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Research Article

Proper D-lucky edge labeling of human chain graphs

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Abstract: This paper determines the proper D-Lucky edge numbers for human chain graphs, circular human chain graphs, strong human chain graphs, and weak human chain graphs.

Keywords: proper D-Lucky edge labeling, human chain graph, circular human chain graph, strong human chain graph, weak human chain graph.

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1. Introduction

In graph theory, graph labeling is one of the best-studied [2]. Lucky edge labeling was studied by many authors, for instance, see [4]. Also, Mirka Miller [3] gave proof for calculating the D-Lucky labeling of a graph. Further, the estimation of D-Lucky edge labeling and proof given by Rajini Ram et.al. see [4]. The concept of a human chain, circular human chain, and strong and weak human chain graph was explained by many authors, for instance, see [1, 5]. Here, we have determined the proper D-Lucky edge number of a human chain, circular human chain, and strong and weak human chain graphs.

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Definition 1. A human chain graph HC is obtained by a path $u_1, u_2, \dots u_{2n+1}$, $n \in N$ joining a cycle of length m and Y-tree, a connected graph that contains no cycle $Y_{m+1}, m \geq 3$ to each u_{2i} for $1 \leq i \leq n$. The vertices of the cycle and Y-tree are $v_1, v_2, \dots v_{(m-1)n}$ and $w_1, w_2, \dots w_{mn}$ respectively.
The vertex and edge sets of HC as follows:
$V(HC) = \{u_i, v_j, w_k : i \leq 2n+1, 1 \leq j \leq (m-1)n, 1 \leq k \leq mn\}$ and $|V| = 2mn + n + 1$
$E(HC) = \{u_i u_{i+1} : 1 \leq i \leq 2n\} \cup \{u_{2i} w_{m(i-1)+1}; u_{2i} v_{(m-1)i}; u_{2i} v_{(m-1)(i-1)+1}; w_{mi} w_{(mi-2)}; 1 \leq i \leq n\} \cup \{w_{mi+j} w_{(mi+j+1)}; v_{(m-1)i+j} v_{(m-1)i+j+1}, 0 < i < n-1, 1 < j < m-2\}$ and $|E| = 2mn + 2n$.

Definition 2. A circular human chain graph CHC is obtained from a cycle $u_1, u_2, \dots u_{2n}$, $n > 2$ by joining a cycle of length m and Y- tree, a connected graph that contains no cycle Y_{m+1}, $m \geq 3$ to each u_{2i}, $1 \leq i \leq n$. The vertices of the cycle and Y-tree Y_{m+1} are $v_1, v_2, \dots v_{(m-1)n}$ and $w_1, w_2, \dots w_{mn}$ respectively. The vertex and edge sets of $CHC_{n,m}$ is $V(CHC) = \{u_i, v_j, w_k : 1 \leq i \leq 2n, 1 \leq j \leq (m-1)n, 1 \leq k \leq mn\}$ with $|V| = 2mn + n$ and $E(CHC) = \{u_i u_{i+1} : 1 \leq i \leq 2n-1\} \cup \{u_1 u_{2n}\} \cup \{u_{2i} w_{m(i-1)+1}; u_{2i} v_{(m-1)i}; u_{2i} v_{(m-1)(i-1)+1}; w_{mi} w_{(mi-2)} : 1 \leq i \leq n\} \cup \{w_{mi+j} w_{(mi+j+1)}; v_{(m-1)i+j} v_{(m-i)+j+1}: 0 < i < n-1, 1 < j < m-2\}$ and
$|E| = 2mn + 2n$.

Definition 3. The Strong human chain graph SHC, $n > 1$, and $m \geq 3$ is obtained from Human chain graph by joining w_{mi} and $w_{m(i+1)-1} : 1 \leq i \leq n-1$ with common vertices in Y-tree. The vertices of $SHC_{n,m}$ are $u_1, u_2, \dots, u_{2n+1}$, $v_1, v_2, \dots, v_{(m-1)n}$, $w_1, w_2, \dots, w_{(m-1)n+1}$ and edges of SHC are
$\{u_i, u_{i+1} : 1 \leq i \leq 2n\} \cup \{u_{2i} w_{m(i-1)+1}; u_{2i} v_{(m-1)i}; u_{2i} v_{(m-1)(i-1)+1} : 1 \leq i \leq n\} \cup \{v_{(m-1)i+j} v_{(m-1)i+j+1} : 0 \leq i \leq n-1, 1 \leq j \leq m-2\} \cup \{w_{mi-m+j} w_{mi-m+j+1} : 1 \leq i \leq n, 1 \leq j \leq m-3\} \cup \{w_{(m-1)i+1} w_{(m-1)i+m-1} : 1 \leq i \leq n-1\} \cup \{w_{m-1} w_{m-2}\}$.

Definition 4. A Weak human chain graph WHC $n \geq 1, m \geq 3$ is obtained from a path $u_1, u_2, \dots u_{n+1}$ by joining the cycle of length m and Y-tree(Y_{m+1}) to each $u_i : 1 \leq i \leq n$. The vertices and edges of WHC as follows $V(WHC) = \{u_1, u_2, \dots u_{n+1}, v_1, v_2, \dots v_{(m-1)n}, w_1, w_2, \dots w_{mn}\}$ and
$E(WHC) = \{u_i, u_{i+1} : 1 \leq i \leq n\} \cup \{u_i w_{m(i-1)+1}; u_i v_{(m-1)(i-1)+1}; u_i v_{(m-1)i} : 1 \leq i \leq n\} \cup \{v_{(m-1)i+j} v_{(m-1)i+j+1} : 0 \leq i \leq n-1\} \cup \{w_{mi+j} w_{mi+j+1} : 0 \leq i \leq n-1, 1 \leq j \leq m-2\} \cup \{w_{mi} w_{mi-2} : 1 \leq i \leq n\}$.

