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Research Article

The elliptic Sombor energy of graphs

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Abstract: The elliptic Sombor index is a topological index based on vertex degree introduced by Gutman. Suppose $G = (V(G), E(G))$ is a finite, connected, and simple graph with $V(G) = \{w_1, w_2, ..., w_p\}$. Suppose $d_G(w_i)$ is the degree of w_i, for $1 \leq i \leq p$. We use $ES(G)$ to represent the Sombor elliptic matrix G which is a $p \times p$ matrix and its (i, j)-entry is equal to $(d_G(w_i) + d_G(w_j))\sqrt{d_G^2(w_i) + d_G^2(w_j)}$ if $w_iw_j \in E(G)$, and zero otherwise. We introduce and investigate the elliptic Sombor energy and elliptic Sombor Estrada index, both base on the eigenvalues of the elliptic Sombor matrix. In addition, we prove some bounds for these new graph invariants.

Keywords: Sombor index, energy of graphs, topoligical indices.

AMS Subject classification: 05C50, 05C07, 05C69

1. Introduction

Let $G = (V(G), E(G))$ be a finite, connected, and simple graph of order $|V(G)| = p$ and size $|E(G)| = q$ where $V(G) = \{w_1, w_2, \ldots, w_p\}$ is the vertex set of G and $E(G)$ denotes the edge set of G. The degree of v_i is the number of first neighbors of the vertex v_i, $i = 1, 2, \ldots, p$ and it is denoted by $d_G(v_i)$. G is called complete graph if every two vertices of it are adjacent and is denoted by K_p.

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Let B be a real bivariate function defined over $\mathbb{R}\times\mathbb{R}$ with condition $B(r,s) = B(s,r)$ for all non-negative real number r and s. The topological index based on vertex degree [11] TI is defined by

$$TI(G) = \sum_{v_iv_j\in E(G)} B(d_G(v_i), d_G(v_j).$$

In [11], Gutman listed 26 types of topological indices based on vertex degree, for instance, the different type of Zagreb indices [2], Randić indices [7], and Sombor index. A molecular graph [5] is a connected graph where its vertices are atoms and its edges are covalent bonds between these atoms. Topological indices contain information on the atom-connectivity molecular refractivity, the nature of atoms, molecular volume, the bond multiplicity, etc.

The Sombor index (briefly SI) was defined by Gutman [11] as follows

$$SO(G) = \sum_{w_iw_j\in E(G)} \sqrt{d_G^2(w_i)+d_G^2(w_j)}, \tag{1.1}$$

and the elliptic Sombor index (briefly ESI) [11, 12] is defined by

$$ESO(G) = \sum_{w_iw_j\in E(G)} (d_G(w_i)+d_G(w_j))\sqrt{d_G^2(w_i)+d_G^2(w_j)}. \tag{1.2}$$

Also, Gutman and Trinajstic in 1972 [13] introduced the Zagreb indices as follow:

$$\begin{aligned} M_1(G) &= \sum_{w\in V(G)} d_G^2(w) = \sum_{wv\in E(G)} (d_G(w)+d_G(v)),\\ M_2(G) &= \sum_{wv\in E(G)} (d_G(w)d_G(v)). \end{aligned}$$

Then Furtula and Gutman in 2015 [9] defined F-index by

$$F(G) = \sum_{w\in V(G)} d_G(w)^3 = \sum_{uv\in E(G)} (d_G(w)^2+d_G(v)^2).$$

In 2004, Li and Zhao [16] defined the general first Zagreb index of a graph G as

$$M_1^{\alpha}(G) = \sum_{w\in V(G)} d_G(w)^{\alpha} = \sum_{wv\in E(G)} (d_G(w)^{\alpha-1}+d_G(v)^{\alpha-1}), \qquad \text{for } \alpha\in\mathbb{R}\setminus\{0,1\}.$$

For especial case, $\alpha = 4$ it is called Y-index and denoted by $Y(G)$, that is, $Y(G) = \sum_{w\in V(G)} d_G(w)^4$. The general Randić index [6] is defined by

$$M_2^{\alpha}(G) = \sum_{wv\in E(G)} (d_G(w)d_G(v))^{\alpha}, \qquad \text{for } \alpha \in \mathbb{R}\setminus\{0,1\},$$

and the (α,β)-Zagreb indices are defined by

$$Z_{(\alpha,\beta)}(G) = \sum_{vw\in E(G)} (d_G(w)^{\alpha}d_G(v)^{\beta} + d_G(w)^{\beta}d_G(v)^{\alpha}), \qquad \text{for } (\alpha,\beta) \neq (0,0).$$

