv \in E(G)} \sqrt{\frac{1}{d_u + d_v}} \qquad (4.1) \\ &\leq 2\Delta\sqrt{\Delta} \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u + d_v}} \qquad (4.2) \\ &= 2\Delta\sqrt{\Delta} ABS(G). \end{aligned}$$

Thus,

$$ABS(G) \geq \frac{1}{2\Delta\sqrt{\Delta}} SO(G). \qquad (4.3)$$

Moreover,

$$\begin{aligned}
SO(G) &= \sum_{uv\in E(G)} \sqrt{d_u^2+d_v^2} \\
&\geq \sqrt{2}\delta \sum_{uv\in E(G)} \sqrt{\frac{d_u+d_v}{d_u+d_v}} \\
&> \sqrt{2}\delta \sum_{uv\in E(G)} \sqrt{\frac{d_u+d_v-2}{d_u+d_v}} \\
&= \sqrt{2}\delta ABS(G)
\end{aligned}$$

Thus,

$$ABS(G) < \frac{1}{\sqrt{2}\delta} SO(G). \tag{4.4}$$

Combining Inequalities 4.3 and 4.4 gives the desires result. □

The above inequalities are not sharp since in the first inequality, equality in 4.1 means that G is regular. In that case $1 < d_u + d_v - 2$ in 4.2. For the second inequality, $d_u + d_v > d_u + d_v - 2$.
Sharp bounds is established in the following result by considering classes of graphs without pendant vertices.

Theorem 10. *Let G be a connected graph with minimum degree $\delta \geq 2$ and maximum degree Δ. Then*

$$\frac{\sqrt{\delta-1}}{\Delta\sqrt{2\Delta}} SO(G) \leq ABS(G) \leq \frac{\sqrt{\Delta-1}}{\delta\sqrt{2\delta}} SO(G).$$

Equality holds for both if and only if G is regular.

Proof. Let G be a graph with minimum degree $\delta \geq 2$. Then $\sqrt{2\delta-2} > 0$. Let $u, v \in V(G)$ such that $uv \in E(G)$. Now, Lemma 2 asserts that $d_u + d_v - 2 > 0$. Hence,

$$\begin{aligned}
SO(G) &= \sum_{uv\in E(G)} \sqrt{d_u^2+d_v^2} \\
&\leq \sqrt{2}\Delta \sum_{uv\in E(G)} \sqrt{\frac{d_u+d_v-2}{d_u+d_v}} \cdot \sqrt{\frac{d_u+d_v}{d_u+d_v-2}} \\
&\leq \frac{\sqrt{2}\Delta\sqrt{2\Delta}}{\sqrt{2\delta-2}} \sum_{uv\in E(G)} \sqrt{\frac{d_u+d_v-2}{d_u+d_v}} \\
&= \frac{\Delta\sqrt{2\Delta}}{\sqrt{\delta-1}} ABS(G).
\end{aligned}$$

Thus,

$$ABS(G) \geq \frac{\sqrt{\delta - 1}}{\Delta\sqrt{2\Delta}} SO(G). \tag{4.5}$$

If equality holds in 4.5, then $\sqrt{2}\Delta = \sqrt{d_u^2 + d_v^2}$, which means that $d_u = d_v = \Delta$ for every $uv \in E(G)$. This further implies that $d_u = \Delta = \delta$ for every $u \in V(G)$ and hence G is regular. Conversely, if G is regular then it can be easily seen that equality holds.

