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Research Article

The forgotten index of hypergraphs and some hypergraph operations

Shashwath S Shetty[†], **K Arathi Bhat**[*]

Department of Mathematics, Manipal Institute of Technology,
Manipal Academy of Higher Education, Manipal, 576104, Karnataka, India
[†]shashwathsshetty01334@gmail.com
[*]arathi.bhat@manipal.edu

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Abstract: Hypergraphs generalize traditional graphs by allowing edges to connect more than two vertices, enabling a richer representation of relationships in complex systems. Forgotten topological index, or simply F-index of a hypergraph, is defined as the sum of cubes of the degrees of all the vertices of the hypergraph. Initially, some sharp bounds for the F-index of hypergraphs in terms of other degree-based topological indices have been obtained. A minimally connected hypergraph is a connected hypergraph such that the removal of any hyperedge disconnects the hypergraph. We have characterized the extremal minimally connected hypergraphs corresponding to the F-index among minimally connected hypergraphs on n vertices. The hyperstar and hyperpath with minimum and maximum F-indices have been studied. The upper and lower bounds for the F-index of the hypergraphs and bipartite hypergraphs are also given. We conclude this article by computing the F-index of join, corona product, and Cartesian product of two hypergraphs.

Keywords: Hyperstar, hyperpath, sunflower, Cartesian product, corona.

AMS Subject classification: 05C65, 05C07, 92E10

1. Introduction

Hypergraphs can model complex social interactions more accurately than usual graphs. In a social network, a hyperedge can represent a group of people involved in an event, discussion, or collaboration, rather than just pairwise connections. This enables a richer representation of the real-world relationships [2]. Hypergraphs are

* *Corresponding Author*

used to represent complex biological data. For example, in protein-protein interaction networks, a hyperedge can represent interactions between multiple proteins simultaneously, providing a more realistic view of cellular processes [13]. A hypergraph model in chemistry is a powerful tool used to analyze polycentric bonds, which involve multiple atoms sharing electrons in complex molecular structures. It extends traditional chemical bonding models by representing these bonds as hyperedges, allowing a more comprehensive understanding of electron distribution and bonding patterns in intricate molecular systems [11].

Topological indices of graphs in chemistry are vital for quantitatively correlating molecular structures with physical and chemical properties, enabling drug design, environmental assessment, and material science advancements. Despite having so many real-world applications [7, 11, 12], degree-based topological indices have been considered only for simple graphs and very recently for graphs with self-loops [16] and for hypergraphs [15, 17]. However, the Wiener index [14, 18], degree-distance index, and Gutman index [3] of hypergraphs have been studied.

A hypergraph $\mathcal{H}$ is an ordered pair $(\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ contains a non-empty finite set of elements called vertices and the edge set $\mathcal{E}$ contains the non-empty subset of the vertex set $\mathcal{V}$, whose elements are called the hyperedges. A walk $w = v_0e_1v_1e_2 \dots e_{t-1}v_{t-1}e_tv_t$ between two vertices v_0 and v_t in a hypergraph is an alternating sequence of vertices and hyperedges (which starts and ends with a vertex) such that v_{i-1}, v_i are contained in e_i, $1 \leq i \leq t$. A path from u to v $(u \neq v)$ in a hypergraph is a walk, where there are no repeated vertices or hyperedges. If the initial and terminal vertices in a path are same, then it is called a cycle. The length of a cycle is the number of hyperedges in it, and is called a k-cycle if the length is equal to k. A hypergraph is said to be connected if there exists a path between every two vertices in the hypergraph. A connected hypergraph is said to be a minimally connected hypergraph if removal of any hyperedge disconnects the hypergraph. Two vertices in a hypergraph are said to be adjacent if there is at least one hyperedge containing both vertices. The degree of a vertex u denoted by d_u in a hypergraph is the total number of hyperedges containing u. A vertex of degree one is called a pendant vertex. If the degree of every vertex in a hypergraph is equal to k, then the hypergraph is said to be a regular hypergraph with regularity k.

The cardinality (number of vertices) of a hyperedge e of $\mathcal{H}$ is called the degree of the hyperedge e in $\mathcal{H}$. In a hypergraph $\mathcal{H}$, if the degree of each hyperedge is equal to r, then $\mathcal{H}$ is an r-uniform hypergraph. A hyperedge e of degree r in a hypergraph having at least two hyperedges is called a pendant hyperedge at a vertex $u \in e$ if $d_u \geq 2$ and all the remaining vertices of e are pendant vertices. If every hyperedge in a minimally connected hypergraph (containing at least two hyperedges) is a pendant hyperedge, then the hypergraph is called a hyperstar. A linear hypergraph is a hypergraph in which any two hyperedges can have at most one vertex in common. Hyperstar is a class of linear minimally connected hypergraphs where it has only one vertex of degree m and all other vertices are of degree 1. Two hyperedges in a hypergraph are said to be adjacent if they have at least one vertex in common. A hyperpath is a minimally connected hypergraph in which each hyperedge is adjacent to at most

two other hyperedges, and k-cycles ($k \geq 3$) are not allowed. A sunflower hypergraph $\mathcal{S}(m, h, r)$ is an r-uniform hypergraph with m hyperedges, each of which contains h vertices of degree m and $r - h$ pendant vertices. A complete hypergraph on n vertices is denoted by $\mathcal{H}_{K_n}$, whose edge set is given by all non-empty subsets of the vertex set. An r-uniform complete hypergraph is denoted by $\mathcal{H}_{K_n}^{(r)}$ and is obtained from $\mathcal{H}_{K_n}$ by removing all those hyperedges of size t, where $t \neq r$. A hypergraph is said to be a bipartite hypergraph if there exists a bi-partition (both are non-empty) of the vertex set such that every hyperedge of the hypergraph has a non-empty intersection with both the partite sets. A hypergraph is said to be a complete bipartite with respect to a given bi-partition if the hypergraph contains all possible hyperedges that have non-empty intersection with both the partite sets, and we denote it by $\mathcal{H}_{K_{s,t}}$, where s, t are the number of vertices in each partite set. From a complete bipartite hypergraph, $\mathcal{H}_{K_{s,t}}$, if we remove all those hyperedges of size t, where $t \neq r$, then we call it an r-uniform complete bipartite hypergraph, and we denote it by $\mathcal{H}_{K_{s,t}}^{(r)}$. For all other undefined terminology in graphs and hypergraphs, the readers can refer to [20] and [4], respectively.
The Sombor index of a hypergraph $\mathcal{H}$ is defined [15] as,

$$SO(\mathcal{H}) = \sum_{e \in \mathcal{E}} \sqrt{\sum_{u \in e} d_u^2}.$$

The first Zagreb index of a Hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ is defined [19] as

$$M_1(\mathcal{H}) = \sum_{u \in \mathcal{V}} d_u^2 = \sum_{e \in \mathcal{E}} \sum_{u \in e} d_u.$$