The ESI has the mathematical properties and chemical applications. Kulli [15] provided ESI on some chemical graphs. Rada et al. [18] studied elliptic Sombor index of benzenoid systems. Also, Espinal et al. [8] studied ESI on some chemical graphs. Chanda and Iyer [4], studied the SI of generalized Siperpiṕski graphs and generalized Mycielskian graphs and obtained some upper and lower bounds for them. Also, Liu [17] investigated multiplicative SI on some graphs such as unicyclic graphs and trees. On the other hand, spectral graph theory using matrix theory and linear algebra plays a fundamental role in study of matrix graphs. Spectral graph theory is interesting to many mathematicians and graph energy is an interesting topic in spectral graph theory. The eigenvalues of matrix corresponding to a topological index are related to most principal invariants of a graph. In a molecular graph, Gutmann [13] defined the relationship between the total electron energy and the graph energy. After than, many matrices were defined based on topological indices. For instance, Zagreb matrix [14], seidel matrix [3], Sombor matrix [10], etc. In [1] Alikhani et al. investigated the elliptic Sombor energy of some graphs such as path graphs, cycle graphs, star graphs, complete bipartite graphs, k-regular graphs, and Petrerson graph.
Motivated as the above works, we investigate the elliptic Sombor energy and elliptic Sombor Estrada index of a graph. In addition, we prove some bounds for these new graph invariants.

2. Elliptic Sombor energy and elliptic Sombor Estrada index

For a simple finite graph G with $V(G) = \{w_1,\dots,w_p\}$ the elliptic Sombor matrix is defined by $ES(G) = [s_{ij}]_{p\times p}$ where

$$s_{ij} = \begin{cases} (d_G(w_i)+d_G(w_j))\sqrt{d_G^2(w_i)+d_G^2(w_j)} & \text{if } w_iw_j \in E(G),\\ 0 & \text{otherwise.}\end{cases}$$

Let I be a $p\times p$ unit matrix. The elliptic Sombor characteristic polynomial $P_{ES(G)}$ is determined as follows

$$P_{ES(G)}(\Lambda) = \det(\Lambda I - ES(G)).$$

All eigenvalues of $ES(G)$ are real becuase $ES(G)$ is a symmetric matrix, and we denoted by $\Lambda_1 \geq \Lambda_2 \geq \cdots \geq \Lambda_p$. The elliptic Sombor energy of G is defined by

$$EES(G) = \sum_{i=1}^{p} |\Lambda_i| \tag{2.1}$$

and the elliptic Sombor Estrada index of G is defined as follows

$$ESE(G) = \sum_{i=1}^{p} e^{\Lambda_i}. \tag{2.2}$$

Example 1. Let $G = K_r$. The (i,j)-entry of $ES(K_r)$ is $2\sqrt{2}(r-1)^2$ if $i \neq j$ and zero otherwise. Then

$$ES(K_r) = \begin{bmatrix} 0 & 2\sqrt{2}(r-1)^2 & 2\sqrt{2}(r-1)^2 & \dots & 2\sqrt{2}(r-1)^2 \\ 2\sqrt{2}(r-1)^2 & 0 & 2\sqrt{2}(r-1)^2 & \dots & 2\sqrt{2}(r-1)^2 \\ 2\sqrt{2}(r-1)^2 & 2\sqrt{2}(r-1)^2 & 0 & \dots & 2\sqrt{2}(r-1)^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 2\sqrt{2}(r-1)^2 & 2\sqrt{2}(r-1)^2 & 2\sqrt{2}(r-1)^2 & \dots & 0 \end{bmatrix}.$$

The elliptic Sombor characteristic polynomial $r_{ES(K_r)}$ is determined by

$$P_{ES(K_r)}(\Lambda) = \det(\Lambda I - ES(K_r)) = \left(\Lambda - 2\sqrt{2}(r-1)^3\right)\left(\Lambda + 2\sqrt{2}(r-1)^2\right)^{r-1}$$

which has a root $2\sqrt{2}(r-1)^3$ and $(r-1)$ roots $-2\sqrt{2}(r-1)^2$. The elliptic Sombor energy of graph K_r is given by

$$EES(K_r) = 4\sqrt{2}(r-1)^3$$

and the elliptic Sombor Estrada index of K_r is obtained as follows

$$ESE(K_r) = e^{2\sqrt{2}(r-1)^3} + (r-1)e^{-2\sqrt{2}(r-1)^2}.$$

3. Some bounds for elliptic Sombor energy

In the following, we study the elliptic Sombor energy and give some bounds for it. Let $N_k = \sum_{i=1}^{p}(\Lambda_i)^k$ be the k-th spectral moment of the elliptic Sombor matrix $ES(G)$, and recall that $N_k = Tr((ES(G))^k)$.

Theorem 1. *Let G represent a graph consisting of p vertices, characterized by its edge set $E(G)$. The elliptic Sombor matrix associated with this graph is denoted as $ES(G)$, and the eigenvalues of this matrix are ordered as $\Lambda_1 \geq \Lambda_2 \geq \cdots \geq \Lambda_p$. Then*

i) $N_1 = \sum_{i=1}^{p} \Lambda_i = 0,$

ii) $N_2 = \sum_{i=1}^{p}(\Lambda_i)^2 = 2\left(M_1^5(G) + 2M_2^2(G) + 2Z_{(3,1)}(G)\right)$,

iii) $N_3 = \sum_{i=1}^{p}(\Lambda_i)^3 = 2\sum_{\Delta}\prod_{vw\in E(\Delta)}(d_G(w)+d_G(v))\sqrt{d_G^2(w)+d_G^2(v)}$,

iv) $N_4 = \sum_{i=1}^{p}(\Lambda_i)^4 = 2\sum_{\Box}\prod_{wv\in E(\Box)}(d_G(w)+d_G(v))\sqrt{d_G^2(w)+d_G^2(v)}$,

where Δ and $\Box$ are a triangle and a cycle of order four in the graph G, respectively.

