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Research Article

A graph-theoretic proof of Cramer's rule

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Abstract: This note contains a new combinatorial proof of Cramer's rule based on the Gessel-Viennot-Lindström Lemma.

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AMS Subject classification: 05A19, 05A05, 05C30, 05C38

1. Introduction

This paper presents a combinatorial proof of Cramer's rule. Such a proof offers a greater understanding of the underlying reasons for the validity of the result, rather than merely explaining the methodology [3, 9, 11]. Numerous concise proofs of Cramer's rule are available on Wikipedia and its associated references [5, 8, 10].
The rule was first published by Gabriel Cramer (1704–1752) in Appendix I of his *Introduction à l'analyse des lignes courbes algébriques* [6], pages 657-659. While Theorem 1.1 is sometimes misattributed-Boyer, Hedman, and others suggest that Colin Maclaurin (1698-1746) was already aware of it by 1729 and included it in his posthumous *Treatise of Algebra* (1748) [4, 7]. As a matter of fact, both Cramer and Maclaurin explicitly solved the 3×3 case, expressing each unknown as a ratio of two sums of six terms. They then sketched how these formulas extend to larger systems; neither, however, used the modern determinant concept, which emerged only in 1771 with Vandermonde [12].
Furthermore, as observed in [2], Maclaurin's method for assigning signs to each summand is flawed. By contrast, Cramer's approach-determining signs via the parity of the associated permutation is correct. Hence, the rule rightfully bears his name. In 1841, Carl Gustav Jacobi (1804-1851) introduced the first formal proof of Cramer's rule in his paper [8]. However, this is not the earliest known demonstration; in 1825, Heinrich Ferdinand Scherk (1798-1885) published a 17-page inductive proof on the

number of unknowns, outlined in [10]. Recently, Doron Zeilberger provided a fully combinatorial proof in [13]. This paper presents a combinatorial proof of Cramer's rule utilizing the Gessel-Viennot-Lindström Lemma.

Let Γ represent a weighted, acyclic directed graph. Consider P_1 as a directed path from vertex X to vertex Y within Γ, and P_2 as another path extending from Y to Z. The concatenation of the two paths, P_1 and P_2, is denoted as $P_1 \bigodot P_2$, which traverses from vertex X to vertex Z. A directed edge is represented by the initial vertex U and the terminal vertex V as $\overrightarrow{UV}$. Let A and B be two fixed subsets of $V(\Gamma)$ both of cardinality n respectively called set of *initial vertices* and set of *final vertices*, where $V(\Gamma)$ is the vertex set of the graph Γ. To these sets, we associate the *path matrix* $M_{AB} = (m_{ij})_{n\times n}$, where $m_{ij} = \sum\limits_{P:A_i\to B_j} w(P)$, with $w(P)$ representing the product of the weights of all edges in the path P. The notation $P : A_i \to B_j$ signifies a directed path that initiates at the vertex A_i and concludes at the vertex B_j. A *path system* $\mathcal{P}$ from A to B consists of a permutation σ and n paths $P_i : A_i \to B_{\sigma(i)}$, with $\text{sgn}(\mathcal{P}) = \text{sgn}(\sigma)$. The *weight* of $\mathcal{P}$ is defined as $w(\mathcal{P}) = \prod_{i=1}^{n} w(P_i)$. We refer to the path system as *vertex-disjoint* if no two paths share a common vertex. Let $VD(\Gamma)$ denote the collection of vertex-disjoint path systems. It is straightforward to observe that $\det(M_{AB}) = \sum\limits_{\mathcal{P}} \text{sgn}(\mathcal{P})w(\mathcal{P})$. However, the Gessel-Viennot-Lindström Lemma provides additional insights.

