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Research Article

Lower bounds on the k-limited packing number of a graph

Michael A. Henning[1], Nader Jafari Rad[2,*]

[1]Department of Mathematics and Applied Mathematics, University of Johannesburg
Auckland Park, 2006 South Africa
mahenning@uj.ac.za

[2]Department of Mathematics, Shahed University, Tehran, Iran
*n.jafarirad@gmail.com

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Abstract: For a given integer $k \geq 1$, a subset S of vertices of a graph G is a k-limited packing if $|N_G[v] \cap S| \leq k$ for all $v \in V(G)$, where $N_G[v]$ denotes the closed neighborhood of a vertex v in G. The k-limited packing number, $L_k(G)$, is the maximum cardinality of a k-limited packing in G. In this paper we present a probabilistic lower bound for the k-limited packing number of a graph. In particular we improve a previous lower bound given in [Discrete Appl. Math. 184 (2015), 146–153]. We also present a randomized algorithm for the k-limited packing number of a graph.

Keywords: k-limited packing number, Probabilistic methods.

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1. Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The order and size of a graph G denoted $n(G)$ and $m(G)$, are $|V(G)|$ and $|E(G)|$, respectively. Two vertices u and v of G are *adjacent* if $uv \in E(G)$, and are called *neighbors.* The *open neighborhood* $N_G(v)$ of a vertex v in G is the set of neighbors of v, while the *closed neighborhood* of v is the set $N_G[v] = \{v\} \cup N_G(v)$. A *packing* (sometimes called a 2-*packing* in the literature) of G is a set S of vertices such that $N_G[u] \cap N_G[v] = \emptyset$ for every two distinct vertices $u, v \in S$, and the *packing number* of G, $\rho(G)$, is the maximum cardinality of a packing in G. An *open packing* of G is a set S of vertices such that $N_G(u) \cap N_G(v) = \emptyset$ for every two distinct vertices $u, v \in S$, and the *open*

* *Corresponding Author*

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packing number of G, $\rho^0(G)$, is the maximum cardinality of an open packing in G. The concept of a packing in graphs is very well studied in the literature, see, for example, in [7, 11].

Limited packings in graphs were introduced by Gallant, Gunther, Hartnell and Rall [4] in 2010 as a generalization of packing in graphs. For a given integer $k \geq 1$, Gallant et al. defined a subset of S of vertices of a graph G to be a *k-limited packing* if $|N_G[v] \cap S| \leq k$ for all $v \in V(G)$. The *k-limited packing number*, $L_k(G)$, is the maximum cardinality of a k-limited packing in G. When $k = 1$, a 1-limited packing is precisely a packing, that is, $L_1(G) = \rho(G)$. We note that if $k > \Delta(G)$ where $\Delta(G)$ denotes the maximum degree among all vertices in G, then $L_k(G) = n(G)$. The concept of limited packing was further studied, for example, in [10, 12].

Total limited packings in graphs were introduced by Moghaddam, Mojdeh and Samadi [8] in 2016. For a given integer $k \geq 1$, Moghaddam et al. defined a subset of S of vertices of a graph G to be a *k-total limited packing* if $|N_G(v) \cap S| \leq k$ for all $v \in V(G)$. The *k-total limited packing number*, $L_{k,t}(G)$, is the maximum cardinality of a k-total limited packing in G. When $k = 1$, a 1-total limited packing is precisely an open packing, that is, $L_{1,t}(G) = \rho^0(G)$. We note that if $k \geq \Delta(G)$, then $L_{k,t}(G) = n(G)$.

We remark that k-limited packing and k-total limited packing is related to multiple domination (also called ℓ-tuple domination in the literature) and multiple total domination (also called ℓ-tuple total domination in the literature) in graphs. For recent books on domination in graphs, we refer the reader to [5, 6].

A powerful tool to obtain bounds for various combinatorial objects is the probabilistic method. We refer the reader to the excellent book by Alon and Spencer [1] on the state of the art on the probabilistic method. Gagarin and Zverovich [3] developed a new probabilistic approach to limited packing number in graphs, resulting in the following lower bound for the k-limited packing number of a graph.

Theorem 1. ([3]) *If G is a graph of order n with maximum degree $\Delta(G) = \Delta \geq k \geq 1$, then*

$$L_k(G) \geq \frac{kn}{(k+1)\sqrt[k]{\binom{\Delta}{k}(\Delta+1)}}.$$

2. Main results

Our contributions in this paper are twofold. Firstly we prove a (new) probabilistic lower bound for the k-limited packing number of a graph that improves the bound given in Theorem 1.