The hyper-Zagreb index of a hypergraph $\mathcal{H}$ is defined [19] as

$$HM_1(\mathcal{H}) = \sum_{e \in \mathcal{E}} \left(\sum_{u \in e} d_u \right)^2.$$

The forgotten index of graphs has been introduced in [10] and the theoretical study has been started in [6]. Subsequently, an immense amount of research has been done on theoretical aspects [1, 5] and its applications [8, 9]. The forgotten index (or simply F-index) of a hypergraph $\mathcal{H}$, denoted by $F(\mathcal{H})$ is defined as

$$F(\mathcal{H}) = \sum_{u \in \mathcal{V}} d_u^3 = \sum_{e \in \mathcal{E}} \sum_{u \in e} d_u^2.$$

The rest of the paper is organized as follows: Section 2 deals with some preliminary results related to the F-index of a hypergraph with other indices. Bounds for the F-index of a minimally connected hypergraph, hyperpath, bipartite hypergraph, and a hypergraph are discussed in Section 3, and in Section 4 we obtain the F-index of few hypergraph operations.

2. Preliminary Results

This section contains a few results that give the relation between the F-index of a hypergraph and other indices.

Proposition 1. *Let $\mathcal{H}$ be a hypergraph with m hyperedges. If $SO(\mathcal{H})$ denotes the Sombor index of the hypergraph $\mathcal{H}$, then*

$$F(\mathcal{H}) \geq \frac{(SO(\mathcal{H}))^2}{m},$$

where the equality holds for a uniform regular hypergraph.

Proof. By using the Cauchy-Schwarz inequality, the above result follows. That is,

$$\begin{aligned}(SO(\mathcal{H}))^2 &= \left(\sum_{e\in\mathcal{E}} \sqrt{\sum_{u\in e} d_u^2}\right)^2 \\ &\leq m \sum_{e\in\mathcal{E}} \sum_{u\in e} d_u^2 = m\ F(\mathcal{H}),\end{aligned}$$

as desired. □

Proposition 2. *Let $\mathcal{H}$ be a hypergraph having m hyperedges, with δ and r being the minimum vertex degree and edge degree, respectively. If $HM_1(\mathcal{H})$ denotes the hyper-Zagreb index of the hypergraph $\mathcal{H}$, then*

$$F(\mathcal{H}) \leq HM_1(\mathcal{H}) - 2m\delta^2 \binom{r}{2},$$

where the equality holds for an r-uniform δ-regular hypergraph.

Proof. We have,

$$\begin{aligned}F(\mathcal{H}) &= \sum_{e\in\mathcal{E}} \sum_{u\in e} d_u^2 \\ &= \sum_{e\in\mathcal{E}} (\sum_{u\in e} d_u^2 + 2 \sum_{u,v\in e} d_u d_v) - 2 \sum_{e\in\mathcal{E}} \sum_{u,v\in e} d_u d_v \\ &= \sum_{e\in\mathcal{E}} (\sum_{u\in e} d_u)^2 - 2 \sum_{e\in\mathcal{E}} \sum_{u,v\in e} d_u d_v \\ &\leq HM_1(\mathcal{H}) - 2 \sum_{e\in\mathcal{E}} \binom{|e|}{2} \delta^2 \\ &\leq HM_1(\mathcal{H}) - 2m \binom{r}{2} \delta^2,\end{aligned}$$

as desired. □

Lemma 1 (Chebyshev's inequality). *Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be real numbers. Then*

$$\frac{1}{n}\sum_{i=1}^{n} a_i b_i \geq \left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} b_i\right),$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$ and $b_1 = b_2 = \cdots = b_n$.

Proposition 3. *Let $\mathcal{H}$ be a hypergraph of order n having m hyperedges with δ and r being the minimum degree of a vertex (among all the vertices) and a hyperedge (among all the hyperedges), respectively. If $M_1(\mathcal{H})$ denotes the first Zagreb index of the hypergraph $\mathcal{H}$, then*

$$F(\mathcal{H}) \leq \frac{1}{n} M_1(\mathcal{H}) rm, \tag{2.1}$$

with equality if and only if $\mathcal{H}$ is an r-uniform δ-regular. Or,

$$F(\mathcal{H}) \leq M_1(\mathcal{H})\delta, \tag{2.2}$$

where the equality holds if and only if $\mathcal{H}$ is δ-regular.

Proof. By using Lemma 1,

$$\begin{aligned}\frac{1}{n}\sum_{u\in\mathcal{V}} d_u^3 &\geq \left(\frac{1}{n}\sum_{u\in\mathcal{V}} d_u^2\right)\left(\frac{1}{n}\sum_{u\in\mathcal{V}} d_u\right)\\ &= \left(\frac{1}{n}M_1(\mathcal{H})\right)\left(\frac{1}{n}\sum_{u\in\mathcal{V}} d_u\right)\end{aligned}$$

By substituting $\sum\limits_{u\in\mathcal{V}} d_u \geq rm$ in the above inequality, we get Equation (2.1), where r is the minimum degree of hyperedge among all hyperedges and m is the total number of hyperedges in the hypergraph, and the equality holds for an r-uniform regular hypergraph. Similarly, by substituting $\sum\limits_{u\in\mathcal{V}} d_u \geq n\delta$ in the above inequality, we get Equation (2.2), where δ is the least degree of a vertex among all vertices and n is the total number of vertices in the hypergraph. □