Definition 5. If $l : V(G) \to N$ is a vertex labeling of a graph G, then the labeling of an edge $uv \in E(G)$ is $l(uv) = l(u) + l(v) + d(u) + d(v)$, where $d(u)$ and $d(v)$ are the degrees and $l(u)$ and $l(v)$ are the labeling of vertices u and v respectively. This labeling is called D-Lucky edge labeling if every pair of adjacent edges are distinct. The D-Lucky edge number is denoted by $\eta_{dle}(G)$, is the least positive integer k for which the graph G has D-Lucky edge labeling with the labels $\{1, 2, 3, \dots, k\}$. A D-Lucky edge labeling is called proper if $l(u) \neq l(v)$ for every pair of adjacent vertices u and v. The proper D-Lucky edge number of a graph G is denoted by $\eta_{pdle}(G)$, is the least positive integer k for which G has a proper D-Lucky edge labeling with the labels $\{1, 2, \dots, k\}$.

2. Main Results

Theorem 1. *For $m \geq 4$, $\eta_{pdle}(HC) = \begin{cases} 4, & n = 1 \\ 5, & n = 2 \\ 6, & n \geq 3 \end{cases}$.*

Proof. According to the D-Lucky edge concept, let G be any graph, let $x \in V(G)$ be adjacent to the same degree vertices $y_1, y_2, \ldots y_n$, then the vertices $y_1, y_2, \ldots y_n$ must required distinct n labels. Also, according to the proper D-Lucky edge concept, let $x \in V(G)$ be adjacent to the same degree vertices $y_1, y_2, \ldots y_n$, then the vertices $y_1, y_2, \ldots y_n$ must require distinct labels to each other and also distinct to the common vertex x. So $n+1$ distinct labels are needed. In the Human chain graph, for $1 \leq i \leq n$, the vertices $u_1, u_{2n+1}, w_{mi}, w_{mi-1}$ are 1-degree, the vertices w_{mi-2} are 3-degree, u_{2i} are 5-degree vertices. The remaining vertices are 2 degrees. Then $N(u_{2i}) = \{u_{2i-1}, u_{2i+1}, w_{m(i-1)+1}, v_{(m-1)(i-1)+1}, v_{(m-1)i}\}$. We consider three cases.

Case (i) $n = 1$.
In this case, the vertices are $u_1, u_2, u_3, w_1, w_2, \ldots, w_m, v_1, v_2, \ldots, v_{(m-1)}$. We can observe that $N(u_2) = \{u_1, u_3, w_1, v_1, v_{(m-1)}\}$. Among these vertices $w_1, v_1, v_{(m-1)}$ are 2-degree vertices. Therefore $l(v_1) \neq l(v_{m-1}) \neq l(w_1) \neq l(u_2)$. Let $l(v_1) = 1$, $l(v_{m-1}) = 3$, $l(w_1) = 4$, and $l(u_2) = 2$. The 1-degree vertices u_1 and u_3 are adjacent to u_2. Therefore $l(u_1) \neq l(u_2) \neq l(u_3)$. Already $l(u_2) = 2$. So $l(u_1) = 1$ and $l(u_3) = 3$. Clearly, we need $\eta_{pdle}(HC) \geq 4$. Now,

$$l(v_2, \ldots v_{(m-2)}) = \begin{cases} 2,3,1,2,3,1\ldots,2,4,1, & \text{for } m \equiv 0 \pmod 3 \\ 2,3,1,2,3,1,\ldots,1,2, & \text{for } m \equiv 1 \pmod 3 \\ 2,3,1,2,3,1,\ldots,2,4, & \text{for } m \equiv 2 \pmod 3. \end{cases}$$

The other vertices of $w_2, w_3, \ldots, w_{m-3}$ are in the sequence 1, 2, 3, 1, 2, 3, ... For the remaining vertices set $l(w_{m-2}, w_{m-1}, \ldots, w_m) = \begin{cases} 1,2,4, & \text{for } m \equiv 0 \pmod 3 \\ 2,1,3, & \text{for } m \equiv 1 \pmod 3 \\ 3,1,4, & \text{for } m \equiv 2 \pmod 3. \end{cases}$

In this case, all the vertices in the graph can be labeled within 4 labels and so the Proper D-Lucky edge labeling number requires at most 4.

Case (ii) $n = 2$.
In this case, the vertices are $u_1, u_2, \ldots u_5, w_1, w_2, \ldots w_m, w_{m+1}, \ldots, w_{2m}, v_1, v_2, \ldots, v_m$, v_{m+1}, $v_{m+2}, \ldots, v_{2(m-1)}$. The vertex u_2 is adjacent to 4 number of 2-degree vertices v_1, v_{m-1}, u_3, and w_1. Therefore $l(v_1) \neq l(v_{m-1}) \neq l(u_3) \neq l(w_1) \neq l(u_2)$. So let $l(v_1) = 1$, $l(u_2) = 2$, $l(v_{m-1}) = 3$, $l(u_3) = 4$, $l(w_1) = 5$. Similarly the vertex u_4, is adjacent to 4 number of 2-degree vertices v_m, v_{m-2}, u_3, and w_{m+1}. Therefore $l(u_4) \neq l(v_m) \neq l(v_{2(m-1)}) \neq l(w_{m+1}) \neq l(u_3)$. So let $l(u_4) = 1$, $l(v_m) = 2$, $l(v_{2(m-1})) = 3$, $l(u_3) = 4$, $l(w_{m+1}) = 5$. Clearly, we need $\eta_{pdle}(HC) \geq 5$. Now, let us allocate the labels of vertices as follows
Let $l(u_1) = 1$, $l(u_5) = 2$,

