Communications in Combinatorics and Optimization
Vol. xx, No. x (xxxx), pp. 1-12
https://doi.org/10.22049/cco.2025.29910.2219

CCO
Commun. Comb. Optim.

Research Article

Interconnections between the different energies of the complements of regular graphs

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Received: 24 July 2024; Accepted: 4 August 2025
Published Online: 14 August 2025

Abstract: The energy of a graph G is determined by the absolute sum of its eigenvalues. Similar to this concept, the distance energy, Harary Energy, Seidel energy, complementary distance energy and reciprocal complementary distance energy are all defined based on the eigenvalues of their respective matrices. In this paper, we study these energies on the complement of a regular graph G in terms of the energy of G. We explore exact relationships among these energies. Recent studies have explored equienergetic graphs concerning the adjacency and distance matrices. In this paper, we provide graphs illustrating the equienergetic properties with respect to six matrices. The results obtained extend some of the existing findings.

Keywords: complement of a graph, Energy, distance energy, Harary energy, complementary distance energy, reciprocal complementary distance energy.

AMS Subject classification: 05C50, 05C76, 05C12

1. Introduction

Molecular matrices play a crucial role in providing structural descriptors for quantitative structure property relationships (QSPR) and quantitative structure activity

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relationships (QSAR) models, as they encode topological information in versatile ways. In the literature, many molecular matrices have been defined, such as the distance matrix, Harary matrix, complementary distance matrix and reciprocal complementary distance matrix.

Structural-property models for the boiling temperature, vaporization enthalpy, standard Gibbs energy of formation, refractive index and density of alkanes are developed using structural descriptors calculated from the CD-matrix [13, 14]. Consequently, the QSPR models can be derived using the complementary distance matrix. The Harary index, which is obtained from the Harary matrix, has been successfully tested in several structure-property relationships. It is used to predict physical properties such as critical pressure, critical temperature, surface tension, boiling point, melting point, heat of evaporation, molar refraction and molar volume of alkanes [16]. Therefore, it is meaningful to explore the mathematical properties and chemical applications of these matrices.

In this article, we consider all the graphs to be simple and connected. The degree of a vertex u_x denoted as d_x, defined as the number of edges which are incident to it. If G is said to be r-regular then each vertex of G is of degree r. The length of the shortest path connecting any two vertices v_x and v_y is the distance between the vertices v_x and v_y and is denoted by d_{xy}. The diameter of G is the maximum distance between any two vertices and is denoted by $diam(G)$.

The *adjacency matrix* of graph G is defined as $A(G) = [a_{xy}]$, in which $a_{xy} = 1$, if v_x is adjacent to v_y otherwise $a_{xy} = 0$. The *distance matrix* is expressed as $D(G) = [d_{xy}]$ if $x \neq y$ and 0 otherwise. The *Harary matrix* [12], which is also known as reciprocal distance matrix [14], is expressed as $H(G) = [hr_{xy}]$ where $hr_{xy} = \frac{1}{d_{xy}}$ if $x \neq y$ and 0 otherwise. The *complementary distance matrix* [13], of a graph G is expressed as $CD(G) = [cd_{xy}]$ where $cd_{xy} = 1 + diam(G) - d_{xy}$ if $x \neq y$ and 0 otherwise. The *reciprocal complementary distance matrix* [13], of a graph G, denoted by $RD(G)$, is expressed as $RD(G) = [rd_{xy}]$ where $rd_{xy} = \frac{1}{1+diam(G)-d_{xy}}$ if $x \neq y$ and 0 otherwise.

The *Seidel matrix* of graph G is defined as $S(G) = [s_{xy}]$, in which $s_{xy} = 1$, if v_x is not adjacent to v_y, $s_{xy} = -1$, if v_x is adjacent to v_y and $s_{xy} = 0$ if $x = y$. The eigenvalues associated with the adjacency, distance, Harary, complementary distance, reciprocal complementary distance matrix and Seidel matrix are known as the adjacency (or A), distance (or D), Harary (or H), complementary distance (or CD), reciprocal complementary distance (or RD) and Seidel (or S) eigenvalues of the graph G, respectively.