Lemma 2 (Pólya–Szego inequality). *Let $0 < a_1 \leq x_i \leq A_1$ and $0 < a_2 \leq y_i \leq A_2$, for $1 \leq i \leq n$. Then*

$$\sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2 \leq \frac{1}{4}\left(\sqrt{\frac{a_1 a_2}{A_1 A_2}} + \sqrt{\frac{A_1 A_2}{a_1 a_2}}\right)\left(\sum_{i=1}^{n} x_i y_i\right)^2.$$

Proposition 4. *Let $\mathcal{H}$ be a hypergraph (with no isolated vertices) of order n. Also, let δ and Δ, respectively, be the minimum and maximum degree of a vertex among all the vertices of the hypergraph. If $M_1(\mathcal{H})$ denotes the first Zagreb index of the hypergraph $\mathcal{H}$, then*

$$F(\mathcal{H}) \geq 2\delta^2\sqrt{\frac{nM_1(\mathcal{H})}{D'}}, \text{ where } D' = \sqrt{\frac{\delta^3}{\Delta^3}} + \sqrt{\frac{\Delta^3}{\delta^3}}$$

and the equality holds for a regular hypergraph (with $\delta \geq 1$).

Proof. It is simple to observe that $0 < \delta \leq d_u \leq \Delta$ and $0 < \delta^2 \leq d_u^2 \leq \Delta^2$, and hence, by using Lemma 2,

$$\sum_{u\in\mathcal{V}}(d_u)^2\sum_{u\in\mathcal{V}}(d_u^2)^2 \leq \frac{1}{4}\left(\sqrt{\frac{\delta\cdot\delta^2}{\Delta\cdot\Delta^2}}+\sqrt{\frac{\Delta\cdot\Delta^2}{\delta\cdot\delta^2}}\right)\left(\sum_{u\in\mathcal{V}} d_u\cdot d_u^2\right)^2$$
$$\implies M_1(\mathcal{H})\sum_{u\in\mathcal{V}} d_u^4 \leq \frac{1}{4}\left(\sqrt{\frac{\delta^3}{\Delta^3}}+\sqrt{\frac{\Delta^3}{\delta^3}}\right)(F(\mathcal{H}))^2$$
$$\implies M_1(\mathcal{H})n\delta^4 \leq \frac{1}{4}\left(\sqrt{\frac{\delta^3}{\Delta^3}}+\sqrt{\frac{\Delta^3}{\delta^3}}\right)(F(\mathcal{H}))^2.$$

By rearrangement of terms in the above inequality, the result follows. □

3. Extremal Bounds

Bounds for the F-index of hypergraphs, minimally connected hypergraphs, and some families of minimally connected hypergraphs are discussed in this section.

Lemma 3. *Let $\mathcal{S}_n$ be a hyperstar on n vertices with m hyperedges. Then,*

$$F(\mathcal{S}_n) = m^3 + n - 1.$$

Proof. A hyperstar on n vertices with m hyperedges contains a vertex of degree m, and all the remaining vertices are pendant. $F(\mathcal{S}_n) = m^3 + (n-1).1^3 = m^3 + n - 1$. □

Proposition 5. *Let $\mathcal{S}_n$ be a hyperstar on n vertices (with at least two hyperedges). Then,*

$$n - 7 \leq F(\mathcal{S}_n) \leq (n-1)(n^2 - 2n + 2).$$

Equality in the lower bound is attained by any hyperstar with two hyperedges, and the upper bound is attained by the star graph on n vertices.

Proof. The proof directly follows from Lemma 3. □

Lemma 4. *Let $\mathcal{P}_n$ be a hyperpath on n vertices and m hyperedges. Then,*

$$n + 7(m-1) \leq F(\mathcal{P}_n) \leq 2(4n - 7),$$

with equality, the lower bound holds when the hyperpath is linear, and the upper bound is attained by any hyperpath (on n vertices and m hyperedges) that has exactly two pendant vertices.

Lemma 5. *Let $\mathcal{S}(m,h,r), r > h$ be an r-uniform sunflower hypergraph with m hyperedges and h seeds. Then, $F(\mathcal{S}(m,h,r)) = m(hm^2 + r - h)$.*

Proof. Each hyperedge in an r-uniform sunflower hypergraph, which contains h seeds, contributes hm^2+r-h to $F(\mathcal{S}(m,h,r))$, and hence it is equal to $m(hm^2+r-h)$. Hence, $F(\mathcal{S}(m,h,r)) = \sum_{u\in\mathcal{V}} d_u^3 = h\cdot m^3 + m(r-h)\cdot 1^3 = m(hm^2+r-h)$. □

Theorem 1. *Let $\mathcal{T}_n$ be a minimally connected hypergraph on n vertices with m hyperedges. Then,*

$$n + 7(m-1) \le F(\mathcal{T}_n) \le nm^3 - m^4 + m,$$

where the equality in the lower bound is attained by the linear hyperpath on n vertices with m hyperedges (or $\mathcal{T}_n$ is a linear minimally connected hypergraph with $\Delta(\mathcal{H}) = 2$) and the upper bound is attained by the sunflower hypergraph $\mathcal{S}(m, n-m, n-m+1)$.

Proof. If the total number of vertices and hyperedges is fixed in $\mathcal{T}_n$, then the variation of the F-index of $\mathcal{T}_n$ is due to the variation in including these n vertices among these m hyperedges. For $m = 1$, the lower bound is clear as the minimally connected hypergraph is always connected. For $m \ge 2$, it is trivial that the lower bound is attained by a linear minimally connected hypergraph. Let $\mathcal{T}_n$ be a linear minimally connected hypergraph (m hyperedges) that contains $t \ge 1$ vertices of degree greater than or equal to 3 and $S = \{v_i \in \mathcal{V}(\mathcal{T}_n) : d(v_i) = \Delta_i \ge 3\}$. Now,, it is important to note that $\mathcal{T}_n$ contains $n - m + 1 + \sum_{i=1}^{t}(\Delta_i - 2)$ pendant vertices and $m - \sum_{i=1}^{t}\Delta_i + t - 1$ vertices of degree two. Therefore,

$$\begin{aligned}
F(\mathcal{T}_n) &= \sum_{i=1}^{t}\Delta_i^3 + \left(m - \sum_{i=1}^{t}\Delta_i + t - 1\right)\cdot 2^3 + \left(n - m + 1 + \sum_{i=1}^{t}(\Delta_i - 2)\right)\cdot 1^3 \\
&= \sum_{i=1}^{t}\Delta_i^3 + 8m - 8\sum_{i=1}^{t}\Delta_i + 8t - 8 + n - m + 1 + \sum_{i=1}^{t}\Delta_i - 2t \\
&= \sum_{i=1}^{t}\Delta_i^3 - 7\sum_{i=1}^{t}\Delta_i + 6t + 7m + n - 7 \\
&> 7m + n - 7 = F(\mathcal{P}_n),
\end{aligned}$$

where $\mathcal{P}_n$ be a linear hyperpath on n vertices with m hyperedges. Also, it is direct that any linear minimally connected hypergraph on n vertices with m hyperedges and $\Delta = 2$ has the same value of F-index (minimum).