$$l(v_2, v_3 \dots v_{(m-2)}) = \begin{cases} 2,3,1,2,3,1 \dots 2,4,1, & \text{for } m \equiv 0 \pmod 3 \\ 2,3,1,2,3,1, \dots 1,2, & \text{for } m \equiv 1 \pmod 3 \\ 2,3,1,2,3,1, \dots, 2,4, & \text{for } m \equiv 2 \pmod 3), \end{cases}$$

$$l(v_{m+1}, v_{m+2} \dots v_{2m-4}, v_{2m-3}) = \begin{cases} 1,3,2,1,3,2 \dots 1,4,2, & \text{if for } m \equiv 0 \pmod 3 \\ 1,3,2,1,3,2, \dots 1,3,2, & \text{if for } m \equiv 1 \pmod 3 \\ 1,3,2,1,3,2, \dots, 4,1, & \text{if for } m \equiv 2 \pmod 3 \end{cases}$$

The vertices $w_2, w_3, \dots w_{m-3}$ are in the sequence 1, 2, 3, 1, 2, 3, ... Also, the vertices $w_{m+2}, w_{m+3}, \dots w_{2m-3}$ are in the sequence $1, 2, 3, 1, 2, 3, \dots$. For the remaining set of vertices set

$$l(w_{m-2}, w_{m-1}, \dots w_m) = l(w_{2m-2}, w_{2m-1}, \dots w_{2m}) = \begin{cases} 1,2,4, & \text{for } m \equiv 0 \pmod 3 \\ 2,1,3, & \text{for } m \equiv 1 \pmod 3 \\ 3,1,4, & \text{for } m \equiv 2 \pmod 3 \end{cases}$$

respectively. However, in this case, the minimum label required for proper D-Lucky edge labeling number is 5. Clearly, $\eta_{pdle}(HC) = 5$ when $n = 2$.

Case (iii). $n \geq 3$.
In this case, the vertices are $u_1, u_2 \dots u_{2n}, u_{2n+1}, w_1, w_2, \dots w_m, w_{m+1}, \dots w_{mn}, v_1, v_2, \dots v_m, v_{m+1}, v_{n(m-1)+2}, \dots v_{2(m-1)}$. Among these vertices set $l(u_1) = 1$, $l(u_{2n+1}) = 1$ for odd n and $l(u_{2n+1}) = 2$ for even n. The vertices u_{2i}, $2 \leq i \leq n-1$ have 5 adjacent same 2-degree vertices. The vertices $[N(u_{2i})]$, $2 \leq i \leq n-1$ required a minimum number of 6 distinct labels $\{1, 2, 3, 4, 5, 6\}$. Clearly, we need $\eta_{pdle}(HC) \geq 6$. Let the labeling pattern be as follows. The vertices $u_2, u_4 \dots u_{2n}$ are labeled in the sequence $2, 1, 2, 1$. The vertices $u_3, u_5 \dots u_{2n-1}$ are labeled in the sequence $4, 5, 4, 5 \dots$. The vertices $w_{m(i-1)+2+j}$ for $1 \leq i \leq n$ and $0 \leq j \leq m-5$, are labeled in the sequence $1, 2, 3, 1, 2, 3 \dots$. For $1 \leq i \leq n$ set

$$l(w_{im-2,} w_{im-1}, w_{im}) = \begin{cases} 1,2,4, & \text{for } m \equiv 0 \pmod 3 \\ 2,1,3, & \text{for } m \equiv 1 \pmod 3 \\ 3,1,4, & \text{for } m \equiv 2 \pmod 3). \end{cases}$$

Let $l(w_1) = l(w_{m(n-1)+1}) = 5$, $l(w_{im+1}) = 6$ for $1 \leq i \leq n-2$, $l(u_{2i}) = 1$, $i = 2, 4, 6, \dots$ and $l(u_{2i}) = 2$, $i = 1, 3, 5, 7 \dots$. For the remaining vertices, the labeling pattern is divided into two cases.
For $i = 2, 4, 6, \dots$ and $j = 1, 2, \dots m-3$, the labeling pattern as follows: $l(v_{(m-1)(i-1)+1}) = 2$, $l(v_{(m-1)i}) = 3$, and

$$l(v_{(m-1)(i-1)+j+1}) = \begin{cases} 1,3,2,1,3,2, \dots 1,4,2, & \text{if for } m \equiv 0 \pmod 3 \\ 1,3,2,1,3,2, \dots 1,3,2 & \text{if for } m \equiv 1 \pmod 3 \\ 1,3,2,1,3,2, \dots, 4,1 & \text{if for } m \equiv 2 \pmod 3). \end{cases}$$

For $i = 1, 3, 5, \dots$ and $j = 1, 2, \dots m-3$, the labeling pattern as follows: $l(v_{(m-1)(i-1)+1}) = 1$, $l(v_{(m-1)i}) = 3$, and

$$l(v_{(m-1)(i-1)+j+1}) = \begin{cases} 2,3,1,2,3,1,\ldots 2,4,1, & \text{for } m \equiv 0 \pmod 3 \\ 2,3,1,2,3,1,\ldots 1,2 & \text{for } m \equiv 1 \pmod 3 \\ 2,3,1,2,3,1,\ldots,2,4, & \text{for } m \equiv 2 \pmod 3). \end{cases}$$

Clearly, we required proper D-Lucky edge labeling number is $\eta_{pdle}(HC) = 6$. Hence, the Proper D-Lucky edge labeling number of the Human chain graph if $m \geq 4$, is $\eta_{pdle}(HC) = 4$ if $n = 1$, $\eta_{pdle}(HC) = 5$ if $n = 2$ and $\eta_{pdle}(HC) = 6$ if $n \geq 3$. □

Example 1. Proper D-Lucky edge labeled Human chain graph for $n = 5, m = 4$ and $k = 4$ is shown in Figure 1.