The concept of A-energy, introduced by I. Gutman in 1978 [5], denotes the absolute sum of all eigenvalues of a graph G, symbolized as $\mathcal{E}_A(G)$. This idea extends to any matrix M associated with a graph G, with a zero trace, defining the M-energy as the absolute sum of its eigenvalues, denoted by $\mathcal{E}_M(G)$. Consequently, we have the H-energy $\mathcal{E}_H(G)$ [4], D-energy $\mathcal{E}_D(G)$ [11], CD-energy $\mathcal{E}_{CD}(G)$ [21], RD-energy $\mathcal{E}_{RD}(G)$ [28] and S-energy $\mathcal{E}_S(G)$ [6] accordingly. Two graphs G_1 and G_2 of the same order are termed *M-equienergetic* if their M-energies exhibit no difference, expressed as $\mathcal{E}_M(G_1) - \mathcal{E}_M(G_2) = 0$. For additional notation and terminology, we follow [2].

Let n_G^+, n_G^- and n_G^0 represents the count of positive, negative and zero eigenvalues of graph G respectively. Some studies on H-energy, CD-energy, RD-energy, D-energy and equienergetic graphs can be seen in [1, 4, 17, 18, 20–22, 28]. Indulal [9, 10], presented an open problem aimed at characterizing or constructing families of graphs that exhibit equienergetic properties concerning both adjacency and distance matrices. Ramane et al. [24] have introduced some families of graphs that demonstrate equienergetic behavior with respect to both the adjacency and distance matrices. Relationships among various graph energies were previously investigated in [29], while the study of multiply equienergetic graphs was presented in [7]. This prompts further exploration into scenarios where different matrices associated with graphs that exhibit equienergetic characteristics.
The main findings rely on the following existing results.

Proposition 1. [24] Consider a graph G that is r-regular and has order n. If the A-eigenvalues of G are $\lambda_k; k = 1, 2, \dots, n$, then

$$\sum_{k=1}^{n} \mid \lambda_k + 2 \mid = \mathcal{E}_A(G) + 2n - 4n_G^- + 2 \sum_{\lambda_k \in (-2,0)} (\lambda_k + 2).$$

Let $\overline{G}$ represent the complement of a graph G.

Theorem 1. [2] *Consider a graph G that is r-regular and has order n. If $r = \lambda_1$, $\lambda_k; k = 2, 3, \dots, n$ are the A-eigenvalues of G, then the complement $\overline{G}$ of G is an $(n-r-1)$-regular graph with the A-eigenvalues $n - r - 1$, $-(1 + \lambda_k); k = 2, 3, \dots, n$.*

Theorem 2. [1] *Consider a graph G that is r-regular and has order n. If $r = \lambda_1$, $\lambda_k; k = 2, 3, \dots, n$ are the A-eigenvalues of G and $diam(G) \leq 2$, then the H-eigenvalues of G are $\frac{1}{2}(n + r - 1)$, $\frac{1}{2}(\lambda_k - 1)$; $k = 2, 3, \dots, n$.*

Theorem 3. [21] *Consider a graph G that is r-regular and has order n. If $r = \lambda_1$, $\lambda_k; k = 2, 3, \dots, n$ are the A-eigenvalues of G and $diam(G) \leq 2$, then the CD-eigenvalues of G are $n + r - 1$, $\lambda_k - 1; k = 2, 3, \dots, n$.*

Theorem 4. [28] *Consider a graph G that is r-regular and has order n. If $r = \lambda_1$, $\lambda_k; k = 2, 3, \dots, n$ are the A-eigenvalues of G and $diam(G) \leq 2$, then the RD-eigenvalues of G are $n - 1 - \frac{r}{2}$, $-(1 + \frac{\lambda_k}{2}); k = 2, 3, \dots, n$.*

Theorem 5. [3] *Consider a graph G that is r-regular and has order n. If $r = \lambda_1$, $\lambda_k; k = 2, 3, \dots, n$ are the A-eigenvalues of G and $diam(G) \leq 2$, then the D-eigenvalues of G are $2n - r - 2$, $-(\lambda_k + 2); k = 2, 3, \dots, n$.*

Theorem 6. [15] *Consider a graph G that is r-regular and has order n, then $diam(G) \leq 2$ or $diam(\overline{G}) \leq 2$.*

Let the order of iterated line graphs $L^k(G)$ of r-regular graph be n_k; $k = 0, 1, 2, \ldots$, where $L^0(G) = G$ and $L^1(G) = L(G)$.