Proof. i) By definition $ES(G)$, (i,i)-entry of $ES(G)$ is equal to zero for $i = 1, 2, \dot{;} p$. Then the trace of $ES(G)$ is zero.
ii) Let $V(G) = \{w_1, \ldots, w_p\}$. The diagonal elements of $(ES(G))^2$ are

$$\begin{aligned} ((ES(G))^2)_{ii} &= \sum_{k=1}^{p} s_{ij}s_{ji} = \sum_{k=1}^{p}(s_{ij})^2 = \sum_{\substack{k\in\{1,2,\ldots,p\}\\ w_iw_k\in E(G)}}(s_{ij})^2 \\ &= \sum_{\substack{k\in\{1,2,\ldots,p\}\\ w_iw_k\in E(G)}}(d_G(w_i)+d_G(w_k))^2(d_G^2(w_i)+d_G^2(w_k)). \end{aligned}$$

Thus,

$$\begin{aligned} Tr((ES(G))^2) &= \sum_{i=1}^{p}\sum_{\substack{j\in\{1,2,\ldots,p\}\\ w_iw_j\in E(G)}}(d_G(w_i)+d_G(w_j))^2(d_G^2(w_i)+d_G^2(w_j)) \\ &= 2\sum_{w_iw_j\in E(G)}(d_G(w_i)+d_G(w_j))^2(d_G^2(w_i)+d_G^2(w_j)) \\ &= 2\sum_{w_iw_j\in E(G)}\{d_G^4(w_i)+d_G^4(w_j)+2d_G^2(w_i)d_G^2(w_j)\} \\ &\quad +2\sum_{w_iw_j\in E(G)}\{2d_G(w_i)d_G(w_j)(d_G^2(w_i)+d_G^2(w_j))\} \\ &= 2\left(M_1^5(G)+2M_2^2(G)+2Z_{(3,1)}(G)\right). \end{aligned}$$

iii) Since the diagonal elements of $(ES(G))^3$ are

$$\begin{aligned} ((ES(G))^3)_{ii} &= \sum_{j=1}^{p} s_{ij}((ES(G))^2)_{ji} = \sum_{\substack{j\in\{1,2,\ldots,p\}\\ w_iw_j\in E(G)}} s_{ij}((ES(G))^2)_{ji} \\ &= \sum_{\substack{j\in\{1,2,\ldots,p\}\\ w_iw_j\in E(G)}}\left(s_{ij}\sum_{\substack{k\in\{1,2,\ldots,p\}\\ w_jw_k\in E(G),\ w_kw_i\in E(G)}} s_{jk}s_{ki}\right). \end{aligned}$$

Hence,

$$
\begin{aligned}
Tr((ES(G))^3) &= \sum_{i=1}^{p} \sum_{\substack{j\in\{1,2,\ldots,p\}\\ w_iw_j\in E(G)}} \left(s_{ij} \sum_{\substack{k\in\{1,2,\ldots,p\}\\ w_jw_k\in E(G),\ w_kw_i\in E(G)}} s_{jk}s_{ki} \right) \\
&= 2\sum_{\Delta} \prod_{ab\in E(\Delta)} (d_G(a)+d_G(b))\sqrt{d_G^2(a)+d_G^2(b)}.
\end{aligned}
$$

iv) Since the diagonal elements of $(ES(G))^4$ are

$$
\begin{aligned}
((ES(G))^3)_{ii} &= \sum_{j=1}^{p} s_{ij}((ES(G))^3)_{ji} = \sum_{\substack{j\in\{1,2,\ldots,p\}\\ w_iw_j\in E(G)}} s_{ij}((ES(G))^3)_{ji} \\
&= \sum_{\substack{j\in\{1,2,\ldots,p\}\\ w_iw_j\in E(G)}} s_{ij} \left[\sum_{\substack{k\in\{1,2,\ldots,p\}\\ w_jw_k\in E(G)}} \left(s_{jk} \sum_{\substack{l\in\{1,2,\ldots,p\}\\ w_kw_l\in E(G),\ w_lw_i\in E(G)}} s_{kl}s_{li} \right) \right].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
Tr((ES(G))^4) &= \sum_{i=1}^{p} \sum_{\substack{j\in\{1,2,\ldots,p\}\\ w_iw_j\in E(G)}} s_{ij} \left[\sum_{\substack{k\in\{1,2,\ldots,p\}\\ w_jw_k\in E(G)}} \left(s_{jk} \sum_{\substack{l\in\{1,2,\ldots,p\}\\ w_kw_l\in E(G),\ w_lw_i\in E(G)}} s_{kl}s_{li} \right) \right] \\
&= 2\sum_{\square} \prod_{ab\in E(\square)} (d_G(a)+d_G(b))\sqrt{d_G^2(a)+d_G^2(b)}.
\end{aligned}
$$