Theorem 2. *If G is a graph of order n with maximum degree $\Delta(G) = \Delta \geq k \geq 1$, then*

$$L_k(G) \geq \frac{kn}{(k+1)\sqrt[k]{\binom{\Delta}{k}(\Delta+1)}}\left(1+\left(1-\frac{1}{\sqrt[k]{\binom{\Delta+1}{k+1}(1+k)}}\right)^{(1+\Delta)}\right).$$

A proof of Theorem 2 is given in Section 3. The bound in Theorem 2 is an improvement of the bound in Theorem 1. Since $\binom{\Delta+1}{k+1} = 1$ when $k = \Delta$ and since

$$\left(1-\left(\frac{1}{\binom{\Delta+1}{k+1}(1+k)}\right)^{\frac{1}{k}}\right) \to 0$$

as $k \to \infty$, an identical proof as that given in [3] yields that the bound of Theorem 2 is asymptotically best possible. We also remark that an improvement of the bound in Theorem 1 is presented in [9]. However in the proof of this result given in [9] they formed a set, namely, $X \cup D$ (see [9]) and claimed that it is a k-limited packing, which is not correct in general. In fact if a vertex outside D is adjacent to more than k vertices of D, then $X \cup D$ is not a k-limited packing. We remark that in [2] a lower bound for the k-limited packing number of a graph for large values of k is established. Our result in Theorem 2 holds for all $k \geq 1$.
Our second contribution is to present a randomized algorithm to find a k-limited packing set whose size satisfies the bound of Theorem 2. We present our randomized algorithm in Section 4.

3. Proof of Theorem 2

In this section, we present a proof of Theorem 2. Let G be a graph of order n with maximum degree $\Delta(G) = \Delta$. For k a positive integer, we note that if $k \geq \Delta + 1$, then $L_k(G) = n(G)$. For a positive integer $t \leq \Delta + 1$, we define

$$\tilde{c}_t = \tilde{c}_t(G) = \binom{\Delta+1}{t}.$$

We present next our key lemma.

Lemma 1. *If G is a graph of order n with maximum degree $\Delta(G) = \Delta \geq k \geq 1$ and if $0 < p < 1$, then there is a k-limited packing set L of G such that*

$$|L| \geq \alpha\left(1+(1-p)^{(1+\Delta)}\right),$$

where $\alpha = pn(1 - p^k\tilde{c}_{k+1})$.

Proof. Let $A \subseteq V(G)$ be a set obtained by choosing each vertex $v \in V(G)$, independently, with probability p, and let $B = \{v \in V(G) : N[v] \cap A = \emptyset\}$ and $B' = \{v \in V(G) : N_G[v] \subseteq B\}$. It is evident that $\deg_{G[B]}(v) = \deg_G(v)$ for every vertex $v \in B'$.

We follow the proof of Theorem 1 given in [3]. For $m \in \{k, \ldots, \Delta\}$, we denote

$$A_m = \{v \in A : |N(v) \cap A| = m\}.$$

For each set A_m, we form a set A'_m in the following way. For every vertex $v \in A_m$, we select $m-(k-1)$ (arbitrary) neighbors from $N_G(v) \cap A$ and add them to A'_m. Thus, $|A'_m| \le (m-k+1)|A_m|$ for each $m \in \{k, \ldots, \Delta\}$. For $m \in \{k+1, \ldots, \Delta\}$, we let

$$B_m = \{v \in V(G) \setminus A : |N_G(v) \cap A| = m\}.$$

For each set B_m, we form a set B'_m in the following way. For every vertex $v \in B_m$, we select $m-k$ (arbitrary) neighbors from $N_G(v) \cap A$ for every vertex $v \in B_m$ and adding them to B'_m. Thus, $|B'_m| \le (m-k)|B_m|$ for each $m \in \{k+1, \ldots, \Delta\}$. Let

$$X = A \setminus \left(\left(\bigcup_{m=k}^{\Delta} A'_m \right) \cup \left(\bigcup_{m=k+1}^{\Delta} B'_m \right) \right).$$

It is proved in [3] that

$$E(|X|) \ge pn - p^{k+1} n \sum_{m=0}^{\Delta-k} (m+1)\tilde{c}_{m+k+1} p^m (1-p)^{\Delta-k-m} = pn(1-p^k \tilde{c}_{k+1}) = \alpha. \quad (3.1)$$

Since for a random variable T, we have $\Pr(T \ge E(T)) > 0$, there is such a subset X such that X is a k-limited packing set in G and $|X| \ge \alpha$.