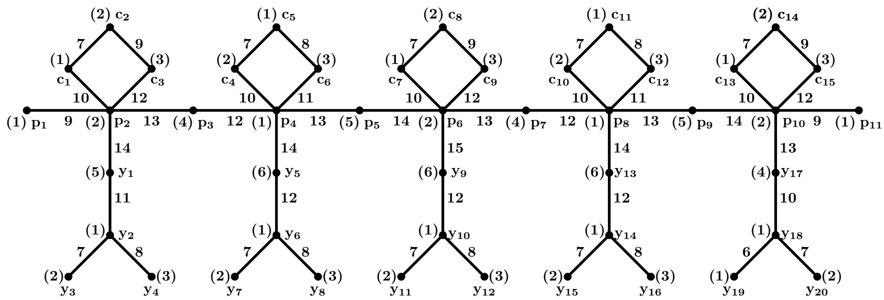

Figure 1. $\eta_{\mathbf{pdle}}(\mathbf{HC}) = \mathbf{6}$

Theorem 2. *For $m \geq 4$, the Proper D-Lucky edge labeling of strong Human chain graph is $\eta_{pdle}(SHC) = 5$ for $n = 2$ and $\eta_{pdle}(SHC) = 6$ for $n \geq 3$.*

Proof. Let (SHC) be the strong human chain graph. For our convenience, let us take $A = \{u_1, u_2, \ldots u_{2n+1}, n \in N\}$, $B = \{v_1, v_2, \ldots v_{(m-1)n}\}$, $C = \{w_1, w_2, \ldots w_{(m-1)n+1}\}$. We consider two cases.

Case (i) $n = 2$.

In this case, the vertices $u_1, u_2, \ldots u_5, w_1, w_2 \ldots w_m \ldots w_{(m-1)n+1}, v_1, v_2, \ldots v_m, v_{m+1}, v_{m+2}, \ldots v_{2(m-1)}$. The vertex u_2, is adjacent to 4 number of 2-degree vertices v_1, v_{m-1}, u_3, and w_1. Therefore $l(v_1) \neq l(v_{m-1}) \neq l(u_3) \neq l(w_1) \neq l(u_2)$. So let $l(v_1) = 1$, $l(u_2) = 2$, $l(v_{m-1}) = 3$, $l(u_3) = 4$, $l(w_1) = 5$. Similarly the vertex u_4 is adjacent to 4 numbers of 2-degree vertices v_m, v_{m-2}, u_3, and w_{m+1}. Therefore $l(u_4) \neq l(v_m) \neq l(v_{2(m-1)}) \neq l(w_{m+1}) \neq l(u_3)$. So let $l(u_4) = 1$, $l(v_m) = 2$, $l(v_{2(m-1)}) = 3$, $l(u_3) = 4$, $l(w_{m+1}) = 5$. Clearly, we need $\eta_{pdle}(SHC) \geq 5$. Now, let us allocate the labels of vertices as follows

Let $l(u_1) = 1, l(u_5) = 2$,

$$l(v_2, v_3 \ldots v_{(m-2)}) = \begin{cases} 2,3,1,2,3,1,\ldots 2,4,1, & \text{for } m \equiv 0 \pmod 3 \\ 2,3,1,2,3,1,\ldots 1,2 & \text{for } m \equiv 1 \pmod 3 \\ 2,3,1,2,3,1,\ldots,2,4, & \text{for } m \equiv 2 \pmod 3), \end{cases}$$

$$l(v_{m+1}, v_{m+2} \dots v_{2m-4}, v_{2m-3}) = \begin{cases} 1,3,2,1,3,2,\dots 1,4,2, & \text{if for } m \equiv 0 \pmod 3 \\ 1,3,2,1,3,2,\dots 1,3,2 & \text{if for } m \equiv 1 \pmod 3 \\ 1,3,2,1,3,2,\dots,4,1 & \text{if for } m \equiv 2 \pmod 3 \end{cases}$$

The vertices $w_2, w_3, \dots w_{m-3}$ are labeled in the sequence $1,2,3,1,2,3,\dots$. Also, the vertices $w_{m+2}, w_{m+3}, \dots w_{(m-1)n-1}$ are labeled in the sequence $1,2,3,1,2,3,\dots$. The remaining vertices are labeled in the following pattern: if $m \equiv 0 \pmod 3$, $l(w_m) = 3$, $l(w_{m-1}) = 2$, $l(w_{m-2}) = 1$, $l(w_{(m-1)n+1}) = 1$, $l(w_{2(m-1)}) = 4$; if $m \equiv 1 \pmod 3$, except $m \neq 4$, $l(w_m) = 1$, $l(w_{m-1}) = 1$, $l(w_{m-2}) = 2$, $l(w_{(m-1)n+1}) = 1$, $l(w_{2(m-1)}) = 4$; if $m \equiv 2 \pmod 3$, except $m \neq 5$, $l(w_m) = 2$, $l(w_{m-1}) = 1$, $l(w_{m-2}) = 3$, $l(w_{(m-1)n+1}) = 1$, $l(w_{2(m-1)}) = 4$; if $m = 4$, $l(w_m) = 2$, $l(w_{m-1}) = 2$, $l(w_{m-2}) = 1$, $l(w_{(m-1)n+1}) = 1$, $l(w_{2(m-1)}) = 3$; and if $m = 5$, $l(w_m) = 4$, $l(w_{m-1}) = 1$, $l(w_{m-2}) = 2$, $l(w_{(m-1)n+1}) = 1$, $l(w_{2(m-1)}) = 3$.