□

Theorem 2. *Let G represent a graph consisting of p vertices. The elliptic Sombor matrix associated with this graph is denoted as $ES(G)$, and the eigenvalues of this matrix are ordered as $\Lambda_1 \geq \Lambda_2 \geq \cdots \geq \Lambda_p$. Then*

$$
\Lambda_1 \leq \sqrt{\frac{2(p-1)}{p}} \sqrt{M_1^5(G) + 2M_2^2(G) + 2Z_{(3,1)}(G)}.
$$

Proof. For $i = 1, 2, \ldots, p$, suppose x_i and y_i are real numbers. The Cauchy-Schwarz inequality leads to

$$
\left(\sum_{i=1}^{p} x_iy_i \right)^2 \leq \left(\sum_{i=1}^{p} x_i^2 \right) \left(\sum_{i=1}^{p} y_i^2 \right). \tag{3.1}
$$

If we consider $x_i = \Lambda_i$, $y_i = 1$, and $2 \leq i \leq p$, in inequality (3.1), we deduce

$$\left(\sum_{i=2}^{p} \Lambda_i\right)^2 \leq (p-1) \sum_{i=2}^{p} \Lambda_i^2. \tag{3.2}$$

Using Theorem 1, we conclude that

$$(-\Lambda_1)^2 \leq (p-1)\left(2\left(M_1^5(G) + 2M_2^2(G) + 2Z_{(3,1)}(G)\right) - \Lambda_1^2\right). \tag{3.3}$$

By solving the last inequality with respect to Λ_1, we get

$$\Lambda_1 \leq \sqrt{\frac{2(p-1)}{p}} \sqrt{M_1^5(G) + 2M_2^2(G) + 2Z_{(3,1)}(G)}.$$

□

Theorem 3. *Let G represent a graph consisting of p vertices, the elliptic Sombor matrix associated with this graph is denoted as $ES(G)$ and $P = \det(ES(G))$. Then*

$$|EES(G)| \geq \sqrt{2\left(M_1^5(G) + 2M_2^2(G) + 2Z_{(3,1)}(G)\right) + P^{\frac{2}{p}}}.$$

Proof. By definition of the elliptic Sombor energy, we get

$$\begin{aligned} |EES(G)|^2 &= \left(\sum_{i=1}^{p} |\Lambda_i|\right)^2 = \sum_{i=1}^{p} |\Lambda_i|^2 + \sum_{\substack{i \neq j \\ 1 \leq i,j \leq p}} |\Lambda_i||\Lambda_j| \\ &= 2\left(M_1^5(G) + 2M_2^2(G) + 2Z_{(3,1)}(G)\right) + \sum_{i \neq j} |\Lambda_i||\Lambda_j|. \end{aligned}$$

For non-negative real number $x_1, \ldots, x_p$, by the arithmetic-geometric mean inequalty, we obtain

$$\frac{1}{p} \sum_{i=1}^{p} x_i \geq \left(\prod_{i=1}^{p} x_i\right)^{\frac{1}{p}}. \tag{3.4}$$

Using (3.4), we deduce

$$\begin{aligned} \frac{1}{p(p-1)} \sum_{1 \leq i \neq j \leq p} |\Lambda_i||\Lambda_j| &\geq \left(\prod_{1 \leq i \neq j \leq p} |\Lambda_i||\Lambda_j|\right)^{\frac{1}{p(p-1)}} = \left(\prod_{i=1}^{p} |\Lambda_i|\right)^{\frac{2}{p}} \\ &= (\det(ES(G)))^{\frac{2}{p}} = P^{\frac{2}{p}}. \end{aligned}$$

Therefore,

$$|EES(G)|^2 \geq 2\left(M_1^5(G) + 2M_2^2(G) + 2Z_{(3,1)}(G)\right) + P^{\frac{2}{p}}.$$

We get the inequality stated in Theorem 3. □

Theorem 4. *Let G represent a graph consisting of p vertices with $ES(G)$ being its elliptic Sombor matrix, $\Lambda_1 \geq \Lambda_2 \geq \cdots \geq \Lambda_p$ are its eigenvalues and $|\Lambda_1|$ and $|\Lambda_p|$ are the maximum and minimum of the absolute value of $\{\Lambda_1, \Lambda_2, \ldots, \Lambda_p\}$. Then*

$$\frac{2\left(M_1^5(G) + 2M_2^2(G) + 2Z_{(3,1)}(G)\right) + p|\Lambda_1||\Lambda_p|}{(|\Lambda_1| + |\Lambda_p|)} \leq EES(G).$$

Equality is true if and only if $|\Lambda_i| = |\Lambda_p|$ or $|\Lambda_i| = |\Lambda_1|$ for $1 \leq i \leq p$.