Now we consider B as a fixed set and can assume that $B' \neq \emptyset$, since B can be viewed as a randomly chosen subset of G. Thus we focus here on the graph $G[B]$. Let $D \subseteq B'$ be a set obtained by choosing each vertex $v \in B'$, independently, with probability p. For $m \in \{k, \ldots, \Delta\}$, we let

$$D_m = \{v \in D : |N_G(v) \cap D| = m\}.$$

For each set D_m, we form a set D'_m in the following way. For every vertex $v \in D_m$, we select $m-(k-1)$ (arbitrary) neighbors from $N_G(v) \cap D$ and add them to D'_m. We note that $|D'_m| \le (m-k+1)|D_m|$ for each $m \in \{k, \ldots, \Delta\}$. For $m \in \{k+1, \ldots, \Delta\}$, we let

$$F_m = \{v \in B \setminus D : |N(v) \cap D| = m\}.$$

For each set F_m, we form a set F'_m by selecting $m-k$ (arbitrary) neighbors from $N_G(v)\cap D$ for every vertex $v\in F_m$. We note that $|F'_m|\le (m-k)|F_m|$ for each $m\in\{k+1,\dots,\Delta\}$. Let

$$Y = D\setminus\left(\left(\bigcup_{m=k}^{\Delta} D'_m\right)\cup\left(\bigcup_{m=k+1}^{\Delta} F'_m\right)\right).$$

By construction, the resulting set Y is a k-limited packing set for $G[B]$. We note that

$$\begin{aligned} E(|Y|) &\ge E(|D|) - \left(\bigcup_{m=k}^{\Delta} E(|D'_m|)\right) - \left(\bigcup_{m=k+1}^{\Delta} E(|F'_m|)\right)\\ &\ge E(|D|) - \left(\bigcup_{m=k}^{\Delta}(m-k+1)E(|D_m|)\right) - \left(\bigcup_{m=k+1}^{\Delta}(m-k)E(|F_m|)\right). \end{aligned}$$

It is evident that $E(|D|)=p|B'|$. With an analogous and similar arguments to that used to establish Inequality (3.1) (given in [3]) we infer that

$$E(|D'_m|)\le p^{m+1}(1-p)^{\Delta-m}c_m|B|$$

and

$$E(|F'_m|)\le p^{m}(1-p)^{\Delta-m+1}c_m|B|.$$

By our earlier observations, we therefore infer that

$$\begin{aligned} E(|Y|) &\ge p|B| - p^{k+1}|B|\sum_{m=0}^{\Delta-k}(m+1)\tilde{c}_{m+k+1}p^m(1-p)^{\Delta-k-m}\\ &= p|B|(1-p^k\tilde{c}_{k+1}). \end{aligned} \tag{3.2}$$

Since for a random variable T, we have $\Pr(T\ge E(T))>0$, there is a subset $Y\subseteq B'$ such that Y is a k-limited packing set in $G[B]$ and $|Y|\ge p|B|(1-p^k\tilde{c}_{k+1})$.
We now return to the graph G and view B and Y as subsets of $V(G)$, where A is chosen randomly and where $|B|$ is a random variable. It is evident that the set $X\cup Y$ is a k-limited packing set for G, since $|N_G[v]\cap(X\cup Y)|\le k$ for all $v\in V(G)$. Moreover,

$$\begin{aligned} E(|X\cup Y|)=E(|X|+|Y|) &= E(|X|)+E(|Y|)\\ &\ge \alpha + p(1-p^k\tilde{c}_{k+1})E(|B|)\\ &\ge \alpha + p(1-p^k\tilde{c}_{k+1})n(1-p)^{1+\Delta}\\ &= \alpha(1+(1-p)^{1+\Delta})\\ &= \alpha(1+1(1-p)^{1+\Delta}). \end{aligned}$$

Therefore, there is a k-limited packing set L such that $|L|\ge\alpha(1+1(1-p)^{1+\Delta})$, as desired. □

Letting

$$p = \left(\frac{1}{\binom{\Delta+1}{k+1}(1+k)}\right)^{\frac{1}{k}},$$

it is proved in [3] that

$$\alpha \geq \frac{kn}{(k+1)\sqrt[k]{\binom{\Delta}{k}(\Delta+1)}}.$$

Thus as a consequence of the above lower bound on α, as an application of Lemma 1 we immediately infer our main result, namely Theorem 2. Recall its statement.

Theorem 2. *If G is a graph of order n with maximum degree $\Delta(G) = \Delta \geq k \geq 1$, then*

$$L_k(G) \geq \frac{kn}{(k+1)\sqrt[k]{\binom{\Delta}{k}(\Delta+1)}}\left(1+\left(1-\frac{1}{\sqrt[k]{\binom{\Delta+1}{k+1}(1+k)}}\right)^{(1+\Delta)}\right).$$

As remarked in Section 2, the bound of Theorem 2 is asymptotically best possible. Since any k-limited packing is a k-total limited packing, we have the following immediate lower bound for the k-total limited packing number of a graph.