It is well known that the maximum degree of a vertex in a hypergraph having m hyperedges is m. In order to maximize $F(\mathcal{T}_n)$, the number of vertices of degree m should be maximum. That is, the number of vertices that are contained in all the m hyperedges should be maximum. Hence, the sunflower hypergraph $\mathcal{S}(m, n-m, n-$

$m+1$) will have the maximum value of F-index among all minimally connected hypergraphs of order n with m hyperedges and is given by

$$F(\mathcal{S}(m, n-m, n-m+1)) = m((n-m)m^2 + (n-m+1) - (n-m)) = nm^3 - m^4 + m,$$

as desired. □

Lemma 6. *Let $n = k + l$ be a partition of a positive integer n into two non-negative integers k and l such that $k(lk^2+1)$ is the maximum. Then*

$$k = s \text{ or } k = s+1,$$

$$\text{where } s = \left\lfloor \frac{1}{4}\left(n + \frac{n^2}{D} + D\right)\right\rfloor \text{ and } D = \sqrt[3]{8 + n^3 + 4\sqrt{4+n^3}}.$$

Proof. Let $n = x + y$ be the partition of a positive integer $n \geq 3$ into two non-negative real numbers x and y. Now the maximization of $x(yx^2+1)$ is the same as that of maximizing $x((n-x)x^2+1) = nx^3 - x^4 + x$, and on differentiating with respect to x and equating it to zero, we get the extremum point. That is,

$$\frac{d}{dx}(nx^3 - x^4 + x) = 3nx^2 - 4x^3 + 1 = 0.$$

$$\implies x = \frac{1}{4}\left(n + \frac{n^2}{\sqrt[3]{8+n^3+4\sqrt{4+n^3}}} + \sqrt[3]{8+n^3+4\sqrt{4+n^3}}\right) \text{ is the extremum}$$

point, as the other two roots of the above polynomial equation are complex.

Since our aim is to partition n into two integers, the floor or ceil of the above real number x will be the (integer) extremum point. □

Theorem 2. *Let $\mathcal{T}_n$ be a minimally connected hypergraph with n vertices. Then,*

$$n \leq F(\mathcal{T}_n) \leq \max_{m=s,s+1}\{nm^3 - m^4 + m\},$$

where $s = \left\lfloor \frac{1}{4}\left(n + \frac{n^2}{D} + D\right)\right\rfloor$, where $D = \sqrt[3]{8+n^3+4\sqrt{4+n^3}}$. Equality in the upper bound is attained by the sunflower hypergraph $\mathcal{S}(m, n-m, n-m+1)$.

Proof. From Theorem 1, it is known that a minimally connected hypergraph on n vertices with m hyperedges has the maximum value of $F(\mathcal{T}_n)$ and is attained by the sunflower hypergraph $\mathcal{S}(m, n-m, n-m+1)$. The value of m for which $nm^3 - m^4 + m$ is maximum can be obtained from Lemma 6. □

Let $\mathcal{H}$ be a hypergraph with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$. If e is an arbitrary hyperedge in $\mathcal{E}$, then we denote the hypergraph obtained from $\mathcal{H}$ on deleting the edge e by $\mathcal{H} - e$.

Lemma 7. *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a hypergraph on n vertices and $e \notin \mathcal{E}$. Then,*

$$F(\mathcal{H} + e) > F(\mathcal{H}).$$

Proof. Let $e \in \mathcal{E}$ be a hyperedge that contains $k \geq 1$ vertices and $\mathcal{V}' = \mathcal{V} \setminus e$. Then, by the definition of the F-index of a hypergraph,

$$F(\mathcal{H}) = \sum_{u \in \mathcal{V}} d_u^3 = \sum_{x \in e} d_x^3 + \sum_{y \in \mathcal{V}'} d_y^3 < \sum_{x \in e} (d_x + 1)^3 + \sum_{y \in \mathcal{V}'} d_y^3 = F(\mathcal{H} + e),$$

as desired. □

Theorem 3. *Let $\mathcal{H} = (\mathcal{V} = U_1 \cup U_2, \mathcal{E})$ be a connected bipartite hypergraph on n vertices with $|U_1| = s$ and $|U_2| = t$. Then,*

$$n \leq F(\mathcal{H}) \leq s\ 2^{3(s-1)}(2^t - 1)^3 + t\ 2^{3(t-1)}(2^s - 1)^3.$$

Equality in the lower bound is attained by the hypergraph, which has a single hyperedge containing all the vertices of the hypergraph, and the upper bound is attained by the complete bipartite hypergraph, $\mathcal{H}_{K_{s,t}}$.

Proof. From Lemma 7 it is direct to note that, among all bipartite hypergraphs, $\mathcal{H} = (U_1 \cup U_2, \mathcal{E})$, the complete bipartite hypergraph, $\mathcal{H}_{K_{s,t}}$ where $|U_1| = s,\ |U_2| = t$ attains the maximum value of F-index. The degree of any vertex $u \in U_1$ in a complete bipartite hypergraph $\mathcal{H}_{K_{s,t}}$ is given by,

$$d_u = (2^t - 1)\left(\binom{s-1}{0} + \binom{s-1}{1} + \cdots + \binom{s-1}{s-1}\right) = (2^t - 1)2^{s-1}.$$

Similarly, for any vertex $v \in U_2$, we have $d_v = (2^s - 1)2^{t-1}$, from which the result follows. □

Theorem 4. *Let $\mathcal{H}^{(r)} = (\mathcal{V} = U_1 \cup U_2, \mathcal{E})$ be an r-uniform connected bipartite hypergraph on n vertices with $|U_1| = s$ and $|U_2| = t$. Then*

$$F(\mathcal{H}^{(r)}) \leq s\ d_u^3 + t\ d_v^3,$$

where $d_u = \sum_{i=0}^{r-2} \binom{s-1}{i}\binom{t}{r-1-i}$ and $d_v = \sum_{i=0}^{r-2} \binom{t-1}{i}\binom{s}{r-1-i}$. Equality in the upper bound is attained by the r-uniform complete bipartite hypergraph, $\mathcal{H}^{(r)}_{K_{s,t}}$.