Case (ii) $n \geq 3$.
In this case, the vertices are $u_1, u_2 \dots u_{2n}, u_{2n+1}, v_1, v_2, \dots v_m, v_{m+1}, v_{n(m-1)+2}, \dots, v_{2(m-1)}, w_1, \dots, w_m, w_{m+1}, \dots w_{mn}, v_1$. Among these vertices, $l(u_1) = 1$, $l(u_{2n+1}) = 1$ for odd n and $l(u_{2n+1}) = 2$ for even n. The vertices u_{2i}, $2 \leq i \leq n-1$ have 5 numbers of adjacent same 2-degree vertices. The vertices $[N(u_{2i})]$, $2 \leq i \leq n-1$ required minimum number of 6 distinct labels $\{1,2,3,4,5,6\}$. Clearly, we need $\eta_{pdle}(SHC) \geq 6$. The vertices $u_2, u_4 \dots u_{2n}$ are labeled in the sequence $2,1,2,1\dots$ The vertices $u_3, u_5 \dots u_{2n-1}$ are labeled in the sequence $4,5,4,5\dots$ The vertices $w_{m(i-1)+2+j}$, $1 \leq i \leq n$ and $0 \leq j \leq m-5$ are labeled in the sequence $1,2,3,1,2,3\dots$
Let $(w_1) = l(w_{m(n-1)+1}) = 5$, $l(w_{im+1}) = 6$, $1 \leq i \leq n-2$. For the remaining vertices, the labeling format is given below:
if $m \equiv 0 \pmod 3$, $l(w_m) = 3$, $l(w_{m-1}) = 2$, $l(w_{m-2}) = 1$, $l(w_{2(m-1)i}) = 4$ for $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$, $l(w_{(m-1)(2i+1)}) = 1$ for $1 \leq i \leq \lfloor \frac{n-1}{2} \rfloor$, $l(w_{2i(m-1)+1}) = 5$ for $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$, $l(w_{(m)(2i+1)-2i}) = 3$ for $1 \leq i \leq \lfloor \frac{n-1}{2} \rfloor$;
if $m \equiv 1 \pmod 3$, $l(w_m) = 1$, $l(w_{m-1}) = 1$, $l(w_{m-2}) = 2$, $l(w_{2(m-1)i}) = 4$ for $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$, $l(w_{(m-1)(2i+1)}) = 2$ for $1 \leq i \leq \lfloor \frac{n-1}{2} \rfloor$, $l(w_{2i(m-1)+1}) = 5$ for $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$, $l(w_{(m)(2i+1)-2i}) = 1$ for $1 \leq i \leq \lfloor \frac{n-1}{2} \rfloor$;
if $m \equiv 2 \pmod 3$, $l(w_m) = 2$, $l(w_{m-1}) = 1$, $l(w_{m-2}) = 3$, $l(w_{2(m-1)i}) = 4$ for $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$, $l(w_{(m-1)(2i+1)}) = 3$ for $1 \leq i \leq \lfloor \frac{n-1}{2} \rfloor$, $l(w_{2i(m-1)+1}) = 5$ for $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$, $l(w_{(m)(2i+1)-2i}) = 2$ for $1 \leq i \leq \lfloor \frac{n-1}{2} \rfloor$.
Let $l(u_{2i}) = 1$ for even i and $l(u_{2i}) = 2$ for odd i.
For the remaining vertices the labeling pattern divided into two cases.
For $i = 2,4,6,\dots$ and $j = 1,2,\dots m-3$, $l(v_{(m-1)(i-1)+1}) = 2$, $l(v_{(m-1)i}) = 3$,

$$l(v_{(m-1)(i-1)+j+1}) = \begin{cases} 1,3,2,1,3,2,\dots 1,4,2, & \text{if for } m \equiv 0 \pmod 3 \\ 1,3,2,1,3,2,\dots 1,3,2 & \text{if for } m \equiv 1 \pmod 3 \\ 1,3,2,1,3,2,\dots,4,1 & \text{if for } m \equiv 2 \pmod 3 \end{cases}$$

For $i = 1,3,5,\dots$ and $j = 1,2,\dots m-3$, $l(v_{(m-1)(i-1)+1}) = 1$, $l(v_{(m-1)i}) = 3$,

$$l(v_{(m-1)(i-1)+j+1}) = \begin{cases} 2,3,1,2,3,1,\dots 2,4,1, & \text{for } m \equiv 0 \pmod 3 \\ 2,3,1,2,3,1,\dots 1,2 & \text{for } m \equiv 1 \pmod 3 \\ 2,3,1,2,3,1,\dots,2,4, & \text{for } m \equiv 2 \pmod 3 \end{cases}$$

Clearly, we required proper D-Lucky edge labeling number is $\eta_{pdle}(SHC) = 6$. Thus, we conclude that the Proper D-Lucky edge labeling number of Strong Human chain graph if $m \geq 4$, is $\eta_{pdle}(SHC) = 5$ for $n = 2$ and $\eta_{pdle}(SHC) = 6$ for $n \geq 3$. $\square$

Example 2. Proper D-Lucky edge labeled Strong Human chain graph for $n = 2, m = 5, k = 5$, is shown in Figure 2.

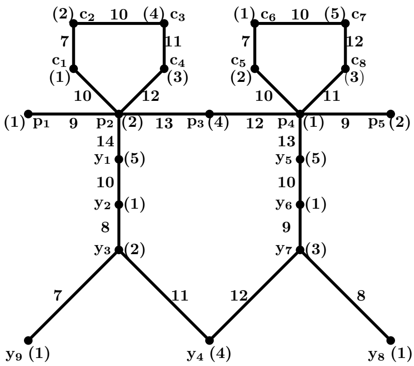

Figure 2. $\eta_{\mathbf{pdle}}(\mathbf{SHC}) = \mathbf{5}$

Theorem 3. *For $n \geq 3$, $\eta_{pdle}(CHC) = 5$ for $m = 3$ and $\eta_{pdle}(CHC) = 6$ for $m \geq 4$.*

Proof. From the structure of circular human chain graph CHC, we notice that $u_1, u_2, \dots, u_{2n}$ be the vertices of cycle C_{2n}, $v_1, v_2, \dots, v_{(m-1)n}$ be the vertices of cycle and $w_1, w_2, \dots, w_{kn}$ be the vertices of Y-tree.

