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Some remarks on the signed total Italian k-domination number of graphs

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Abstract: Let $k \geq 1$ be an integer, and let G be a finite and simple graph with vertex set $V(G)$. Volkmann [Signed total Italian k-domination in graphs, Commun. Comb. Optim. 6 (2021), 171–183], defined the signed total Italian k-dominating function (STIkDF) on a graph G as a function $f : V(G) \to \{-1, 1, 2\}$ satisfying the conditions that $\sum_{x \in N(v)} f(x) \geq k$ for each vertex $v \in V(G)$, where $N(v)$ is the neighborhood of v, and every vertex u for which $f(u) = -1$ is adjacent to at least one vertex v for which $f(v) = 2$ or adjacent to two vertices w and z with $f(w) = f(z) = 1$. The weight of an STIkDF f is $w(f) = \sum_{v \in V(G)} f(v)$. The signed total Italian k-domination number $\gamma^k_{stI}(G)$ of G is the minimum weight of an STIkDF on G. In this paper we continue the study of the signed total Italian k-domination number. We present new bounds on $\gamma^k_{stI}(G)$, and we determine the signed total Italian k-domination number of some complete p-partite graphs. Furthermore, we show that the difference $\gamma^k_{stR}(G) - \gamma^k_{stI}(G)$ can be arbitrarily large, where $\gamma^k_{stR}(G)$ is the signed total Roman k-domination number.

Keywords: signed total Italian k-dominating function, signed total Italian k-domination number.

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1. Introduction

For notation and graph theory terminology, we in general follow Haynes, Hedetniemi and Slater [6]. Specifically, let G be a graph with vertex set $V(G) = V$ and edge set $E(G) = E$. The integers $n = n(G) = |V(G)|$ and $m = m(G) = |E(G)|$ are the *order* and the *size* of the graph G, respectively. The *open neighborhood* of vertex v is $N_G(v) = N(v) = \{u \in V(G) | uv \in E(G)\}$, and the *closed neighborhood* of v is $N_G[v] = N[v] = N(v) \cup \{v\}$. The *degree* of a vertex v is $d_G(v) = d(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta(G) = \delta$ and $\Delta(G) = \Delta$, respectively. A graph G is *regular* or r*-regular* if $\Delta(G) = \delta(G) = r$. Let K_n be the *complete graph*, and let $K_{n_1,n_2,\ldots,n_p}$ denote the *complete p-partite graph* with partite sets $X_1, X_2, \ldots, X_p$ such that $|X_i| = n_i$ for $1 \le i \le p$.

In this paper we continue the study of Roman dominating functions in graphs and digraphs (see, for example, the survey papers [1–4]). For a subset $S \subseteq V(G)$ of vertices of a graph G and a function $f : V(G) \to \mathbb{R}$, we define $f(S) = \sum_{x\in S} f(x)$.

If $k \ge 1$ is an integer, then Volkmann [9] defined the *signed total Roman k-dominating function* (STRkDF) on a graph G as a function $f : V(G) \longrightarrow \{-1, 1, 2\}$ such that $f(N(v)) \ge k$ for every $v \in V(G)$, and every vertex u for which $f(u) = -1$ is adjacent to a vertex v for which $f(v) = 2$. The weight of an STRkDF f on a graph G is $\omega(f) = \sum_{v\in V(G)} f(v)$. The *signed total Roman k-domination number* $\gamma^k_{stR}(G)$ of G is the minimum weight of an STRkDF on G. A $\gamma^k_{stR}(G)$-function is a signed total Roman k-dominating function on G of weight $\gamma^k_{stR}(G)$. The special case $k = 1$ was introduced and investigated in [8].