Proof. The maximum value of F-index among all r-uniform bipartite hypergraphs is attained by the r-uniform complete bipartite hypergraph $\mathcal{H}^{(r)}_{K_{s,t}}$. For $u \in U_1$, the number of ways of choosing an r element subset e from $U_1 \cup U_2$ such that $u \in e$ and $e \cap U_2 \neq \emptyset$ is given by $\sum_{i=0}^{r-2} \binom{s-1}{i}\binom{t}{r-1-i}$. Hence, the degree of a vertex $u \in U_1$ in $\mathcal{H}^{(r)}_{K_{s,t}}$ is given by $\sum_{i=0}^{r-2} \binom{s}{i}\binom{t}{r-1-i}$. Similarly, for some $v \in U_2$, the number of ways of choosing an $r-1$ element subset e' from $U_1 \cup U_2$ such that $e' \cap U_1 \neq \emptyset$ is $\sum_{i=0}^{r-2} \binom{t-1}{i}\binom{s}{r-1-i}$ and hence the proof. □

Theorem 5. *Let $\mathcal{H}^{(r)} = (\mathcal{V}, \mathcal{E})$ be a connected r-uniform hypergraph on n vertices. Then*

$$F(\mathcal{H}) \leq n\binom{n-1}{r-1}^3.$$

Equality in the upper bound is attained by the r-uniform complete hypergraph, $\mathcal{H}^{(r)}_{K_n}$.

Proof. The expression for the F-index of the r-uniform complete hypergraph $\mathcal{H}^{(r)}_{K_n}$ can be computed by observing that the degree of every vertex in $\mathcal{H}^{(r)}_{K_n}$ is equal to $\binom{n-1}{r-1}$. Also, the r-uniform complete hypergraph has the maximum value of F-index among the class of r-uniform hypergraphs, which follows from Lemma 7. □

Theorem 6. *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a connected hypergraph on n vertices. Then,*

$$n \leq F(\mathcal{H}) \leq n\ 2^{3(n-1)}.$$

Equality in the upper bound is attained by the complete hypergraph, $\mathcal{H}_{K_n}$.

Proof. The proof follows by noting that the degree of every vertex in $\mathcal{H}_{K_n}$ is equal to 2^{n-1}. □

4. F-Index of Hypergraph Operations

In this section, we discuss the forgotten index of a few hypergraph operations.

4.1. Join

Let $\mathcal{H}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ and $\mathcal{H}_2 = (\mathcal{V}_2, \mathcal{E}_2)$ be two r-uniform hypergraphs. Then (r-uniform) join of $\mathcal{H}_1$ and $\mathcal{H}_2$ denoted by $\mathcal{H}_1 +^r \mathcal{H}_2$ has the vertex set $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ and the edge set is given by $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}^*$, where

$$\mathcal{E}^* = \{e \subseteq \mathcal{V} : |e| = r, e \cap \mathcal{V}_i \neq \emptyset \text{ for } i = 1, 2\}.$$

Since we have considered not only uniform hypergraphs in our study, we generalize the join operation to non-uniform hypergraphs as follows:
Let $\mathcal{H}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ and $\mathcal{H}_2 = (\mathcal{V}_2, \mathcal{E}_2)$ be two hypergraphs. Then the join of $\mathcal{H}_1$ and $\mathcal{H}_2$ denoted by $\mathcal{H}_1 + \mathcal{H}_2$ has the vertex set $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ and the edge set is given by $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}^\dagger$, where

$$\mathcal{E}^\dagger = \{e \subseteq \mathcal{V} : e \cap \mathcal{V}_i \neq \emptyset \text{ for } i = 1, 2\}.$$

Theorem 7. *Let $\mathcal{H}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ and $\mathcal{H}_2 = (\mathcal{V}_2, \mathcal{E}_2)$ be two r-uniform hypergraphs with $|\mathcal{V}_1| = n_1, |\mathcal{V}_2| = n_2, |\mathcal{E}_1| = m_1$ and $|\mathcal{E}_2| = m_2$. If $M_1(\mathcal{H})$ denotes the first Zagreb index of the hypergraph $\mathcal{H}$, then*

$$F(\mathcal{H}_1 +^r \mathcal{H}_2) = F(\mathcal{H}_1) + F(\mathcal{H}_2) + n_1 D_1^3 + n_2 D_2^3 + 3D_1(M_1(\mathcal{H}_1) + D_1 r m_1) + 3D_2(M_1(\mathcal{H}_2) + D_2 r m_2),$$

where $D_1 = \sum_{i=0}^{r-2} \binom{n_1-1}{i}\binom{n_2}{r-1-i}$ and $D_2 = \sum_{i=0}^{r-2} \binom{n_2-1}{i}\binom{n_1}{r-1-i}$.