Case (i) $m = 3$.

Here the maximum degree vertices of CHC are u_{2i} for $1 \leq i \leq n$. That is $d(u_{2i}) = 5$ for $1 \leq i \leq n$. Here $N(u_{2i}) = \{u_{2i-1}, u_{2i+1}, w_{m(i-1)+1}, v_{(m-1)(i-1)+1}, v_{(m-1)i} : 1 \leq i \leq n\}$. These vertices $\{u_{2i-1}, u_{2i+1}, v_{(m-1)(i-1)+1}, v_{(m-1)i}, 1 \leq i \leq n\}$ are same 2 degrees. That is $d(u_{2i-1}) = d(u_{2i+1}) = d(v_{(m-1)(i-1)+1}) = d(v_{(m-1)i})$. Therefore $l(u_{2i-1}) \neq l(u_{2i+1}) \neq l(v_{(m-1)(i-1)+1}) \neq l(v_{(m-1)i})$. So Proper D-Lucky edge labeling number for these 4 vertices together with u_{2i} required 5 distinct labels. Let it be $\{1,2,3,4,5\}$. Clearly, we have $\eta_{pdle}(CHC) \geq 5$. Define the labeling pattern as follows: $l(u_1) = 5$, $l(u_{2n}) = 1$ for even n and $l(u_{2n}) = 3$ for odd n,

$$l(u_3, u_5 \dots u_{2n-1}) = \begin{cases} 4,5,4,5,\dots,5,4, & \text{if } n \text{ is even} \\ 4,5,4,5,\dots,4,2, & \text{if } n \text{ is odd,} \end{cases}$$

$$l(u_2, u_4 \ldots u_{2n}) = \begin{cases} 2,1,2,1,\ldots,2,1, & \text{if } n \text{ is even} \\ 2,1,2,1,\ldots,1,3, & \text{if } n \text{ is odd,} \end{cases}$$

$$l(w_{im-2,} w_{im-1}, w_{im}) = \begin{cases} 1,2,3, & \text{if } i = 1,3,5,\ldots,n-1 \text{ and } n \text{ is even} \\ 1,2,3, & \text{if } i = 1,3,5,\ldots,n-2 \text{ and if } n \text{ is odd,} \end{cases}$$

$$l(w_{im-2}, w_{im-1}, w_{im}) = \begin{cases} 5,1,2, & \text{if } i = 2,4,6,\ldots,n \text{ and } n \text{ is even} \\ 5,1,2, & \text{if } i = 2,4,6,\ldots,n-1 \text{ and if } n \text{ is odd,} \end{cases}$$

$$l(v_1) = 1,\ l(v_{(m-1)(i-1)+1}) = \begin{cases} 1, & \text{for } i = 1,3,5,\ldots,n \text{ and } n \text{ is odd} \\ 1, & \text{for } i = 1,3,5,\ldots,n-1 \text{ and } n \text{ is even,} \end{cases}$$

$$l(v_{(m-1)(i-1)+1}) = \begin{cases} 2, & \text{for } i = 2,4,6,\ldots,n-3 \text{ and } n \text{ is odd} \\ 1, & \text{for } i = 2,4,6,\ldots,n \text{ and } n \text{ is even,} \end{cases}$$

$$l(v_{(m-1)(i)}) = \begin{cases} 3, & \text{for } i = 1,2,3,\ldots,n-1 \text{ and } n \text{ is odd} \\ 3, & \text{for } i = 1,2,3,\ldots,n \text{ and } n \text{ is even.} \end{cases}$$

If n is odd, set $l(v_{(m-1)(i)}) = 4$ and $l(v_{(m-1)(i-1)+1}) = 5$, for $i = n-1$. The above pattern shows that $\eta_{pdle}(CHC) = 5$.

Case (ii) $m \geq 4$.
Here the maximum degree vertices of CHC are u_{2i}, $1 \leq i \leq n$. That is $d(u_{2i}) = 5$ for $1 \leq i \leq n$ and $N(u_{2i}) = \{u_{2i-1}, u_{2i+1}, w_{m(i-1)+1}, v_{(m-1)(i-1)+1}, v_{(m-1)i} : 1 \leq i \leq n\}$. These vertices $\{u_{2i-1}, u_{2i+1}, v_{(m-1)(i-1)+1}, v_{(m-1)i}, w_{m(i-1)+1} : 1 \leq i \leq n\}$ are the same 2 degrees. That is $d(u_{2i-1}) = d(u_{2i+1}) = d(v_{(m-1)(i-1)+1}) = d(v_{(m-1)i}) = d(w_{m(i-1)+1})$. Therefore $l(u_{2i-1}) \neq l(u_{2i+1}) \neq l(v_{(m-1)(i-1)+1}) \neq l(v_{(m-1)i}) \neq l(w_{m(i-1)+1})$. So Proper D-Lucky edge labeling number for these 5 vertices together with u_{2i} required 6 distinct labels. Let it be $\{1,2,3,4,5,6\}$. Hence $\eta_{pdle}(CHC) \geq 6$.
Define the labeling pattern as follows.
$l(u_1) = 6$, $l(u_{2n}) = 1$ for even n and $l(u_{2n}) = 3$ for odd n,

$$l(u_3, u_5 \ldots u_{2n-1}) = \begin{cases} 4,5,4,5,\ldots 4, & \text{if } n \text{ is even} \\ 4,5,4,5,\ldots 5, & \text{if } n \text{ is odd,} \end{cases}$$

$$l(u_2, u_4 \ldots u_{2n}) = \begin{cases} 2,1,2,1,\ldots 2,1 & \text{if } n \text{ is even} \\ 2,1,2,1,\ldots 1,3, & \text{if } n \text{ is odd.} \end{cases}$$