If $k \ge 1$ is an integer, then Volkmann [10] defined the *signed total Italian k-dominating function* (STIkDF) on a graph G as a function $f : V(G) \to \{-1, 1, 2\}$ satisfying the conditions that $f(N(v)) \ge k$ for each vertex $v \in V(G)$, and every vertex u for which $f(u) = -1$ is adjacent to at least one vertex v for which $f(v) = 2$ or adjacent to two vertices w and z with $f(w) = f(z) = 1$. The weight of an STIkDF f is $w(f) = \sum_{v\in V(G)} f(v)$. The *signed total Italian k-domination number* $\gamma^k_{stI}(G)$ of G is the minimum weight of an STIkDF on G. A $\gamma^k_{stI}(G)$-function is a signed total Italian k-dominating function on G of weight $\gamma^k_{stI}(G)$. If $\delta(G) \ge 2$ or $k \ge 2$, then we note that the second condition in the definition of an STIkDF is superfluous. For an STRkDF or STIkDF f on G, let $V_i = V_i(f) = \{v \in V(G) : f(v) = i\}$ for $i = -1, 1, 2$. A signed total Roman k-dominating or signed total Italian k-dominating function $f : V(G) \longrightarrow \{-1, 1, 2\}$ can be represented by the ordered partition (V_{-1}, V_1, V_2) of $V(G)$.

For an integer $\ell \ge 1$, Kulli [7] called a subset D of vertices of a graph G a *total ℓ-dominating set* if every vertex $x \in V(G)$ has at least ℓ neighbors in D. The *total ℓ-domination number* $\gamma_{t\ell}(G)$ is the minimum cardinality of a total ℓ-dominating set of G. The special case $\ell = 1$ is the usual *total domination number* $\gamma_t(G)$, introduced by Cockayne, Dawes and Hedetniemi [5].

The signed total Italian k-domination number exists when $\delta(G) \ge \frac{k}{2}$. The definitions lead to $\gamma^k_{stI}(G) \le \gamma^k_{stR}(G)$. Therefore each lower bound of $\gamma^k_{stI}(G)$ is a lower bound of $\gamma^k_{stR}(G)$, and each upper bound of $\gamma^k_{stR}(G)$ is also an upper bound of $\gamma^k_{stI}(G)$.

In this paper we present different sharp bounds on $\gamma^k_{stI}(G)$. In addition, we determine the signed total Italian k-domination number of some complete p-partite graphs. Furthermore, we show that the difference $\gamma^k_{stR}(G) - \gamma^k_{stI}(G)$ can be arbitrarily large. We make use of the following known results.

Proposition 1. *[10]* If $k \ge 1$ and $n \ge 2$ are integers such that $2n - 2 \ge k$, then it holds:

(a) If $k \ge n$, then $\gamma^k_{stI}(K_n) = k + 2$.

(b) If $k \le n - 1$ and $n - k$ is odd, then $\gamma^k_{stI}(K_n) = k + 1$.

(c) If $k \leq n-1$ and $n-k$ is even, then $\gamma_{stI}^{k}(K_n) = k+2$.

Proposition 2. *[10]* If G is an r-regular graph of order n with $r \geq \frac{k}{2}$, then $\gamma_{stI}^{k}(G) \geq \frac{kn}{r}$.

2. Bounds on the signed total Italian k-domination number

We start with a sharp upper bound on the signed total Italian k-domination number of a graph, depending on the order and the minimum degree.

Theorem 1. If $k \geq 1$ is an integer, and G a graph of order n with minimum degree $\delta \geq k+2$, then

$$\gamma_{stI}^{k}(G) \leq n - 2\left\lfloor \frac{\delta - k}{2} \right\rfloor.$$

Proof. Define $t = \lfloor (\delta - k)/2 \rfloor$. Let $A = \{u_1, u_2, \ldots, u_t\}$ be a set of t vertices of G. Define the function $f : V(G) \longrightarrow \{-1, 1, 2\}$ by $f(x) = -1$ for $x \in A$ and $f(x) = 1$ for $x \in V(G) \setminus A$. Then

$$f(N(w)) \geq -t + (\delta - t) = \delta - 2t = \delta - 2\left\lfloor \frac{\delta - k}{2} \right\rfloor \geq k$$

for each vertex $w \in V(G)$. Since $\delta \geq 3$, we observe that every vertex is adjacent to at least two vertices of weight 1. Therefore f is an STIkDF on G of weight $n - 2t$ and thus $\gamma_{stI}^{k}(G) \leq n + 2t$. □

With the help of Proposition 1 we will demonstrate that Theorem 1 is sharp. If $G = K_n$, then $\delta = n - 1$. We assume that $1 \leq k \leq n - 3$.