Proof. For $u \in \mathcal{V}_1$ and $v \in \mathcal{V}_2$, let d_u and d_v be the degrees of the vertices u in $\mathcal{H}_1$ and v in $\mathcal{H}_2$, respectively. Also, let d'_u and d'_v be the degrees of the vertices u and v in $\mathcal{H}_1 +^r \mathcal{H}_2$. If x is an arbitrary vertex in $\mathcal{V}$, by the definition of F-index, we have $F(\mathcal{H}_1 +^r \mathcal{H}_2)$

$$\begin{aligned}
&= \sum_{x \in \mathcal{V}_1 \cup \mathcal{V}_2} (d'_x)^3 = \sum_{u \in \mathcal{V}_1} (d'_u)^3 + \sum_{v \in \mathcal{V}_2} (d'_v)^3 \\
&= \sum_{u \in \mathcal{V}_1} (d_u + D_1)^3 + \sum_{v \in \mathcal{V}_2} (d_v + D_2)^3 \\
&\text{where } D_1 = \sum_{i=0}^{r-2} \binom{n_1-1}{i}\binom{n_2}{r-1-i} \text{ and } D_2 = \sum_{i=0}^{r-2} \binom{n_2-1}{i}\binom{n_1}{r-1-i} \\
&= \sum_{u \in \mathcal{V}_1} (d_u^3 + D_1^3 + 3d_u^2 D_1 + 3d_u D_1^2) + \sum_{v \in \mathcal{V}_2} (d_v^3 + D_2^3 + 3d_v^2 D_2 + 3d_v D_2^2) \\
&= F(\mathcal{H}_1) + n_1 D_1^3 + 3D_1 M_1(\mathcal{H}_1) + 3D_1^2 \sum_{u \in \mathcal{V}_1} d_u + F(\mathcal{H}_2) + n_2 D_2^3 + 3D_2 M_1(\mathcal{H}_2) + 3D_2^2 \sum_{v \in \mathcal{V}_2} d_v \\
&= F(\mathcal{H}_1) + n_1 D_1^3 + 3D_1 M_1(\mathcal{H}_1) + 3D_1^2 r m_1 + F(\mathcal{H}_2) + n_2 D_2^3 + 3D_2 M_1(\mathcal{H}_2) + 3D_2^2 r m_2,
\end{aligned}$$

as desired. □

Corollary 1. *Let $\mathcal{H}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ and $\mathcal{H}_2 = (\mathcal{V}_2, \mathcal{E}_2)$ be two r-uniform hypergraphs with $|\mathcal{V}_1| = n_1, |\mathcal{V}_2| = n_2, |\mathcal{E}_1| = m_1$ and $|\mathcal{E}_2| = m_2$. Then,*

$$F(\mathcal{H}_1 + \mathcal{H}_2) = F(\mathcal{H}_1) + F(\mathcal{H}_2) + n_1 D_1^3 + n_2 D_2^3 + 3D_1(M_1(\mathcal{H}_1) + D_1 r m_1) + 3D_2(M_1(\mathcal{H}_2) + D_2 r m_2),$$

where $D_1 = 2^{n_1-1}(2^{n_2} - 1)$ and $D_2 = 2^{n_2-1}(2^{n_1} - 1)$.

Theorem 8. *Let $\mathcal{H}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ and $\mathcal{H}_2 = (\mathcal{V}_2, \mathcal{E}_2)$ be two hypergraphs with $|\mathcal{V}_1| = n_1, |\mathcal{V}_2| = n_2, |\mathcal{E}_1| = m_1$ and $|\mathcal{E}_2| = m_2$. If Δ_1 and Δ_2 are the maximum degrees in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, then*

$$F(\mathcal{H}_1 + \mathcal{H}_2) \leq F(\mathcal{H}_1) + F(\mathcal{H}_2) + n_1 D_1^3 + n_2 D_2^3 + 3D_1(M_1(\mathcal{H}_1) + D_1 n_1 \Delta_1) \\ +3D_2(M_1(\mathcal{H}_2) + D_2 n_2 \Delta_2),$$

where $D_1 = 2^{n_1-1}(2^{n_2} - 1)$ and $D_2 = 2^{n_2-1}(2^{n_1} - 1)$. The equality in the above holds if and only if both $\mathcal{H}_1$ and $\mathcal{H}_2$ are regular.

Note 1. By replacing Δ_1, Δ_2 by δ_1 and δ_2, respectively, in the inequality of Theorem 8, we get the lower bound for $F(\mathcal{H}_1 + \mathcal{H}_2)$, and the equality in the lower bound holds if and only if both $\mathcal{H}_1$ and $\mathcal{H}_2$ are regular.

4.2. Cartesian Product

Let $\mathcal{H}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ and $\mathcal{H}_2 = (\mathcal{V}_2, \mathcal{E}_2)$ be two hypergraphs. The Cartesian product of $\mathcal{H}_1$ and $\mathcal{H}_2$ denoted by $\mathcal{H}_1 \Box \mathcal{H}_2$ has the vertex set $\mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2$ and the edge set is given by $\mathcal{E} = \mathcal{E}_1^* \cup \mathcal{E}_2^*$, where

$$\mathcal{E}_1^* = \{e_1 \times \{v\} : e_1 \in \mathcal{E}_1 \text{ and } v \in \mathcal{V}_2\}$$

and

$$\mathcal{E}_2^* = \{\{u\} \times e_2 : e_2 \in \mathcal{E}_2 \text{ and } u \in \mathcal{V}_1\}.$$

If $|\mathcal{V}_1| = n_1, |\mathcal{V}_2| = n_2, |\mathcal{E}_1| = m_1$ and $|\mathcal{E}_2| = m_2$, then the total number of hyperedges in the Cartesian product of $\mathcal{H}_1$ and $\mathcal{H}_2$ is given by,

$$|\mathcal{E}| = |\mathcal{E}_1^*| + |\mathcal{E}_2^*| = m_1 n_2 + m_2 n_1.$$

Example 1. Let $\mathcal{H}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ and $\mathcal{H}_2 = (\mathcal{V}_2, \mathcal{E}_2)$ be two hypergraphs with $\mathcal{V}_i, i = 1, 2$ and $\mathcal{E}_i, i = 1, 2$ defined as follows: $\mathcal{V}_1 = \{1, 2, 3, 4\}$, $\mathcal{V}_2 = \{a, b, c\}$, $\mathcal{E}_1 = \{\{1, 2, 3\}, \{3, 4\}\}$ and $\mathcal{E}_2 = \{\{a, b, c\}\}$. For the sake of simplicity, here we write the ordered pair (x, y) as xy. $\mathcal{V}(\mathcal{H}_1 \Box \mathcal{H}_2) = \{1a, 1b, 1c, 2a, 2b, 2c, 3a, 3b, 3c, 4a, 4b, 4c\}$

$$\mathcal{E}_1^* = \{\{1a, 2a, 3a\}, \{1b, 2b, 3b\}, \{1c, 2c, 3c\}, \{3a, 4a\}, \{3b, 4b\}, \{3c, 4c\}\}$$

and

$$\mathcal{E}_2^* = \{\{1a, 1b, 1c\}, \{2a, 2b, 2c\}, \{3a, 3b, 3c\}, \{4a, 4b, 4c\}\}.$$