The vertices $w_{m(i-1)+2+j}$ for $1 \leq i \leq n$ and $0 \leq j \leq m-5$, are labeled in the sequence $1,2,3,1,2,3\ldots$.

$$\text{For } 1 \leq i \leq n,\ l(w_{im-2}, w_{im-1}, w_{im}) = \begin{cases} 1,2,4 & \text{for } m \equiv 0 \pmod 3 \\ 2,1,3 & \text{for } m \equiv 1 \pmod 3 \\ 3,1,4 & \text{for } m \equiv 2 \pmod 3. \end{cases}$$

Let $(w_1) = 5$, $l(w_{m(n-1)+1}) = 4$ if n is odd, $l(w_{m(n-1)+1}) = 5$ if n is even, $l(w_{im+1}) = 6$, $1 \leq i \leq n-2$, and

$$l(v_1) = 1,\ l(v_{(m-1)(i-1)+1}) = \begin{cases} 1, & \text{for } i = 1,3,5,\ldots n \text{ and } n \text{ is odd} \\ 1, & \text{for } i = 1,3,5,\ldots n-1 \text{ and } n \text{ is even,} \end{cases}$$

$$l(v_{(m-1)(i-1)+1}) = \begin{cases} 2, & \text{for } i = 2,4,6,\dots n-1 \text{ and } n \text{ is odd} \\ 2, & \text{for } i = 2,4,6,\dots n \text{ and } n \text{ is even,} \end{cases}$$

$$l(v_{(m-1)(i)}) = \begin{cases} 3, & \text{for } i = 1,2,3,\dots n-1 \text{ and } n \text{ is odd} \\ 3, & \text{for } i = 1,2,3,\dots n \text{ and } n \text{ is even.} \end{cases}$$

If n is odd and $i = n$, $l(v_{(m-1)(i)}) = 2$. Let $l(u_{2i}) = 1$ for even i and $l(u_{2i}) = 2$ for odd i. For the remaining vertices the labeling pattern divided into two cases.
For $i = 2,4,6,\dots$ and $j = 1,2,\dots m-3$,

$$l(v_{(m-1)(i-1)+j+1}) = \begin{cases} 1,3,2,1,3,2\dots,1,4,2 & \text{if for } m \equiv 0 \pmod 3 \\ 1,3,2,1,3,2,\dots,1,3,2 & \text{if for } m \equiv 1 \pmod 3 \\ 1,3,2,1,3,2,\dots 4,1 & \text{if for } m \equiv 2 \pmod 3, \end{cases}$$

$l(v_{(m-1)(i-1)+1}) = 1$, $l(v_{(m-1)i}) = 3$, and for $i = 1,3,5,\dots$ and $j = 1,2,\dots m-3$

$$l(v_{(m-1)(i-1)+j+1}) = \begin{cases} 2,3,1,2,3,1\dots,2,4,1 & \text{for } m \equiv 0 \pmod 3 \\ 2,3,1,2,3,1,\dots,1,2 & \text{for } m \equiv 1 \pmod 3 \\ 2,3,1,2,3,1,\dots 2,4 & \text{for } m \equiv 2 \pmod 3. \end{cases}$$

It follows that $\eta_{pdle}(CHC) = 6$ and the proof is complete.

□

Example 3. Proper D-Lucky edge labeled Circular Human chain graph for $n = 3, m = 3, k = 3$, is shown in Figure 3

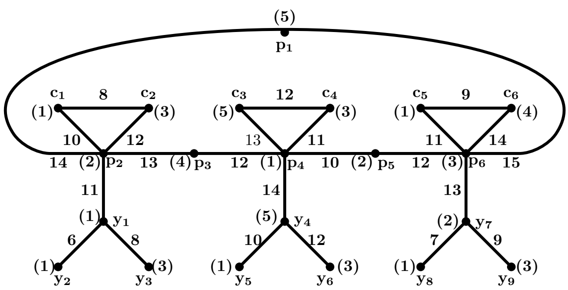

Figure 3. $\eta_{pdle}(CHC) = 5$

Theorem 4. *For $n \geq 3$, $\eta_{pdle}(WHC) = 3$ for $m = 3$ and $\eta_{pdle}(WHC) = 4$ for $m \geq 4$.*

Proof. Let WHC be a weak human chain graph. Let the vertices WHC are as follows: $A = \{u_1, u_2, \dots u_{n+1}, n \in N\}$, $B = \{v_1, v_2, \dots v_{m-1}\}$, $C = \{w_1, w_2, \dots w_{m-1}\}$.
Case (i) $m = 3$.
From the structure of WHC, the degree of vertices $u_2, u_3, \dots, u_n$ are same. Also, for $i = 1,2,\dots n-1$, the vertices u_i and u_{i+1} are adjacent to each other. Since u_2 and u_3 are adjacent, $l(u_2) \neq l(u_3)$. Similarly, up to u_{n-1}, u_n are adjacent

$l(u_3) \neq l(u_4), \ldots, l(u_{n-1}) \neq l(u_n)$. The labeling of the vertices $u_2, u_3, \ldots u_{n+1}$ are in the sequence $2, 3, 1, 2, 3, 1, \ldots$, $l(v_2) = 2$, $l(v_1) = 1$, $l(u_1) = 3$,

$$l(v_{(m-1)i}) = \begin{cases} 3, & i \not\equiv 0 \pmod 3 \\ 2, & i \equiv 0 \pmod 3), \end{cases}$$

$$l(v_{(m-2)i}) = \begin{cases} 1, & i \not\equiv 1 \pmod 3 \\ 2, & i \equiv 1 \pmod 3), \end{cases}$$

$$l(w_{(im+1)}) = \begin{cases} 2, & i \not\equiv 1 \pmod 3 \\ 1, & i \equiv 1 \pmod 3), \end{cases}$$

$l(w_{im+1}) = 1$ for $i = 1, 4, 7, 10$ and $l(w_{im+1}) = 2$ for $i = 0, 2, 5, 8$. Hence $\eta_{pdle}(WHC) = 3$ for $m = 3$.