Proof. Assume that $(x_1, \ldots, x_p)$ and $(y_1, \ldots, y_p)$ are positive real numbers, satisfy the condition $rx_i \leq y_i \leq Rx_i$ for $i = 1, 2, \dot{;} p$ and some real constants r, R. The Diaz-Metcalf inequality yields

$$\sum_{i=1}^{p}(y_i^2 + rRx_i^2) \leq (r + R)\sum_{i=1}^{p} x_i y_i$$

and equality holds if and only if $y_i = Rx_i$ or $y_i = rx_i$ for $1 \leq i \leq p$. Putting $y_i = |\Lambda_i|$, $x_i = 1$, $r = |\Lambda_p|$, and $R = |\Lambda_1|$ in Diaz-Metcalf inequality, it follows that

$$\sum_{i=1}^{p} |\Lambda_i|^2 + p|\Lambda_1||\Lambda_p| \leq (|\Lambda_1| + |\Lambda_p|)EES(G).$$

From Theorem 1, we conclude that

$$2\left(M_1^5(G) + 2M_2^2(G) + 2Z_{(3,1)}(G)\right) + p|\Lambda_1||\Lambda_p| \leq (|\Lambda_1| + |\Lambda_p|)EES(G).$$

This completes the proof. □

Theorem 5. *Let G represent a graph consisting of p vertices with $ES(G)$ being its elliptic Sombor matrix. Then*

$$EES(G) \leq \sqrt{2p\left(M_1^5(G) + 2M_2^2(G) + 2Z_{(3,1)}(G)\right)}.$$

Proof. Taking $x_i = 1$ and $y_i = |\Lambda_i|$ in Cauchy-Schwarz inequality (3.1), we arrive at

$$(EES(G))^2 = \left(\sum_{i=1}^{p} |\Lambda_i|\right)^2 \leq p\left(\sum_{i=1}^{p} |\Lambda_i|^2\right) = 2p\left(M_1^5(G) + 2M_2^2(G) + 2Z_{(3,1)}(G)\right).$$

By simplifying it, we derive the required result. □

Theorem 6. *Let G represent a graph consisting of p vertices with $ES(G)$ being its elliptic Sombor matrix. Then*

$$EES(G) \leq \sqrt{2(p-1)\left(M_1^5(G) + 2M_2^2(G) + 2Z_{(3,1)}(G)\right) + pP^{\frac{2}{p}}}.$$

Proof. Let $z_1, \ldots, z_p$ be non-negative real numbers. We have

$$p\left(\frac{1}{p}\sum_{j=1}^{p} z_j - \left(\prod_{j=1}^{p} z_j\right)^{\frac{1}{p}}\right) \leq p\sum_{j=1}^{p} z_j - \left(\sum_{j=1}^{p}\sqrt{z_j}\right)^2. \tag{3.5}$$

Inserting $z_j = \Lambda_j^2$ in (3.5), we infer

$$p\left(\frac{1}{p}\sum_{j=1}^{p} \Lambda_j^2 - \left(\prod_{j=1}^{p} \Lambda_j^2\right)^{\frac{1}{p}}\right) \leq p\sum_{j=1}^{p} \Lambda_j^2 - \left(\sum_{j=1}^{p}|\Lambda_j|\right)^2.$$

Using Theorem 1, we infer

$$\begin{aligned}&2\left(M_1^5(G) + 2M_2^2(G) + 2Z_{(3,1)}(G)\right) - pP^{\frac{2}{p}}\\ &\leq 2p\left(M_1^5(G) + 2M_2^2(G) + 2Z_{(3,1)}(G)\right) - (EES(G))^2.\end{aligned}$$

By simplifying it, we deduce the required result. □

Theorem 7. *Let G represent a graph consisting of p vertices with $ES(G)$ being its elliptic Sombor matrix such that the maximum of $\{|\Lambda_1|, |\Lambda_2|, \ldots, |\Lambda_p|\}$ is greater or equal to 1. Then*

$$e^{-\sqrt{2\left(M_1^5(G)+2M_2^2(G)+2Z_{(3,1)}(G)\right)}} \leq EES(G) \leq e^{\sqrt{2\left(M_1^5(G)+2M_2^2(G)+2Z_{(3,1)}(G)\right)}}.$$

Proof. By definition of the $EES(G)$ and using the arithmetic-geometric mean inequality, we get

$$EES(G) = \sum_{j=1}^{p}|\Lambda_j| = p\left(\frac{1}{p}\sum_{j=1}^{p}|\Lambda_j|\right) \geq p\left(\prod_{j=1}^{p}|\Lambda_j|\right)^{\frac{1}{p}}.$$