Theorem 3. *If G is a graph of order n with maximum degree $\Delta(G) = \Delta \geq k \geq 1$, then*

$$L_{k,t}(G) \geq \frac{kn}{(k+1)\sqrt[k]{\binom{\Delta}{k}(\Delta+1)}}\left(1+\left(1-\frac{1}{\sqrt[k]{\binom{\Delta+1}{k+1}(1+k)}}\right)^{(1+\Delta)}\right).$$

4. A randomized algorithm

Gagarin and Zverovich [3] presented a randomized algorithm, namely Algorithm 1 in [3], to find a k-limited packing set whose size satisfies the bound of Theorem 1. In this section we develop the randomized Algorithm 1 (presented in [3]), and present a randomized algorithm to find a k-limited packing set whose size satisfies the bound of Theorem 2.

Our algorithm can be implemented to run in $O(n^2)$ time. In Line 2 to Line 15, this algorithm constructs a k-limited packing set by recursively removing unwanted vertices from the initially constructed set A. Lines 1–15 can be implemented in $O(n^2)$

time, as they are precisely the lines of Algorithm 1 of [3]. In Line 16 the algorithm forms the graph $G[B]$ by removing each vertex v such that $N[v] \cap A \neq \emptyset$ from G, and this can be done in $O(n+m)$ steps. In Lines 17–20, it forms the subset B' in $O(n+m)$ steps. In Lines 22–34 the algorithm mimics the computations of Lines 2-15 to constructs a k-limited packing set by recursively removing unwanted vertices from the initially constructed set D, and as it was seen it can be done in $O(n^2)$ steps. Finally in Line 35, the algorithm does an extension of the preliminary k-limited packing set $X \cup Y$. For this purpose, checking whether $X \cup Y$ is maximal or extending $X \cup Y$ to a maximal k-limited packing can be done in $O(n+m)$ time, since it examines the vertices of $V(G) \setminus (X \cup Y)$ one by one to decide whether to add them to the set $X \cup Y$ or not.

Algorithm 1 Randomized k-limited packing

Input: A graph G and an integer k with $1 \le k \le \Delta$.
Output: A k-limited packing set L of G.

1: Compute $p = \left(\frac{1}{\tilde{c}_{k+1}(1+k)}\right)^{\frac{1}{k}}$
2: Initialize $A = \emptyset$, $D = \emptyset$ and $B' = \emptyset$
3: **for** each vertex $v \in V(G)$ **do**
4:     with probability p, decide whether $v \in A$ or $v \notin A$.
5: **end for**
6: **for** each vertex $v \in V(G)$ **do**
7:     Compute $r = |N_G(v) \cap A|$
8:     **if** $v \in A$ and $r \ge k$ **then**
9:         Remove any $r-k+1$ vertices of $N_G(v) \cap A$ from A
10:     **end if**
11:     **if** $v \notin A$ and $r > k$ **then**
12:         Remove any $r-k$ vertices of $N_G(v) \cap A$ from A
13:     **end if**
14: **end for**
15: Put $X = A$
16: Form a set B by removing each vertex v such that $N[v] \cap A \neq \emptyset$ from G
17: **for** each vertex $v \in B$ **do**
18:     **if** $N[v] \subseteq B$ **then**
19:         $B' = B' \cup \{v\}$
20:     **end if**
21: **end for**
22: **for** each vertex $v \in B'$ **do**
23:     with probability p, decide whether $v \in D$ or $v \notin D$.
24: **end for**
25: **for** each vertex $v \in B$ **do**
26:     Compute $r = |N_G(v) \cap D|$
27:     **if** $v \in D$ and $r \ge k$ **then**
28:         Remove any $r-k+1$ vertices of $N_G(v) \cap D$ from D

    end if
    if $v \notin D$ and $r > k$ **then**
        Remove any $r-k$ vertices of $N_G(v) \cap D$ from D
    end if
end for
Put $Y = D$
Extend $X \cup Y$ to a maximal k-limited packing L
Return L

5. Concluding Remarks

In Lemma 1 we have shown that a graph G of order n with $\Delta(G) = \Delta \geq k \geq 1$ has a k-limited packing set of cardinality at least $\alpha(1 + (1-p)^{(1+\Delta)})$, where $0 < p < 1$ and

$$\alpha = pn\left(1 - p^k \binom{\Delta+1}{k+1}\right).$$

It would be interesting to study if this lower bound can be further improved to $\alpha(1 + s(1-p)^{(1+\Delta)})$ for each integer s. If this can be proved, then the bound of Theorem 2 will be improved to

$$L_k(G) \geq \frac{kn}{(k+1)\sqrt[k]{\binom{\Delta}{k}(\Delta+1)}}\left(1 + s\left(1 - \frac{1}{\sqrt[k]{\binom{\Delta+1}{k+1}(1+k)}}\right)^{(1+\Delta)}\right)$$

for each integer s.

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