For the second inequality, we have

$$\begin{aligned}
SO(G) &= \sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2} \\
&\geq \sqrt{2}\delta \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u + d_v}} \cdot \sqrt{\frac{d_u + d_v}{d_u + d_v - 2}} \\
&\geq \frac{\sqrt{2}\delta\sqrt{2\delta}}{\sqrt{2\Delta - 2}} \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u + d_v}} \\
&= \frac{\delta\sqrt{2\delta}}{\sqrt{\Delta - 1}} ABS(G).
\end{aligned}$$

Hence,

$$ABS(G) \leq \frac{\sqrt{\Delta - 1}}{\delta\sqrt{2\delta}} SO(G). \tag{4.6}$$

If equality holds in 4.6, then $\sqrt{2}\delta = \sqrt{d_u^2 + d_v^2}$, which means that $d_u = d_v = \delta$ fro every $uv \in E(G)$. Hence $d_u = \delta = \Delta$ for every $u \in V(G)$ and conclude that G is regular. Conversely, if G is regular then it can be easily seen that equality holds. □

Corollary 3. *Let G be a connected graph with minimum degree $\delta \geq 2$ and maximum degree Δ. If G is not regular, then*

$$ABS(G) < \frac{\sqrt{\Delta - 1}}{\delta\sqrt{2\delta}} SO(G).$$

Comparing with Theorem 9, we have

$$\frac{\sqrt{\Delta - 1}}{\delta\sqrt{2\delta}} SO(G) \leq \frac{1}{\sqrt{2\delta}} SO(G),$$

which implies that $\sqrt{\Delta - 1} \leq \sqrt{\delta}$. Thus, if G is a graph with $\Delta = \delta + 1$, then Corollary 3 is better than Theorem 9.

Next, we will look at the relationship between the ABS index with the modified Sombor index. Our result is established in the following theorem.

Theorem 11. *Let G be a connected graph of order at least three with minimum degree δ and maximum degree Δ. Then*

$$\frac{\delta}{\sqrt{\Delta}} SO^m(G) \leq ABS(G) < \sqrt{2}\Delta SO^m(G).$$

Proof. Let $u, v \in V(G)$ such that $uv \in E(G)$. Then by Lemma 2, $d_u + d_v - 2 \geq 1$. Thus,

$$\begin{aligned}
SO^m(G) &= \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u^2 + d_v^2}} \\
&\leq \frac{1}{\sqrt{2}\delta} \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v}{d_u + d_v}} \\
&\leq \frac{\sqrt{2\Delta}}{\sqrt{2}\delta} \sum_{uv \in E(G)} \sqrt{\frac{1}{d_u + d_v}} \\
&\leq \frac{\sqrt{\Delta}}{\delta} \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u + d_v}} \\
&= \frac{\sqrt{\Delta}}{\delta} ABS(G).
\end{aligned}$$

Hence,

$$ABS(G) \geq \frac{\delta}{\sqrt{\Delta}} SO^m(G). \tag{4.7}$$

Moreover,

$$\begin{aligned}
SO^m(G) &= \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u^2 + d_v^2}} \\
&\geq \frac{1}{\sqrt{2}\Delta} \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v}{d_u + d_v}} \\
&> \frac{1}{\sqrt{2}\Delta} \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u + d_v}} \\
&= \frac{1}{\sqrt{2}\Delta} ABS(G).
\end{aligned}$$

Hence,

$$ABS(G) < \sqrt{2}\Delta SO^m(G). \tag{4.8}$$

Combining Inequalities 4.7 and 4.8 gives the desired result. □

The above bounds are not sharp since in the lower bound, when $\sqrt{2}\delta = \sqrt{d_u^2 + d_v^2}$, then G will be regular and in that case $1 < d_u + d_v - 2$. For the upper bound, we use the fact that $d_u + d_v > d_u + d_v - 2$.
We now consider graphs without pendant vertices.

Theorem 12. *Let G be a connected graph with minimum degree $\delta \geq 2$ and maximum degree Δ. Then*

$$\frac{\delta\sqrt{2\delta - 2}}{\sqrt{\Delta}} SO^m(G) \leq ABS(G) \leq \frac{\Delta\sqrt{2\Delta - 2}}{\sqrt{\delta}} SO^m(G).$$

Equality holds for both if and only if G is regular.