Theorem 9. *Let $\mathcal{H}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ and $\mathcal{H}_2 = (\mathcal{V}_2, \mathcal{E}_2)$ be two hypergraphs with $|\mathcal{V}_1| = n_1$ and $|\mathcal{V}_2| = n_2$. If Δ_1(resp. δ_1) and Δ_2(resp. δ_2) are the maximum (resp. minimum) degree in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, then*

$$F(\mathcal{H}_1 \Box \mathcal{H}_2) \leq n_2 F(\mathcal{H}_1) + n_1 F(\mathcal{H}_2) + 3n_1 n_2 \Delta_1 \Delta_2 (\Delta_1 + \Delta_2),$$

and

$$F(\mathcal{H}_1 \Box \mathcal{H}_2) \geq n_2 F(\mathcal{H}_1) + n_1 F(\mathcal{H}_2) + 3n_1 n_2 \delta_1 \delta_2 (\delta_1 + \delta_2).$$

Equality in the above holds if and only if both $\mathcal{H}_1$ and $\mathcal{H}_2$ are regular.

Proof. Let d_u and d_v, respectively, be the degrees of the vertices u in $\mathcal{H}_1$ and v in $\mathcal{H}_2$. We denote the degree of a vertex (u,v) in $\mathcal{H}_1 \Box \mathcal{H}_2$ by d_{uv}, and it is very important to observe that $d_{uv} = d_u + d_v$. Hence,

$$\begin{aligned} F(\mathcal{H}_1 \Box \mathcal{H}_2) &= \sum_{(u,v)\in\mathcal{V}_1\times\mathcal{V}_2} d_{uv}^3 \\ &= \sum_{(u,v)\in\mathcal{V}_1\times\mathcal{V}_2} (d_u+d_v)^3 \\ &= \sum_{(u,v)\in\mathcal{V}_1\times\mathcal{V}_2} (d_u^3 + d_v^3 + 3d_u^2 d_v + 3d_v^2 d_u) \\ &= n_2 \sum_{u\in\mathcal{V}_1} d_u^3 + n_1 \sum_{v\in\mathcal{V}_2} d_v^3 + 3 \sum_{(u,v)\in\mathcal{V}_1\times\mathcal{V}_2} d_u d_v (d_u + d_v) \\ &\le n_2 F(\mathcal{H}_1) + n_1 F(\mathcal{H}_2) + 3n_1 n_2 \Delta_1 \Delta_2 (\Delta_1 + \Delta_2). \end{aligned}$$

The lower bound can also be obtained similarly, and it is clear that the equality in the lower bound as well as the upper bound holds if and only if both $\mathcal{H}_1$ and $\mathcal{H}_2$ are regular. □

4.3. Corona Product

Let $\mathcal{G} = (\mathcal{U} = \{u_1, \ldots, u_{n_1}\}, \mathcal{E}')$ and $\mathcal{H} = (\mathcal{V} = \{v_1, \ldots, v_{n_2}\}, \mathcal{E})$ be two hypergraphs of order n_1 and n_2 respectively, also $\{\mathcal{H}_i = (\mathcal{V}_i = \{v_1^{(i)}, \ldots, v_{n_2}^{(i)}\}, \mathcal{E}_i) : 1 \le i \le n_1\}$ be the collection of n_1 copies of $\mathcal{H}$ and $\mathcal{W}_i = \mathcal{V}_i \cup \{u_i\}$. Now, corona product ($r$-uniform, $r \ge 2$) $\mathcal{G} \circ^r \mathcal{H}$ of two hypergraphs $\mathcal{G}$ and $\mathcal{H}$ has the vertex set $\bigcup_{i=1}^{n_1} \mathcal{W}_i$ and the edge set $\bigcup_{i=1}^{n_1} \left(\mathcal{E}_i \cup \mathcal{E}_i^\dagger\right) \cup \mathcal{E}'$, where

$$\mathcal{E}_i^\dagger = \{e \subseteq \mathcal{W}_i : u_i \in e \text{ and } |e| = r \ge 2\}.$$

Example 2. Let $\mathcal{G} = (\mathcal{U}, \mathcal{E}')$ and $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be two hypergraphs, where $\mathcal{U} = \{1,2,3\}$, $\mathcal{E}' = \{\{1,2\},\{1,2,3\}\}$, $\mathcal{V} = \{a,b,c,d\}$ and $\mathcal{E} = \{\{a,b\},\{a,b,d\},\{b,c,d\}\}$. Let $\mathcal{V}_i = \{a^{(i)}, b^{(i)}, c^{(i)}, d^{(i)}\}, 1 \le i \le 3$ and $\mathcal{E}_i = \{\{a^{(i)}, b^{(i)}\}, \{a^{(i)}, b^{(i)}, d^{(i)}\}, \{b^{(i)}, c^{(i)}, d^{(i)}\}\}$. The (3-uniform) corona product $\mathcal{G} \circ^3 \mathcal{H}$ of $\mathcal{G}$ and $\mathcal{H}$ has the vertex set $\bigcup_{i=1}^{3} \mathcal{W}_i$ where $\mathcal{W}_i = \mathcal{V}_i \cup \{i\}, 1 \le i \le 3$ and the edge set $\bigcup_{i=1}^{3} (\mathcal{E}_i \cup \mathcal{E}_i^\dagger) \cup \mathcal{E}'$, where $\mathcal{E}_i^\dagger = \{\{1, a^{(i)}, b^{(i)}\}, \{1, a^{(i)}, c^{(i)}\}, \{1, a^{(i)}, d^{(i)}\}, \{1, b^{(i)}, c^{(i)}\}, \{1, b^{(i)}, d^{(i)}\}, \{1, c^{(i)}, d^{(i)}\}\}$.