By virtue the inequality $p\left(\sum_{j=1}^{p}\frac{1}{a_j}\right)^{-1} \geq \left(\prod_{j=1}^{p}|\Lambda_j|\right)^{\frac{1}{p}}$ for some positive numbers $a_1, \ldots .a_p$, we get

$$p\left(\prod_{j=1}^{p}|\Lambda_j|\right)^{\frac{1}{p}} \geq p^2\left(\sum_{j=1}^{p}\frac{1}{|\Lambda_j|}\right)^{-1} \geq p^2\left(\sum_{j=1}^{p}\frac{1}{|\Lambda_j|}\sum_{j=1}^{p}|\Lambda_j|\right)^{-1}.$$

Let $y_1 \leq y_2 \leq \cdots \leq y_p$ and $z_1 \leq z_2 \leq \cdots \leq z_p$ be real numbers. The Chebishev's inequality, implies that

$$\left(\sum_{j=1}^{p} y_j\right)\left(\sum_{j=1}^{p} z_j\right) \leq p\sum_{j=1}^{p} y_j z_j.$$

Taking $y_j = \frac{1}{|\Lambda_j|}$ and $z_j = |\Lambda_j|$ in Chebishev's inequality, one gets

$$\sum_{j=1}^{p} \frac{1}{|\Lambda_j}| \sum_{j=1}^{p} |\Lambda_j| \leq p^2.$$

Then

$$\begin{aligned} p\left(\prod_{j=1}^{p} |\Lambda_j|\right)^{\frac{1}{p}} &\geq 1 \geq \frac{1}{\sum_{j=1}^{p} e^{|\Lambda_j|}} \geq \frac{1}{\sum_{j=1}^{p}\sum_{k\geq 0}\frac{|\Lambda_j|^k}{k!}} = \frac{1}{\sum_{k\geq 0}\frac{1}{k!}\left(\sum_{j=1}^{p}|\Lambda_j|^k\right)} \\ &\geq \frac{1}{\sum_{k\geq 0}\frac{1}{k!}\left(\sum_{j=1}^{p}|\Lambda_j|^2\right)^{\frac{k}{2}}} \\ &= \frac{1}{\sum_{k\geq 0}\frac{1}{k!}\left(2\left(M_1^5(G)+2M_2^2(G)+2Z_{(3,1)}(G)\right)\right)^{\frac{k}{2}}} \\ &= e^{-\sqrt{\left(2\left(M_1^5(G)+2M_2^2(G)+2Z_{(3,1)}(G)\right)\right)}}. \end{aligned}$$

Therefore, we have

$$EES(G) \geq e^{-\sqrt{2\left(M_1^5(G)+2M_2^2(G)+2Z_{(3,1)}(G)\right)}}.$$

On the other hand,

$$\begin{aligned} EES(G) &= \sum_{j=1}^{p} |\Lambda_j| \leq \sum_{j=1}^{p} e^{|\Lambda_j|} = \sum_{j=1}^{p}\sum_{k\geq 0}\frac{|\Lambda_j|^k}{k!} = \sum_{k\geq 0}\frac{1}{k!}\left(\sum_{j=1}^{p}|\Lambda_j|^k\right) \\ &\leq \sum_{k\geq 0}\frac{1}{k!}\left(\sum_{j=1}^{p}|\Lambda_j|^2\right)^{\frac{k}{2}} \\ &= \sum_{k\geq 0}\frac{1}{k!}\left(2\left(M_1^5(G)+2M_2^2(G)+2Z_{(3,1)}(G)\right)\right)^{\frac{k}{2}} \\ &= e^{\sqrt{2\left(M_1^5(G)+2M_2^2(G)+2Z_{(3,1)}(G)\right)}}. \end{aligned}$$

□

4. Some bounds for elliptic Sombor Estrada index

In the following, for a given graph G we obtain some bounds for the elliptic Sombor Estrada index.

Theorem 8. *Let G represent a graph consisting of p vertices with $ES(G)$ being its elliptic Sombor matrix. Then*

$$\begin{aligned} ESE(G) \geq\ & p + M_1^5(G) + 2M_2^2(G) + 2Z_{(3,1)}(G) \\ & + \frac{1}{3}\sum_{\Delta}\prod_{ab\in E(\Delta)}(d_G(a)+d_G(b))\sqrt{d_G^2(a)+d_G^2(b)} \\ & + \frac{1}{12}\sum_{\Box}\prod_{ab\in E(\Box)}(d_G(a)+d_G(b))\sqrt{d_G^2(a)+d_G^2(b)}. \end{aligned}$$

Proof. Since $e^x \geq 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$, we deduce

$$\begin{aligned} ESE(G) &= \sum_{i=1}^{p} e^{\Lambda_i} \geq \sum_{i=1}^{p}\left(1 + (\Lambda_i) + \frac{(\Lambda_i)^2}{2} + \frac{(\Lambda_i)^3}{6} + \frac{(\Lambda_i)^4}{24}\right) \\ &= p + N_1 + \frac{N_2}{2} + \frac{N_3}{6} + \frac{N_4}{24}. \end{aligned}$$

Replacing N_1, N_2, N_3, and N_4 from Theorem 1 in the above inequality, we conclude the required inequality.