Proposition 2. [25] *Consider a graph G that is $r(\geq 3)$-regular and has order n_0, then the graphs $L^k(G)$ for every $k > 1$ have same degree and have all negative eigenvalues equal to -2 and $\mathcal{E}_A(L^k(G)) = 4(n_k - n_{k-1})$.*

Theorem 7. [15] *Consider a graph G that is r-regular and has order $n(\geq 8)$, then $diam(\overline{L^k(G)}) = 2$ for every $k \geq 1$.*

Let $n^-_{G_1}$ and $n^-_{G_2}$ represent the number of negative A-eigenvalues of the graphs G_1 and G_2, respectively.

Proposition 3. [23] *Consider the graphs G_k; $k = 1, 2$ that are r_k-regular and both have same order n. If $G_k; k = 1, 2$ are the A-equienergetic graphs which have no A-eigenvalues in the interval $(-1, 0)$, then the graphs $\overline{G_k}; k = 1, 2$ are A-equienergetic if and only if $r_1 + n^-_{G_1} = r_2 + n^-_{G_2}$.*

Theorem 8. [19] *Consider the graphs G_k; $k = 1, 2$ that are r_k-regular and both have same order n, then the graphs $L^k(G_1)$ and $L^k(G_2)$ are S-equienergetic for every $k \geq 1$.*

2. Main Results

If one aims to fully investigate a regular graph G, then Theorem 6 guarantees that it suffices to consider G under the condition that $diam(\overline{G}) \leq 2$. This leads us to discuss the following explicit relation among $\mathcal{E}_{RD}(\overline{G})$, $\mathcal{E}_A(G)$ and $\mathcal{E}_D(\overline{G})$:

Theorem 9. *Consider a graph G that is r-regular and has order n. If $r = \lambda_1$, $\lambda_k; k = 2, 3, \ldots, n$ are the A-eigenvalues of G and $diam(\overline{G}) \leq 2$, then*

$$2\mathcal{E}_{RD}(\overline{G}) = 2n - 2n^+_G - 2 \sum_{\lambda_k \in (0,1)} (\lambda_k - 1) + \mathcal{E}_A(G) = \mathcal{E}_D(\overline{G}).$$

Proof. Let $\eta_1, \eta_2, \ldots, \eta_n$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the RD and A-eigenvalues of G, respectively. If $diam(\overline{G}) \leq 2$, then by Theorems 4 and 1, the RD-eigenvalues of

$\overline{G}$ are $\frac{1}{2}(n+r-1)$, $\frac{1}{2}(\lambda_k-1)$; $k=2,3,\cdots,n$. Therefore, the RD-energy of $\overline{G}$ is

$$\begin{aligned}
\mathcal{E}_{RD}(\overline{G}) &= \sum_{k=1}^{n}|\eta_k| = \frac{1}{2}\left(n+r-1+\sum_{k=2}^{n}|\lambda_k-1|\right) \\
2\mathcal{E}_{RD}(\overline{G}) &= n+\sum_{k=1}^{n}|\lambda_k-1| \\
&= n+\sum_{\lambda_k\leq 1}(-\lambda_k+1)+\sum_{\lambda_k\geq 1}(\lambda_k-1) \\
&= n+\sum_{\lambda_k\leq 1}-\lambda_k+n_\lambda[\lambda_n,1]+\sum_{\lambda_k>1}\lambda_k-n_\lambda(1,\lambda_1] \\
&= n_\lambda[\lambda_n,1]-n_\lambda(1,\lambda_1]+\sum_{\lambda_k\leq 0}|\lambda_k|+\sum_{\lambda_j\in(0,1]}-\lambda_k+\sum_{\lambda_k>1}\lambda_k, \qquad (2.1)
\end{aligned}$$

where $n_\lambda(I)$ denotes the count of A-eigenvalues of G that are contained in the interval I. The A-energy $\mathcal{E}_A(G)$ of a graph G can be expressed as

$$\mathcal{E}_A(G) = \sum_{\lambda_k\leq 0}|\lambda_k|+\sum_{\lambda_k\in(0,1]}\lambda_k+\sum_{\lambda_k>1}\lambda_k. \qquad (2.2)$$