Lemma 1 (Gessel-Viennot-Lindström [1]). *Let Γ be a weighted, acyclic digraph and M_{AB} be the path matrix of Γ. Then $\det(M_{AB}) = \sum\limits_{\mathcal{P}\in VD(\Gamma)} sgn(\mathcal{P})w(\mathcal{P})$.*

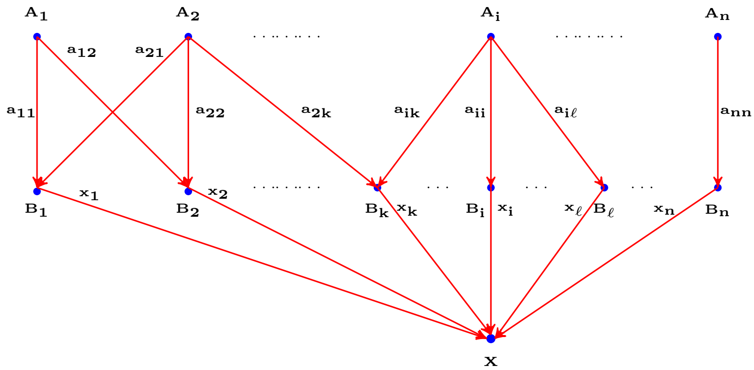
Note that the sum is 0 if no path system exists from A to B. We now present an almost visual demonstration of Cramer's rule for solving a system of linear equations. Consider the following system of equations:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad \vdots \qquad \ddots \qquad \vdots \qquad &\;\; \vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}$$

This system can be expressed in matrix form as $AX = B$, where $A = (a_{ij})_{n\times n}$ represents the $n \times n$ matrix, $X = (x_1, \cdots, x_n)^T$ is the column vector of the unknowns, and $B = (b_1, \cdots, b_n)^T$ is the column vector of constants. Let A_i (for $i = 1, \cdots, n$) denote the matrix obtained by substituting the i-th column of A with the column vector B.

Theorem 1 (Cramer's rule [6]). *For the system $AX = B$, consisting of n linear* *equations with n unknowns and* $\det(A) \neq 0$*, Cramer's rule states that*

$$x_i = \frac{\det(A_i)}{\det(A)}, \quad (i = 1, \cdots, n).$$



Proof. Our objective is to demonstrate that $x_i \det(A) = \det(A_i)$ for every $i \in [n]$. Consider the directed graph Γ illustrated in Figure 1. The graph Γ is a weighted digraph having directed edge from A_i to B_j with weight a_{ij} for each $i, j \in [n]$ and the weight of the edge $\overrightarrow{B_i X}$ is x_i, for each $i \in [n]$. Let $A = \{A_1, \cdots, A_n\}$ represent the initial set of vertices, while $B = \{B_1, \cdots, B_{i-1}, X, B_{i+1}, \cdots, B_n\}$ denotes the terminal set of vertices in Γ. The weight associated with the edge connecting vertex A_i to vertex B_j in the graph Γ is denoted as a_{ij}. Furthermore, the weight of the edge from vertex B_i to vertex X is represented by x_i. It is important to note that

$$\sum_{P: A_j \to X} w(P) = \sum_{k=1}^{n} a_{jk} x_k, \text{ for all } j \in [n].$$

Consequently, the i-th column of the path matrix M_{AB} in the graph Γ can be expressed as follows:

$$\begin{pmatrix} \sum\limits_{k=1}^{n} a_{1k} x_k \\ \sum\limits_{k=1}^{n} a_{2k} x_k \\ \vdots \\ \sum\limits_{k=1}^{n} a_{nk} x_k \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

Furthermore, it is evident that the column $C_j, j \in [n] \setminus \{i\}$ of the path matrix M_{AB} is represented as:

$$\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix}.$$

Thus, the path matrix M_{AB} can be formulated as:

$$\begin{pmatrix} a_{11} & \cdots & \sum_{k=1}^{n} a_{1k}x_k & \cdots & a_{1n} \\ a_{21} & \cdots & \sum_{k=1}^{n} a_{2k}x_k & \cdots & a_{2n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{(n-1)1} & \cdots & \sum_{k=1}^{n} a_{(n-1)k}x_k & \cdots & a_{(n-1)n} \\ a_{n1} & \cdots & \sum_{k=1}^{n} a_{nk}x_k & \cdots & a_{nn} \end{pmatrix} = A_i.$$