□

Theorem 9. *Let G represent a graph consisting of p vertices with $ES(G)$ being its elliptic Sombor matrix. Then*

$$ESE(G) \leq p - 1 + e^{\sqrt{2\left(M_1^5(G)+2M_2^2(G)+2Z_{(3,1)}(G)\right)}}.$$

Proof. Let $\Lambda_1, \ldots, \Lambda_\alpha$ be positive elliptic Sombor eigenvalue of $ES(G)$ and $\Lambda_{\alpha+1}, \ldots, \Lambda_p$ be non-positive elliptic Sombor eigenvalue of $ES(G)$. Function $f(x) = e^x$ is a monotonically increase on $(-\infty, +\infty)$. Then

$$\begin{aligned} ESE(G) &= \sum_{j=1}^{p} e^{\Lambda_j} \leq (p-\alpha) + \sum_{j=1}^{\alpha} e^{\Lambda_j} = (p-\alpha) + \sum_{j=1}^{\alpha}\sum_{r\geq 0}\frac{(\Lambda_j)^r}{r!} \\ &= p + \sum_{r\geq 1}\frac{1}{r!}\sum_{j=1}^{\alpha}(\Lambda_j)^r \leq p + \sum_{r\geq 1}\frac{1}{r!}\left(\sum_{j=1}^{\alpha}(\Lambda_j)^2\right)^{\frac{r}{2}} \\ &\leq p + \sum_{r\geq 1}\frac{1}{r!}\left(\sum_{j=1}^{p}(\Lambda_j)^2\right)^{\frac{r}{2}} = p - 1 + \sum_{r\geq 0}\frac{1}{r!}\left(\sum_{j=1}^{p}(\Lambda_j)^2\right)^{\frac{r}{2}} \\ &= p - 1 + e^{\sqrt{2\left(M_1^5(G)+2M_2^2(G)+2Z_{(3,1)}(G)\right)}}. \end{aligned}$$

□

Theorem 10. *Let G represent a graph consisting of p vertices with $ES(G)$ being its elliptic Sombor matrix. Then*

$$\begin{aligned} ESE(G) \geq & \sqrt{2n\left(M_1^5(G)+2M_2^2(G)+2Z_{(3,1)}(G)\right)} \\ & +\sqrt{\frac{2n}{3}\sum_{\Delta}\prod_{ab\in E(\Delta)}(d_G(a)+d_G(b))\sqrt{d_G^2(a)+d_G^2(b)}} \\ & +\sqrt{\frac{p}{6}\sum_{\square}\prod_{ab\in E(\square)}(d_G(a)+d_G(b))\sqrt{d_G^2(a)+d_G^2(b)}} \\ & +M_1^5(G)+2M_2^2(G)+2Z_{(3,1)}(G). \end{aligned}$$

Proof. Suppose that $\Lambda_1,\dots,\Lambda_p$ is the elliptic Sombor eigenvalue of $ES(G)$. Since $e^x \geq 1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}$, we get

$$\begin{aligned} (ESE(G))^2 &= \sum_{j=1}^{p}\sum_{k=1}^{p} e^{\Lambda_j+\Lambda_k} \\ &\geq \sum_{j=1}^{p}\sum_{k=1}^{p}\left(1+\Lambda_j+\Lambda_k+\frac{(\Lambda_j+\Lambda_k)^2}{2}+\frac{(\Lambda_j+\Lambda_k)^3}{6}+\frac{(\Lambda_j+\Lambda_k)^4}{24}\right) \\ &= \sum_{j=1}^{p}\sum_{k=1}^{p}\left(1+\Lambda_j+\Lambda_k+\frac{\Lambda_j^2+\Lambda_k^2}{2}+\Lambda_j\Lambda_k+\frac{\Lambda_j^3+\Lambda_k^3}{6}\right) \\ &\quad +\sum_{j=1}^{p}\sum_{k=1}^{p}\left(\frac{1}{2}\Lambda_j\Lambda_k(\Lambda_j+\Lambda_k)+\frac{\Lambda_j^4+\Lambda_k^4}{24}+\frac{1}{4}\Lambda_j^2\Lambda_k^2+\frac{1}{2}\Lambda_j\Lambda_k(\Lambda_j^2+\Lambda_k^2)\right). \end{aligned}$$

Applying Theorem 1, we have

$$\sum_{j=1}^{p}\sum_{k=1}^{p}(\Lambda_j+\Lambda_k)=0,$$

$$\sum_{j=1}^{p}\sum_{k=1}^{p}\Lambda_j\Lambda_k=\left(\sum_{j=1}^{p}\Lambda_j\right)^2=0,$$

$$\sum_{j=1}^{p}\sum_{k=1}^{p}\frac{\Lambda_j^2+\Lambda_k^2}{2}=p\sum_{j=1}^{p}\Lambda_j^2=2n\left(M_1^5(G)+2M_2^2(G)+2Z_{(3,1)}(G)\right),$$