Case (ii) $m \geq 3$.

In this case, the vertices are $u_1, u_2 \ldots u_{n+1}, w_1, w_2, \ldots w_m, w_{m+1}, \ldots w_{mn}, v_1, v_2, \ldots v_m, v_{m+1}, \ldots v_{n(m-1)}$. The vertices u_{i+1}, $1 \leq i \leq n-1$ have 3 adjacent same 2-degree vertices. The vertices in $N(u_{i+1})$ for, $1 \leq i \leq n-1$ required a minimum number of 4 distinct labels $\{1, 2, 3, 4\}$. Hence $\eta_{pdle}(WHC) \geq 4$. Let the labeling pattern as follows. Let $l(u_{n+1}) = 1$; the vertices $u_2, u_3 \ldots u_n$ are labeled in the sequence $2, 3, 4, \ldots$; $l(u_1) = 3$; $w_{m(i-1)+2+j}$ for $1 \leq i \leq n$ and $0 \leq j \leq m-5$, are labeled in the sequence $1, 2, 3, 1, 2, 3 \ldots$; for $1 \leq i \leq n$,

$$l(w_{im-2}, w_{im-1}, w_{im}) = \begin{cases} 1, 2, 4, & \text{for } m \equiv 0 \pmod 3) \\ 2, 1, 3, & \text{for } m \equiv 1 \pmod 3) \\ 3, 1, 4 & \text{for } m \equiv 2 \pmod 3). \end{cases}$$

Let $l(w_1) = l(w_{m(n-1)+1}) = 5$, $l(w_{im+1}) = 6$ for $1 \leq i \leq n-2$.

$$\text{Let } l(u_{(i+1)}) = \begin{cases} 3, & \text{for } i \equiv 0 \pmod 3) \\ 4, & \text{for } i \equiv 1 \pmod 3) \\ 2, & \text{for } i \equiv 2 \pmod 3), \end{cases}$$

$l(u_{2i}) = 1$ for $i = 2, 4, 6 \ldots$ and $l(u_{2i}) = 2$, $i = 1, 3, 5 \ldots$. For the remaining vertices, the labeling pattern divided into two cases.

For $i = 2, 4, 6, \ldots$ and $j = 1, 2, \ldots m-3$, $l(v_{(m-1)(i-1)+1}) = 2$, $l(v_{(m-1)i}) = 3$ and

$$l(v_{(m-1)(i-1)+j+1}) = \begin{cases} 1, 3, 2, 1, 3, 2 \ldots, 1, 4, 2 & \text{for } m \equiv 0 \pmod 3) \\ 1, 3, 2, 1, 3, 2, \ldots, 1, 3, 2 & \text{for } m \equiv 1 \pmod 3) \\ 1, 3, 2, 1, 3, 2, \ldots, 4, 1 & \text{for } m \equiv 2 \pmod 3) \end{cases}$$

For $i = 1, 3, 5, \ldots$ and $j = 1, 2, \ldots m-3$, $l(v_{(m-1)(i-1)+1}) = 1$, $l(v_{(m-1)i}) = 3$ and

$$l(v_{(m-1)(i-1)+j+1}) = \begin{cases} 2, 3, 1, 2, 3, 1 \ldots, 2, 4, 1 & \text{for } m \equiv 0 \pmod 3) \\ 2, 3, 1, 2, 3, 1, \ldots, 1, 2 & \text{for } m \equiv 1 \pmod 3) \\ 2, 3, 1, 2, 3, 1, \ldots, 2, 4 & \text{for } m \equiv 2 \pmod 3). \end{cases}$$

Thus $\eta_{pdle}(WHC) = 4$ for $m \geq 4$. □

Example 4. Proper D-Lucky edge labeled Weak Human chain graph for $n = 5, m = 6, k = 6$, is shown in Figure 4.

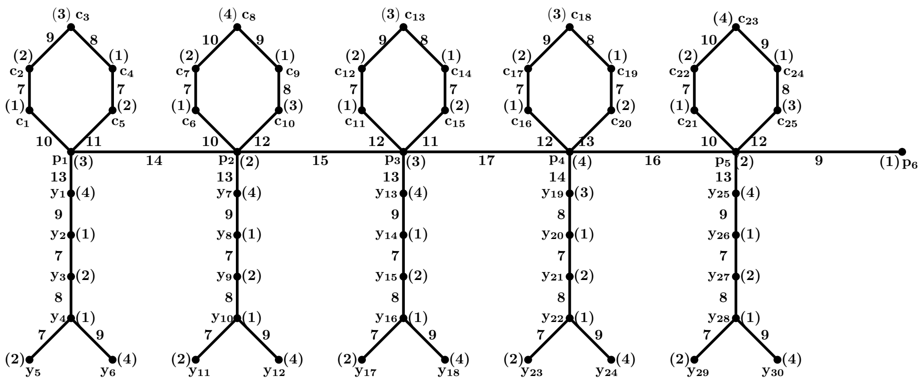

Figure 4. $\eta_{\mathbf{pdle}}(\mathbf{WHC}) = \mathbf{4}$

3. Conclusion

This paper determines the proper D-lucky edge numbers for human chain graphs, circular human chain graphs, strong human chain graphs, and weak human chain graphs.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

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