The inertia of G can be expressed as

$$n = n_\lambda[\lambda_n,1]+n_\lambda(1,\lambda_1] = n_\lambda(0,1]+n_\lambda(1,\lambda_1]+n^0+n^-. \qquad (2.3)$$

By equalities (2.2) and (2.3), the equality (2.1) becomes

$$\begin{aligned}
2\mathcal{E}_{RD}(\overline{G}) &= n+n+\mathcal{E}_A(G)-2n_\lambda(1,\lambda_1]-2\sum_{\lambda_k\in(0,1]}\lambda_k \\
&= 2n+\mathcal{E}_A(G)+2n_\lambda(0,1]-2n_G^+-2\sum_{\lambda_k\in(0,1]}\lambda_k \\
&= 2n+\mathcal{E}_A(G)-2n_G^++2n_\lambda(0,1]-2\sum_{\lambda_k\in(0,1]}(\lambda_k-1)-2n_\lambda(0,1] \\
&= 2n+\mathcal{E}_A(G)-2n_G^+-2\sum_{\lambda_k\in(0,1)}(\lambda_k-1).
\end{aligned}$$

The equality $2\mathcal{E}_{RD}(\overline{G})=\mathcal{E}_D(\overline{G})$ follows directly from the interrelations between D-eigenvalues and RD-eigenvalues by Theorems 5, 4 and 1 which concludes the proof. □

The following are useful factors when discussing immediate results based on Theorem 9.

1. $\sum\limits_{\lambda_k\in(0,1)}(\lambda_k - 1) = 0$ if only if there are no A-eigenvalues λ_k in the open interval $(0, 1)$.

2. For each A-eigenvalue $\lambda_k \in (0,1)$ of a graph G, we get $-\sum\limits_{\lambda_k\in(0,1)}(\lambda_k - 1) > 0$ and $n_G^- + \sum\limits_{\lambda_k\in(0,1)}(\lambda_k - 1) > 0$.

With help of these, we arrive at the following results:

Corollary 1. *Consider a graph G that is r-regular and has order n with diam$(\overline{G}) \leq 2$, then*

$$n - n_G^+ + \frac{1}{2}\mathcal{E}_A(G) \leq \mathcal{E}_{RD}(\overline{G}) < n + \frac{1}{2}\mathcal{E}_A(G).$$

The left side of the inequality attains equality if and only if G does not possess any A-eigenvalues within the interval $(0, 1)$.

Let $n_{G_1}^+$ and $n_{G_2}^+$ represent the number of positive A-eigenvalues in G_1 and G_2, respectively.

Corollary 2. *Consider the graphs G_k; $k = 1, 2$ that are r_k-regular and both have same order n with diam$(\overline{G_k}) \leq 2$. If $G_k; k = 1, 2$ are A-equienergetic graphs, then their complements $\overline{G_k}; k = 1, 2$ are RD-equienergetic if and only if*

$$n_{G_1}^+ + \sum_{\lambda_k^{'}\in(0,1)}(\lambda_k^{'} - 1) = n_{G_2}^+ + \sum_{\lambda_k^{''}\in(0,1)}(\lambda_k^{''} - 1).$$

In particular, when $G_k; k = 1, 2$ do not possess any A-eigenvalues within the interval $(0, 1)$, the graphs $\overline{G_k}; k = 1, 2$ are RD-equienergetic if and only if $n_{G_1}^+ = n_{G_2}^+$.

The equality $\mathcal{E}_D(\overline{G}) = 2\mathcal{E}_{RD}(\overline{G})$ in Theorem 10 implies that if there exist graphs meeting the conditions of Theorem 10 and are D-equienergetic, then they are also RD-equienergetic and vice versa. Based on this observation, we derive the following result:

Corollary 3. *Consider the graphs G_k; $k = 1, 2$ that are r_k-regular and both have same order n with diam$(\overline{G_k}) \leq 2$. Then $\overline{G_k}; k = 1, 2$ are RD-equienergetic graphs if and only if they are D-equienergetic graphs.*

The following theorem presents an explicit relation among $\mathcal{E}_H(\overline{G})$, $\mathcal{E}_A(G)$ and $\mathcal{E}_{CD}(\overline{G})$.