$$\sum_{j=1}^{p}\sum_{k=1}^{p}\frac{\Lambda_j^3+\Lambda_k^3}{6}=\frac{p}{3}\sum_{j=1}^{p}\Lambda_j^3=\frac{2n}{3}\sum_{\Delta}\prod_{ab\in E(\Delta)}(d_G(a)+d_G(b))\sqrt{d_G^2(a)+d_G^2(b)},$$

$$\sum_{j=1}^{p}\sum_{k=1}^{p}\frac{1}{2}\Lambda_j\Lambda_k(\Lambda_j+\Lambda_k)=0,$$

$$\sum_{j=1}^{p}\sum_{k=1}^{p}\frac{\Lambda_j^4+\Lambda_k^4}{24}=\frac{p}{12}\sum_{j=1}^{p}\Lambda_j^3=\frac{p}{6}\sum_{\square}\prod_{ab\in E(\square)}(d_G(a)+d_G(b))\sqrt{d_G^2(a)+d_G^2(b)},$$

$$\sum_{j=1}^{p}\sum_{k=1}^{p}\frac{1}{4}\Lambda_j^2\Lambda_k^2=\frac{1}{4}\left(\sum_{j=1}^{p}\Lambda_j^2\right)^2=\left(M_1^5(G)+2M_2^2(G)+2Z_{(3,1)}(G)\right)^2,$$
$$\sum_{j=1}^{p}\sum_{k=1}^{p}\frac{1}{2}\Lambda_j\Lambda_k(\Lambda_j^2+\Lambda_k^2)=0.$$

Combining the above equations, we obtain the inequality stated in Theorem 10. □

Theorem 11. *Let G represent a graph consisting of p vertices with $ES(G)$ being its elliptic Sombor matrix. Then*

$$ESE(G)\geq\sqrt{p^2+4\left(M_1^5(G)+2M_2^2(G)+2Z_{(3,1)}(G)\right)}. \tag{4.1}$$

Proof. Suppose that $\Lambda_1,\ldots,\Lambda_p$ is the elliptic Sombor eigenvalue of $ES(G)$. We get

$$(ESE(G))^2=\sum_{j=1}^{p}\sum_{k=1}^{p}e^{\Lambda_j+\Lambda_k}=\sum_{j=1}^{p}e^{2\lambda_j}+2\sum_{j<k}e^{\Lambda_j+\Lambda_k}.$$

Also, we deduce

$$\begin{aligned}\sum_{j<k}e^{\Lambda_j+\Lambda_k} &\geq p(p-1)\left(\prod_{j<k}e^{\Lambda_j}e^{\Lambda_k}\right)^{\frac{2}{p(p-1)}}=p(p-1)\left(\prod_{j=1}^{p}e^{\Lambda_j}\right)^{\frac{2}{p}}\\ &= p(p-1)e^{\frac{2}{p}\sum_{j=1}^{p}\Lambda_j}=p(p-1),\end{aligned}$$

and

$$\sum_{j=1}^{p}e^{2\lambda_j}=\sum_{j=1}^{p}\sum_{r\geq 0}\frac{(2\lambda_j)^r}{r!}=p+2\sum_{j=1}^{p}\Lambda_j^2+\sum_{j=1}^{p}\sum_{r\geq 3}\frac{(2\lambda_j)^r}{r!}.$$

We consider a multiplier $\beta\in[0,8]$ and we find

$$\begin{aligned}\sum_{j=1}^{p}e^{2\lambda_j} &\geq p+4\left(M_1^5(G)+2M_2^2(G)+2Z_{(3,1)}(G)\right)+\beta\sum_{j=1}^{p}\sum_{r\geq 3}\frac{(2\lambda_j)^r}{r!}\\ &= (1-\beta)p+(4-\beta)\left(M_1^5(G)+2M_2^2(G)+2Z_{(3,1)}(G)\right)+\beta ESE(G).\end{aligned}$$

Therefore,

$$(ESE(G))^2\geq(p-\beta)p+(4-\beta)\left(M_1^5(G)+2M_2^2(G)+2Z_{(3,1)}(G)\right)+\beta ESE(G).$$

By solving the above inequality with respect to $ESE(G)$, we deduce

$$ESE(G) \geq \frac{\beta}{2} + \sqrt{(\frac{\beta}{2} - p)^2 + (4-\beta)\left(M_1^5(G) + 2M_2^2(G) + 2Z_{(3,1)}(G)\right)}. \tag{4.2}$$

Function $f(x) = \frac{x}{2} + \sqrt{(\frac{x}{2} - p)^2 + (4-x)\left(M_1^5(G) + 2M_2^2(G) + 2Z_{(3,1)}(G)\right)}$ is a monotonically decreasing function with respect to x on $[0,8]$ for $p \geq 2$ and $p_2 \geq 1$. Setting $\beta = 0$ into (4.2), we get (4.1). □

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- Conflict of interest/Competing interests
  We declare that we do not have any commercial or associative interest that represents a conflict of interest in connection with the work submitted.
- Consent for publication
  The author agree to publications.
- Availability of data and materials
  All data generated or analysed during this study are included in this published article.
- Authors' contributions
  All authors contributed equally in the preparation of this manuscript.

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