Case 1. Let k and n be even. Then $n - k$ is even and

$$\left\lfloor \frac{\delta - k}{2} \right\rfloor = \left\lfloor \frac{n - 1 - k}{2} \right\rfloor = \frac{n - k - 2}{2}.$$

Therefore Proposition 1 (c) leads to

$$\gamma_{stI}^{k}(K_n) = k + 2 = n - n + k + 2 = n - 2\left\lfloor \frac{\delta - k}{2} \right\rfloor.$$

Case 2. Let k be even and n be odd. Then $n - k$ is odd and

$$\left\lfloor \frac{\delta - k}{2} \right\rfloor = \left\lfloor \frac{n - 1 - k}{2} \right\rfloor = \frac{n - k - 1}{2}.$$

Therefore Proposition 1 (b) leads to

$$\gamma_{stI}^k(K_n) = k + 1 = n - n + k + 1 = n - 2\left\lfloor \frac{\delta - k}{2} \right\rfloor.$$

Case 3. Let k and n be odd. Then $n - k$ is even and

$$\left\lfloor \frac{\delta - k}{2} \right\rfloor = \left\lfloor \frac{n - 1 - k}{2} \right\rfloor = \frac{n - k - 2}{2}.$$

Therefore Proposition 1 (c) leads to

$$\gamma_{stI}^k(K_n) = k + 2 = n - n + k + 2 = n - 2\left\lfloor \frac{\delta - k}{2} \right\rfloor.$$

Case 4. Let k be odd and n be even. Then $n - k$ is odd and

$$\left\lfloor \frac{\delta - k}{2} \right\rfloor = \left\lfloor \frac{n - 1 - k}{2} \right\rfloor = \frac{n - k - 1}{2}.$$

Therefore Proposition 1 (b) leads to

$$\gamma_{stI}^k(K_n) = k + 1 = n - n + k + 1 = n - 2\left\lfloor \frac{\delta - k}{2} \right\rfloor.$$

In all these examples we have equality in the inequality of Theorem 1.

Theorem 2. Let $k \geq 1$ be an integer, and let G be a graph of order n with minimum degree $\delta \geq k + s$ with an integer $s \geq 0$. If $f = (V_{-1}, V_1, V_2)$ is an STIkDF on G, then $V_1 \cup V_2$ is a total $\lceil \frac{2k+s}{3} \rceil$-dominating set of G.

Proof. Suppose on the contrary, that there exists a vertex v with at most $\lceil \frac{2k+s}{3} \rceil - 1$ neighbors in $V_1 \cup V_2$. Then v has at least

$$\delta(G) - \left(\left\lceil \frac{2k+s}{3} \right\rceil - 1\right) \geq k + s - \left(\left\lceil \frac{2k+s}{3} \right\rceil - 1\right)$$

neighbors in V_{-1}. Hence the definition implies the contradiction

$$\begin{aligned} k &\leq f(N(v)) \leq 2\left(\left\lceil \frac{2k+s}{3} \right\rceil - 1\right) - \left(k + s - \left\lceil \frac{2k+s}{3} \right\rceil - 1\right) \\ &= 3\left\lceil \frac{2k+s}{3} \right\rceil - 3 - k - s \leq \frac{3(2k+s+2)}{3} - 3 - k - s = k - 1. \end{aligned}$$

Consequently, $V_1 \cup V_2$ is a total $\lceil \frac{2k+s}{3} \rceil$-dominating set. $\square$

Corollary 1. Let $k \geq 1$ be an integer, and let G be a graph of order n with minimum degree $\delta \geq k+s$ with an integer $s \geq 0$. If $f = (V_{-1}, V_1, V_2)$ is a $\gamma^k_{stI}(G)$-function, then $\gamma^k_{stI}(G) \geq 2\gamma_{t\lceil \frac{2k+s}{3} \rceil} + |V_2| - n$.

Proof. Since $\gamma^k_{stI}(G) = |V_1| + 2|V_2| - |V_{-1}|$ and $n = |V_1| + |V_2| + |V_{-1}|$, it follows from Theorem 2 that

$$\begin{aligned} \gamma^k_{stI}(G) &= |V_1| + 2|V_2| - |V_{-1}| = 2|V_1| + 3|V_2| - n \\ &= 2|V_1 \cup V_2| + |V_2| - n \geq 2\gamma_{t\lceil \frac{2k+s}{3} \rceil} + |V_2| - n. \end{aligned}$$

□

In Section 3 we will demostrate that Theorem 2 as well as Corollary 1 are sharp.