Proof. Let G be a graph with minimum degree $\delta \geq 2$. Then $\sqrt{2\delta - 2} > 0$. Let $u, v \in V(G)$ such that $uv \in E(G)$. Now, Lemma 2 asserts that $d_u + d_v - 2 > 0$. Hence,

$$\begin{aligned} SO^m(G) &= \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u^2 + d_v^2}} \\ &\leq \frac{1}{\sqrt{2}\delta} \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u + d_v}} \cdot \sqrt{\frac{d_u + d_v}{d_u + d_v - 2}} \\ &\leq \frac{\sqrt{2\Delta}}{\sqrt{2}\delta\sqrt{2\delta - 2}} \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u + d_v}} \\ &= \frac{\sqrt{\Delta}}{\delta\sqrt{2\delta - 2}} ABS(G). \end{aligned}$$

Thus,

$$ABS(G) \geq \frac{\delta\sqrt{2\delta - 2}}{\sqrt{\Delta}} SO^m(G). \tag{4.9}$$

Suppose equality holds in 4.9. Then $\sqrt{2}\delta = \sqrt{d_u^2 + d_v^2}$ which leads to $d_u = d_v = \delta$ for every $uv \in E(G)$. Consequently, $d_u = \delta = \Delta$ for every $u \in V(G)$ and thus G is regular. Conversely, if G is regular then it can be easily seen that equality holds.
We will show the second inequality. Now,

$$\begin{aligned} SO^m(G) &= \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u^2 + d_v^2}} \\ &\geq \frac{1}{\sqrt{2}\Delta} \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u + d_v}} \cdot \sqrt{\frac{d_u + d_v}{d_u + d_v - 2}} \\ &\geq \frac{\sqrt{2\delta}}{\sqrt{2}\Delta\sqrt{2\Delta - 2}} \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u + d_v}} \end{aligned}$$

$$= \frac{\sqrt{\delta}}{\Delta\sqrt{2\Delta - 2}} ABS(G).$$

Thus,

$$ABS(G) \leq \frac{\Delta\sqrt{2\Delta - 2}}{\sqrt{\delta}} SO^m(G). \tag{4.10}$$

If equality holds in 4.10, then $\sqrt{2}\Delta = \sqrt{d_u^2 + d_v^2}$ which leads to $d_u = d_v = \Delta$ for every $uv \in E(G)$. Consequently, $d_u = \Delta = \delta$ for every $u \in V(G)$ and therefore G is regular. Conversely, if G is regular then it can be easily seen that equality holds. $\square$

Corollary 4. *Let G be a connected graph with minimum degree $\delta \geq 2$ and maximum degree Δ. If G is not regular, then*

$$ABS(G) < \frac{\Delta\sqrt{2\Delta - 2}}{\sqrt{\delta}} SO^m(G).$$

Comparing with Theorem 11, we have

$$\frac{\Delta\sqrt{2\Delta - 2}}{\sqrt{\delta}} SO^m(G) \leq \sqrt{2}\Delta SO^m(G),$$

which implies that $\sqrt{\Delta - 1} \leq \sqrt{\delta}$. Thus, if G is a graph with $\Delta = \delta + 1$, then Corollary 4 is better than Theorem 11.

It is interesting to note that if G has no pendant vertices, then for any pair of adjacent vertices u and v in G, $d_u d_v \geq d_u + d_v$. Hence, in this case, $ABC(G) \leq ABS(G)$. Moreover, $ABS(G) = ABC(G)$ if and only if G is 2-regular.

5. Conclusion

In this work, we have shown some bounds of ABS index in terms of ABC index, first Zagreb index, second Zagreb index, harmonic index, Sombor index, and modified Sombor index with minimum degree and maximum degree of a graph. For connected graphs of order at least 3, we have established sharp bounds of ABS index with ABC index in terms of minimum degree δ and maximum degree Δ. For relations between ABS index with other TIs, we were able to establish extremal results for classes of graph without pendant vertices. We have shown that regular graphs correspond to extremal graphs. Moreover, for graphs that are not regular, better bounds are attained when $\Delta = \delta + 1$.

Readers may consider working on extremal results and bounds of ABS index with other distance-based topological indices.

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