Theorem 10. *Consider a graph G that is r-regular and has order n. If $r = \lambda_1$, $\lambda_k; k = 2, 3, \ldots, n$ are the A-eigenvalues of G and diam$(\overline{G}) \leq 2$, then*

$$2\mathcal{E}_H(\overline{G}) = 4n - 2r - 4 - 4n_G^- + 2\sum_{\lambda_k\in(-2,0)}(\lambda_k + 2) + \mathcal{E}_A(G) = \mathcal{E}_{CD}(\overline{G}).$$

Proof. Let $\sigma_1, \sigma_2, \dots, \sigma_n$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the H and A-eigenvalues of G, respectively. If $diam(\overline{G}) \leq 2$, then by Theorems 1 and 2, the H-eigenvalues of $\overline{G}$ are $\frac{1}{2}(2n - r - 2)$, $\frac{1}{2}(-\lambda_k - 2)$; $k = 2, 3, \dots, n$. Therefore, the H-energy of $\overline{G}$ is

$$\begin{aligned}
\mathcal{E}_H(\overline{G}) &= \sum_{k=1}^{n} |\sigma_k| = \frac{1}{2}\left(2n - r - 2 + \sum_{k=2}^{n} |-\lambda_k - 2|\right) \\
2\mathcal{E}_H(\overline{G}) &= 2n - 2r - 4 + \sum_{k=1}^{n} |-\lambda_k - 2| \\
&= 2n - 2r - 4 + \mathcal{E}_A(G) + 2n - 4n_G^- + 2 \sum_{\lambda_k \in (-2,0)} (\lambda_k + 2) \\
&\quad \text{by Proposition 1} \\
&= 4n - 2r - 4 - 4n_G^- + 2 \sum_{\lambda_k \in (-2,0)} (\lambda_k + 2) + \mathcal{E}_A(G).
\end{aligned}$$

The equality $2\mathcal{E}_H(\overline{G}) = \mathcal{E}_{CD}(\overline{G})$ follows directly from the interrelations between H-eigenvalues and CD-eigenvalues from Theorems 2, 3 and 1 which concludes the proof. □

The following are useful factors when discussing immediate results based on Theorem 10.

1. $\sum_{\lambda_k \in (-2,0)} (\lambda_k + 2) = 0$ if only if there are no A-eigenvalues λ_k in the open interval $(-2, 0)$.
2. For each A-eigenvalue $\lambda_k \in (-2, 0)$ of a graph G, we get $\sum_{\lambda_k \in (-2,0)} (\lambda_k + 2) > 0$ and $2n_G^- - \sum_{\lambda_k \in (-2,0)} (\lambda_k + 2) > 0$.

With help of these, we arrive at the following results.

Corollary 4. *Consider a graph G that is r-regular and has order n with $diam(\overline{G}) \leq 2$, then*

$$2n - r - 2 - 2n_G^- + \frac{1}{2}\mathcal{E}_A(G) \leq \mathcal{E}_H(\overline{G}) < 2n - r - 2 + \frac{1}{2}\mathcal{E}_A(G).$$

The left side of the inequality attains equality if and only if G does not possess any A-eigenvalues within the interval $(-2, 0)$.

Now, we proceed to provide a characterization to construct H-equienergetic graphs by using Theorem 10. Let the A-eigenvalues of regular graphs G_1 and G_2 be denoted by $\lambda_k^{'}; k = 1, 2, \dots, n$ and $\lambda_k^{''}; k = 1, 2, \dots, n$ respectively. Also, let $n_{G_1}^-$ and $n_{G_2}^-$ represent the number of negative A-eigenvalues in G_1 and G_2 respectively.

Corollary 5. *Consider the graphs G_k; $k=1,2$ that are r_k-regular and both have same order n with $diam(\overline{G_k}) \leq 2$. If $G_k; k=1,2$ are A-equienergetic graphs, then their complements $\overline{G_k}; k=1,2$ are H-equienergetic if and only if*

$$r_1 + 2n^-_{G_1} - \sum_{\lambda^{'}_k \in(-2,0)} (\lambda^{'}_k + 2) = r_2 + 2n^-_{G_2} - \sum_{\lambda^{''}_k \in(-2,0)} (\lambda^{''}_k + 2).$$

In particular, when $G_k; k=1,2$ do not possess any A-eigenvalues within the interval $(-2,0)$, the graphs $\overline{G_k}; k=1,2$ are H-equienergetic if and only if $r_1 + 2n^-_{G_1} = r_2 + 2n^-_{G_2}$.