3. Special classes of graphs

In this section we determine the signed total Italian k-domination number for some complete p-partite graphs.

Example 1. *Let $k \geq 1$ and $p \geq 3$ be integers, and let $G = K_{n_1,n_2,\dots,n_p}$ be a complete p-partite graph with $n_1 \leq n_2 \leq \ldots \leq n_p$ and $n_p \geq 2$.*

(a) If $p \geq k+1$, then $\gamma^k_{stI}(G) = k+1$.

(b) If $p \leq k$, then $\gamma^k_{stI}(G) \geq k+2$.

(c) If $p \leq k \leq 2p-2$, then $\gamma^k_{stI}(G) = k+2$.

(d) If $k \geq 2p-1$, then $\gamma^k_{stI}(G) \geq k+3$.

Proof. Let $X_1, X_2, \dots, X_p$ be the partite sets of G with $|X_i| = n_i$ for $1 \leq i \leq p$, and let f be a $\gamma^k_{stI}(G)$-function.
(a) Assume that $p \geq k+1$. If we suppose that $f(X_i) \leq 0$ for each $1 \leq i \leq p$, then we obtain the contradiction

$$f(N(v)) = \sum_{i=2}^{p} f(X_i) \leq 0$$

when $v \in X_1$. Hence there exists a partite set, say X_1, with $f(X_1) \geq 1$. If $v \in X_1$, then we deduce that

$$\gamma^k_{stI}(G) = f(X_1) + f(N(v)) \geq k+1.$$

For the converse we consider two cases.

Case 1. Assume that $n_1 \geq 2$. We will define the function g such that $g(X_1) = g(X_2) = \ldots = g(X_{k+1}) = 1$ and $g(X_i) = 0$ for $k+1 \leq i \leq p$ as follows. Let

$X_i = \{x_1, x_2, \ldots, x_t\}$. First let $i \in \{1, 2, \ldots, k+1\}$. If $t = 2s$ for an integer $s \geq 1$, then define $g(x_1) = 2$, $g(x_2) = g(x_3) = \ldots = g(x_s) = 1$ and $g(x_{s+1}) = g(x_{s+2}) = \ldots = g(x_{2s}) = -1$. If $t = 2s+1$ for an integer $s \geq 1$, then define $g(x_1) = g(x_2) = \ldots = g(x_s) = -1$ and $g(x_{s+1}) = g(x_{s+2}) = \ldots = g(x_{2s+1}) = 1$. In both cases we observe that $g(X_i) = 1$.
Second let $i \in \{k+2, k+3, \ldots, p\}$. If $t = 2s$ for an integer $s \geq 1$, then define $g(x_1) = g(x_2) = \ldots = g(x_s) = 1$ and $g(x_{s+1}) = g(x_{s+2}) = \ldots = g(x_{2s}) = -1$. If $t = 2s+1$ for an integer $s \geq 1$, then define $g(x_1) = 2$, $g(x_2) = g(x_3) = \ldots = g(x_s) = 1$ and $g(x_{s+1}) = g(x_{s+2}) = \ldots = g(x_{2s+1}) = -1$. In both cases we note that $g(X_i) = 0$. Therefore $\gamma^k_{stI}(G) \leq k+1$ and so $\gamma^k_{stI}(G) = k+1$ when $n_1 \geq 2$.

Case 2. Assume that $n_1 = 1$. Let $n_1 = n_2 = \ldots = n_s = 1$ for an integer $1 \leq s \leq p-1$ and $n_{s+1} \geq 2$.
If $s \leq k+1$, then define $g(X_1) = g(X_2) = \ldots = g(X_{k+1}) = 1$ and $g(X_i) = 0$ for $k+2 \leq i \leq p$ as in Case 1.
Next let $s \geq k+2$. Define $g(X_1) = g(X_2) = \ldots = g(X_{k+1}) = 1$. Then $|X_{k+2}| = |X_{k+3}| = \ldots = |X_s| = 1$. If $s-k-1 = 2t$ is even, then it is no problem to define g such that $g(X_{k+2}) + g(X_{k+3}) + \ldots + g(X_s) = 0$ and $g(X_i) = 0$ for $i \geq s+1$. If $s-k-1 = 2t+1$ is odd, then define g such that $g(X_{k+2}) + g(X_{k+3}) + \ldots + g(X_s) = -1$, $g(X_{s+1}) = 1$ and $g(X_i) = 0$ for $i \geq s+2$. In all cases g is an STIkDF on G of weight $k+1$. Therefore $\gamma^k_{stI}(G) \leq k+1$ and so $\gamma^k_{stI}(G) = k+1$ also in the last case.
(b) Assume that $p \leq k$. If we suppose that $f(X_i) \leq 1$ for each $1 \leq i \leq p$, then we obtain the contradiction