According to Lemma 1, it follows that $\det(A_i) = \sum_{\mathcal{P} \in VD(\Gamma)} \operatorname{sgn}(\mathcal{P}) w(\mathcal{P})$. From Figure 1, it is evident that the set $\mathcal{P} = \{P_1, \cdots, P_n\}$ constitutes a vertex disjoint path system in the induced graph $\Gamma \setminus \{X\}$, with the initial vertex set being $\{A_1, \cdots, A_n\}$ and the terminal vertex set being $\{B_1, \cdots, B_n\}$ if and only if $\bar{\mathcal{P}} = \{P_1, \cdots, P_{i-1}, P_i \odot \overrightarrow{B_i X}, P_{i+1}, \cdots, P_n\}$ forms a vertex disjoint path system in the graph Γ, where $A = \{A_1, \cdots, A_n\}$ and $B = \{B_1, \cdots, B_{i-1}, X, B_{i+1}, \cdots, B_n\}$ represent the initial and terminal vertex sets of Γ, respectively. Furthermore, it is important to observe that $w(\bar{\mathcal{P}}) = x_i w(\mathcal{P})$ and $\operatorname{sgn}(\bar{\mathcal{P}}) = \operatorname{sgn}(\mathcal{P})$. Consequently, we have

$$\left(\sum_{\mathcal{P} \in VD(\Gamma)} \operatorname{sgn}(\mathcal{P}) w(\mathcal{P}) \right) = x_i \left(\sum_{\mathcal{P} \in VD(\Gamma \setminus \{X\})} \operatorname{sgn}(\mathcal{P}) w(\mathcal{P}) \right)$$
$$\Rightarrow \det(A_i) = x_i \det(A).$$

This concludes the proof.

Example 1. Here we explain the idea of the proof for the case $n = 3$. Consider the graph Γ in Figure 1.

We aim to demonstrate that $\det(A_1) = x_1 \det(A)$. Let us define the sets $A = \{A_1, A_2, A_3\}$ and $B = \{X, B_2, B_3\}$ as the initial and terminal sets of vertices in

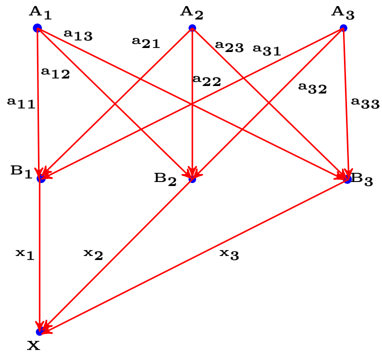


Figure 1. Γ is a weighted digraph having edge weight a_{ij} for each directed edge A_i to B_j and x_i for each edge B_i to X.

the graph Γ, respectively. It is straightforward to observe that $w(\bar{\mathcal{P}}) = x_1 w(\mathcal{P})$ and $\operatorname{sgn}(\bar{\mathcal{P}}) = \operatorname{sgn}(\mathcal{P})$, where $\bar{\mathcal{P}}$ and $\mathcal{P}$ represent vertex-disjoint path systems in the graphs Γ and $\Gamma \setminus \{X\}$, respectively. Consequently, we have the following relationship:

$$\left(\sum_{\mathcal{P}\in VD(\Gamma)} \operatorname{sgn}(\mathcal{P})w(\mathcal{P})\right) = x_1 \left(\sum_{\mathcal{P}\in VD(\Gamma\setminus\{X\})} \operatorname{sgn}(\mathcal{P})w(\mathcal{P})\right)$$
$$\Rightarrow \det(A_1) = x_1 \det(A).$$

□

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