Similarly to Corollary 4 and Corollary 5, we have the following results:

Corollary 6. *Consider a graph G that is r-regular and has order n with diameter $diam(\overline{G}) \leq 2$, then*

$$4n - 2r - 4 - 4n^-_G + \mathcal{E}_A(G) \leq \mathcal{E}_H(\overline{G}) < 4n - 2r - 4 + \mathcal{E}_A(G).$$

The left side of the inequality attains equality if and only if G does not possess any A-eigenvalues within the interval $(-2,0)$.

Corollary 7. *Consider the graphs G_k; $k=1,2$ that are r_k-regular and both have same order n with $diam(\overline{G_k}) \leq 2$. If $G_k; k=1,2$ are A-equienergetic graphs, then their complements $\overline{G_k}; k=1,2$ are H-equienergetic if and only if*

$$r_1 + 2n^-_{G_1} - \sum_{\lambda^{'}_k \in(-2,0)} (\lambda^{'}_k + 2) = r_2 + 2n^-_{G_2} - \sum_{\lambda^{''}_k \in(-2,0)} (\lambda^{''}_k + 2).$$

In particular, when $G_k; k=1,2$ do not possess any A-eigenvalues within the interval $(-2,0)$, the graphs $\overline{G_k}; k=1,2$ are H-equienergetic if and only if $r_1 + 2n^-_{G_1} = r_2 + 2n^-_{G_2}$.

The equality $\mathcal{E}_{CD}(\overline{G}) = 2\mathcal{E}_H(\overline{G})$ in Theorem 10 implies that if there exist graphs meeting the conditions of Theorem 10 and are H-equienergetic, then they are also CD-equienergetic and vice versa. Based on this observation, we derive the following result.

Corollary 8. *Consider the graphs G_k; $k=1,2$ that are r_k-regular and both have same order n with $diam(\overline{G_k}) \leq 2$, then $\overline{G_k}; k=1,2$ are H-equienergetic graphs if and only if they are CD-equienergetic graphs.*

Theorem 11. *Consider the graphs G_j; $j=1,2$ that are $r(\geq 3)$-regular and both have same order $n(\geq 8)$, then the graphs $\overline{\mathcal{L}^k(G_j)}$; $j=1,2$ are A-equienergetic, H-equienergetic and CD-equienergetic for all $k>1$.*

Proof. Consider the graphs G_j; $j = 1, 2$ which are $r(\geq 3)$-regular and both have same order n, then the graphs $L^k(G_j)$, for all $k > 1$, exhibit no A-eigenvalues in the interval $(-2, 0)$ and are A-equienergetic as per Proposition 2. Moreover, both graphs $\overline{L^k(G_j)}$ have the same order and degree. According to Proposition 3, the graphs $\overline{L^k(G_j)}$; $j = 1, 2$, are also A-equienergetic. If $n \geq 8$, then by Theorem 7, the graphs $\overline{L^k(G_j)}$; $j = 1, 2$, both have diameter 2. Consequently, as per Corollary 5, these graphs are H-equienergetic. Now, by Corollary 8, these graphs are CD-equienergetic. Thus, the graphs $\overline{L^k(G_j)}$; $j = 1, 2$, are both A-equienergetic, H-equienergetic and CD-equienergetic for all $k > 1$. □

Remark 1. In the papers [17, 26], the authors presented H-equienergetic and CD-equienergetic graphs $\overline{L^k(G_j)}$; $j = 1, 2$, for the r-regular graphs G_j of order n, with the constraint $r \leq \frac{n-1}{2}$. However, Theorem 11 does not impose any such restriction.