$$f(N(v)) = \sum_{i=2}^{p} f(X_i) \leq p-1 \leq k-1$$

when $v \in X_1$. Hence there exists a partite set, say X_1, with $f(X_1) \geq 2$. If $v \in X_1$, then we deduce that

$$\gamma^k_{stI}(G) = f(X_1) + f(N(v)) \geq k+2.$$

(c) Assume that $k = p+s$ with $s \leq p-2$. Define the function g such that $g(X_1) = g(X_2) = \ldots = g(X_{s+2}) = 2$ and $g(X_i) = 1$ for $s+3 \leq i \leq p$. Then g is an STIkDF on G of weight $2(s+2) + (p-s-2) = p+s+2 = k+2$ and therefore $\gamma^k_{stI}(G) \leq k+2$ and so $\gamma^k_{stI}(G) = k+2$ in this case.
(d) Assume that $k \geq 2p-1$. If we suppose that $f(X_i) \leq 2$ for each $1 \leq i \leq p$, then we obtain the contradiction

$$f(N(v)) = \sum_{i=2}^{p} f(X_i) \leq 2p-2 \leq k-1$$

when $v \in X_1$. Hence there exists a partite set, say X_1, with $f(X_1) \geq 3$. If $v \in X_1$, then we deduce that

$$\gamma^k_{stI}(G) = f(X_1) + f(N(v)) \geq k+3.$$

□

If $n_1 = n_2 = \ldots = n_p = s \geq 2$ in Example 1, then the graph G is $s(p-1)$-regular of order $n = ps$. Since $k \leq p-1$, we observe that

$$\gamma^k_{stI}(G) = k+1 = \left\lceil \frac{k(p-1)+k}{p-1} \right\rceil = \left\lceil \frac{kps}{s(p-1)} \right\rceil = \left\lceil \frac{kn}{s(p-1)} \right\rceil.$$

Thus this is an example with equality in the inequality of Proposition 2.

Let $n_1 = n_2 = \ldots = n_p = 2$ and $k = p-1$ in Example 1. If $X_i = \{a_i, b_i\}$, then the function g with $g(a_i) = 2$ and $g(b_i) = -1$ for $1 \leq i \leq p$ is a $\gamma^k_{stI}(G)$-function of weight $p = k+1$ with $|V_2| = p$. Furthermore, $\delta(G) = 2(p-1) = k+s = k+p-1$, $\lceil \frac{2k+s}{3} \rceil = p-1$ and $\gamma_{t(p-1)}(G) = p$. If $g = (V_{-1}, V_1, V_2)$, then we observe that $V_1 \cup V_2 = V_2$ is a total $\lceil \frac{2k+s}{3} \rceil$-dominating set of G. Thus Theorem 2 is sharp. In addition, it follows from Corollary 1 and Example 1 that

$$p = k+1 = \gamma^k_{stI}(G) \geq 2\gamma_{t\lceil \frac{2k+s}{3} \rceil} + |V_2| - n(G) = 2p + p - 2p = p.$$

So this is an example with equality in the inequality of Corollary 1.