Example 1. The regular iterated line graphs, denoted as $G_1 = \mathcal{L}^k(K_{n,n} \times K_{n-1})$ and $G_2 = \mathcal{L}^k(K_{n-1,n-1} \times K_n)$, have the same order and degree. Here, the symbol $\times$ represents the Cartesian product. These graphs exhibit A-equienergetic properties, sharing an equal number of negative A-eigenvalues that lie outside the interval $(-2, 0)$ for all $k \geq 1$ and $n \geq 5$ [27]. According to Theorem 7, the diameter of both $\overline{G_1}$ and $\overline{G_2}$ is 2. Consequently, these graphs are H-equienergetic for all $n \geq 5$ and $k \geq 1$ by Corollary 5. Furthermore, they are also A-equienergetic for all $n \geq 4$ and $k \geq 0$ [27]. Thus, these graphs exhibit both A-equienergetic and H-equienergetic properties for all $n \geq 5$ and $k \geq 1$. Also, the graphs G_1 and G_2 have the same number of positive A-eigenvalues which do not have any eigenvalues in the interval $(0, 1)$ for all $k \geq 1$ and $n \geq 6$ [27]. Therefore, the graphs $\overline{G_1}$ and $\overline{G_2}$ are RD-equienergetic for all $n \geq 6$ and $k \geq 1$ by Corollary 2. Hence, for all $n \geq 6$ and $k \geq 1$, the graphs $\overline{G_1}$ and $\overline{G_2}$ are A-equienergetic, H-equienergetic, CD-equienergetic, RD-equienergetic and D-equienergetic.

Theorem 12. *Consider the graphs G_j; $j = 1, 2$ that are $r(\geq 4)$-regular and both have same order $n(\geq 8)$, then the graphs $\overline{\mathcal{L}^k(G_j)}$; $j = 1, 2$ are D-equienergetic and RD-equienergetic for all $k > 1$.*

Proof. If the graphs G_j; $j = 1, 2$ are $r(\geq 4)$-regular, then as mentioned in the Remark 3.5 of [24], for all $k > 1$, the graphs $L^k(G_j)$; $j = 1, 2$, exhibit the same count of positive A-eigenvalues, each at least 2. Additionally, these graphs are A-equienergetic and share the same degree, as indicated by Proposition 2. For all $n \geq 8$, in accordance with Theorem 7, the graphs $\overline{L^k(G_j)}$; $j = 1, 2$, have the diameter 2. Consequently, following Corollary 2, these graphs are RD-equienergetic for all $k > 1$. Now, by applying Corollary 3, these graphs are also D-equienergetic. □

Remark 2. In the papers [8, 22], the authors presented D-equienergetic and RD-equienergetic graphs $\overline{L^k(G_j)}; j = 1, 2$, for the r-regular graphs G_j of order n, with the constraint $r \leq \frac{n-1}{2}$. However, Theorem 12 does not impose any such condition.

Note 1. Given any graph G, it is well known that $\mathcal{E}_S(G) = \mathcal{E}_S(\overline{G})$. Therefore, If G_j; $j = 1, 2$ are $r(\geq 4)$-regular graphs, both of same order $n(\geq 8)$, then by Theorems 11, 12 and 8 the graphs $\overline{\mathcal{L}^k(G_j)}$; $j = 1, 2$ are A-equienergetic, D-equienergetic, H-equienergetic, CD-equienergetic, RD-equienergetic and S-equienergetic for all $k > 1$.

There are certain classes of graphs with the diameter of G and diameter of $\overline{G}$ is at most 2 such as strongly regular graphs and the graphs $K_n \times K_m; n > 2, m > 2$. On such classes of graphs, we present the following explicit relations among various energies:

Theorem 13. *Consider a graph G that is regular and has order n. If $diam(G) \leq 2$ and $diam(\overline{G}) \leq 2$, then*

$$\mathcal{E}_D(\overline{G}) = \mathcal{E}_{CD}(G) = 2\mathcal{E}_H(G) = 2\mathcal{E}_{RD}(\overline{G}) = 2n - 2n_G^+ - 2 \sum_{\lambda_i \in (0,1)} (\lambda_i - 1) + \mathcal{E}_A(G).$$

Proof. If $diam(G) \leq 2$ and $diam(\overline{G}) \leq 2$, then the proof directly follows from the interconnections among the eigenvalues of the involved matrices and Theorem 9. □

Conclusion

In this paper, we have studied the energy concept pertaining to matrices A, D, H, CD and RD of regular graphs, examining their interconnections. This investigation could be expanded to uncover clear energy relationships within non-regular graphs, especially by examining $n(\geq 5)$ matrices associated with graphs.

Acknowledgements: The author R. S. Naikar is thankful to Karnataka Science and Technology Promotion Society, Bengalore, India for providing fellowship No.DST/KSTePS/Ph.D. Fellowship/MAT-01:2022-23/1017.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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