Example 2. *Let $k \geq 1$ and $p \geq 3$ be integers, and let $G = K_{n_1,n_2,\ldots,n_p}$ be a complete p-partite graph with $n_1 = n_2 = \ldots = n_p = q \geq 2$. If $2 \leq t \leq 2q$ is an integer such that $(t-1)p - (t-2) \leq k \leq tp - t$, then $\gamma^k_{stI}(G) = k+t$.*

Proof. If we suppose that $f(X_i) \leq t-1$ for each $1 \leq i \leq p$, then we obtain the contradiction

$$f(N(v)) = \sum_{i=2}^{p} f(X_i) \leq (p-1)(t-1) = p(t-1) - t + 1 \leq k-1$$

when $v \in X_1$. Hence there exists a partite set, say X_1, with $f(X_1) \geq t$. If $v \in X_1$, then we deduce that

$$\gamma^k_{stI}(G) = f(X_1) + f(N(v)) \geq k+t.$$

Now let $k = (t-1)p - (t-2) + s$ with an integer $0 \leq s \leq p-2$. Define the function g such that $g(X_1) = g(X_2) = \ldots = g(X_{s+2}) = t$ and $g(X_i) = t-1$ for $s+3 \leq i \leq p$. Then g is an STIkDF on G of weight

$$t(s+2) + (p-s-2)(t-1) = p(t-1) + s + 2 = k+t.$$

Therefore $\gamma^k_{stI}(G) \leq k+t$ and so $\gamma^k_{stI}(G) = k+t$. This completes the proof. □

If we choose $k = t(p-1)$ in Example 2, then it follows from Proposition 2 that

$$k + t = \gamma^k_{stI}(G) \geq \frac{kn(G)}{r} = \frac{kpq}{q(p-1)} = \frac{kp}{p-1} = \frac{t(p-1)p}{p-1} = tp = k + t$$

and therefore equality in Proposition 2.

The next example will demonstrate that the the difference $\gamma^k_{stR}(G) - \gamma^k_{stI}(G)$ can be arbitrarily large.

Example 3. *Let F be an arbitrary graph of order $t \geq 2$, and for each vertex $v \in V(F)$ add a vertex-disjoint copy of a complete graph K_s with $s \geq k+4$ such that $s-k$ is odd and identify the vertex v with one vertex of the added complete graph. Let H be the resulting graph. Furthermore, let $H_1, H_2, \ldots, H_t$ be the added copies of K_s. For $i = 1, 2, \ldots, t$, let v_i be the vertex of H_i that is identified with a vertex of F.*
First we construct an STIkDF on H as follows. For each $i = 1, 2, \ldots, t$ let $f_i : V(H_i) \to \{-1, 1, 2\}$ be the STIkDF on the complete graph defined in Proposition 1 such that $f_i(v_i) \geq 1$. As shown in Proposition 1, we have $\omega(f_i) = k+1$. Now let $f : V(H) \to \{-1, 1, 2\}$ be the function defined by $f(v) = f_i(v)$ for for each $v \in V(H_i)$ for $i = 1, 2, \ldots, t$. Then f is an STIkDF of H of weight $t(k+1)$ and thus $\gamma^k_{stI}(H) \leq t(k+1)$.
Now let g be a $\gamma^k_{stR}(H)$-function. We show that $g(H_i) \geq k+2$ for each $1 \leq i \leq t$. If $g(x) = -1$ for at most one $x \in V(H_i)$, then $g(V(H_i)) \geq s-2 \geq k+2$. Hence assume that there exist at least two vertices $x, y \in V(H_i)$ such that $g(x) = g(y) = -1$. This implies that there exists a vertex $w \in V(H_i)$ with $g(w) = 2$. If $w \neq v_i$, then we deduce that $g(V(H_i)) = g(w) + g(N(w)) \geq 2 + k$. Next we assume that $w = v_i$ and $g(a) \in \{-1, 1\}$ for $a \in V(H_i) \setminus \{w\}$. If z is a vertex of $V(H_i)$ with $g(z) = 1$, then assume that z has exactly j neighbors of weight 1 and $s-j-2$ neighbors of weight -1. We deduce that

$$g(N(z)) = 2 + j - (s - 2 - j) = 4 + 2j - s \geq k,$$

and since $s-k$ is odd, it follows that

$$g(N(z)) = 4 + 2j - s \geq k + 1.$$

Thus $g(V(H_i)) = g(z) + g(N(z)) \geq k+2$, and we obtain

$$\gamma^k_{stR}(H) = g(V(H)) = \sum_{i=1}^{t} g(V(H_i) \geq t(k+2).$$

Consequently, we observe that

$$\gamma^k_{stR}(H) - \gamma^k_{stI}(H) \geq t(k+2) - t(k+1) = t.$$

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