Theorem 10. *Let $\mathcal{G} = (\mathcal{U}, \mathcal{E}')$ and $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be two r-uniform ($r \ge 2$) hypergraphs with $|\mathcal{U}| = n_1, |\mathcal{V}| = n_2, |\mathcal{E}'| = m_1$ and $|\mathcal{E}| = m_2$. If $M_1(\mathcal{H})$ denotes the first Zagreb index of the hypergraph $\mathcal{H}$ and Δ_1 (resp. Δ_2) denotes the maximum degree of a vertex among all vertices of $\mathcal{G}$ (resp. $\mathcal{H}$), then*

$$\begin{aligned} F(\mathcal{G} \circ^r \mathcal{H}) = F(\mathcal{G}) + n_1 F(\mathcal{H}) + n_1(P_1^3 + n_2 P_2^3) + 3(P_1 M_1(\mathcal{G}) + n_1 P_2 M_1(\mathcal{H})) \\ + 3r(P_1^2 m_1 + n_1 P_2^2 m_2), \end{aligned}$$

where $P_1 = \binom{n_2}{r-1}$ *and* $P_2 = \binom{n_2-1}{r-2}$.

Proof. Let $\mathcal{H}_i = (\mathcal{V}_i, \mathcal{E}_i), 1 \leq i \leq n_1$ be the n_1 copies of the hypergraph, $\mathcal{H}$ and d_u,d_v be the degrees of the vertices u of $\mathcal{G}$ and v of $\mathcal{H}$ respectively, and d'_u, d'_v respectively be the degrees of the vertices u and v in $\mathcal{G} \circ^r \mathcal{H}$. For an arbitrary vertex x of $\mathcal{G} \circ^r \mathcal{H}$, d'_x denote the degree of the vertex x. Hence

$$\begin{aligned}
\sum_{x \in \bigcup\limits_{i=1}^{n_1} \mathcal{W}_i} d'^3_x &= \sum_{u \in \mathcal{U}} d'^3_u + \sum_{i=1}^{n_1} \sum_{v \in \mathcal{V}_i} d'^3_v \\
&= \sum_{u \in \mathcal{U}} (d_u + P_1)^3 + \sum_{i=1}^{n_1} \sum_{v \in \mathcal{V}_i} (d_v + P_2)^3 \\
&= \sum_{u \in \mathcal{U}} (d_u^3 + P_1^3 + 3d_u P_1(d_u + P_1)) + \sum_{i=1}^{n_1} \sum_{v \in \mathcal{V}_i} (d_v^3 + P_2^3 + 3P_2 d_v(P_2 + d_v)) \\
&= \sum_{u \in \mathcal{U}} (d_u^3 + P_1^3 + 3d_u P_1(d_u + P_1)) + n_1 \sum_{v \in \mathcal{V}_i} (d_v^3 + P_2^3 + 3P_2 d_v(P_2 + d_v))
\end{aligned}$$

$$\begin{aligned}
&= F(\mathcal{G}) + n_1 P_1^3 + 3P_1 M_1(\mathcal{G}) + 3P_1^2 \sum_{u \in \mathcal{U}} d_u + n_1 F(\mathcal{H}) \\
&+ n_1 n_2 P_2^3 + 3n_1 P_2 M_1(\mathcal{H}) + 3n_1 P_2^2 \sum_{v \in \mathcal{V}} d_v \\
&= F(\mathcal{G}) + n_1 P_1^3 + 3P_1 M_1(\mathcal{G}) + 3P_1^2 r m_1 + n_1 F(\mathcal{H}) \\
&+ n_1 n_2 P_2^3 + 3n_1 P_2 M_1(\mathcal{H}) + 3n_1 P_2^2 r m_2,
\end{aligned}$$

as desired. □

Corollary 2. *Let* $\mathcal{G} = (\mathcal{U}, \mathcal{E}')$ *and* $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ *be two hypergraphs with* $|\mathcal{U}| = n_1, |\mathcal{V}| = n_2, |\mathcal{E}'| = m_1$ *and* $|\mathcal{E}| = m_2$. *If* $M_1(\mathcal{H})$ *denotes the first Zagreb index of the hypergraph* $\mathcal{H}$, *then*

$$\begin{aligned}
F(\mathcal{G} \circ^r \mathcal{H}) \leq F(\mathcal{G}) + n_1 F(\mathcal{H}) + n_1(P_1^3 + n_2 P_2^3) + 3(P_1 M_1(\mathcal{G}) + n_1 P_2 M_1(\mathcal{H})) \\
+ 3(P_1^2 n_1 \Delta_1 + n_1 P_2^2 n_2 \Delta_2),
\end{aligned}$$

where $P_1 = \binom{n_2}{r-1}$ *and* $P_2 = \binom{n_2-1}{r-2}$. *The equality in the above holds if both* $\mathcal{G}$ *and* $\mathcal{H}$ *are regular.*

The generalized (not uniform) corona product of two hypergraphs $\mathcal{G}$ and $\mathcal{H}$ denoted by $\mathcal{G} \circ \mathcal{H}$ has the same vertex set as that of $\mathcal{G} \circ^r \mathcal{H}$ and the edge set $\bigcup\limits_{i=1}^{n_1} \left(\mathcal{E}_i \cup \mathcal{E}_i^{\dagger'}\right) \cup \mathcal{E}'$, where

$$\mathcal{E}_i^{\dagger'} = \{e \subseteq \mathcal{W}_i : u_i \in e \text{ and } e \cap \mathcal{V}_i \neq \emptyset\}.$$

Corollary 3. *Let $\mathcal{G} = (\mathcal{U}, \mathcal{E}')$ and $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be two hypergraphs with $|\mathcal{U}| = n_1, |\mathcal{V}| = n_2, |\mathcal{E}'| = m_1$ and $|\mathcal{E}| = m_2$. If $M_1(\mathcal{H})$ denotes the first Zagreb index of the hypergraph $\mathcal{H}$, then*

$$F(\mathcal{G} \circ \mathcal{H}) \leq F(\mathcal{G}) + n_1 F(\mathcal{H}) + n_1(P_1^3 + n_2 P_2^3) + 3(P_1 M_1(\mathcal{G}) + n_1 P_2 M_1(\mathcal{H})) + 3(P_1^2 n_1 \Delta_1 + n_1 P_2^2 n_2 \Delta_2),$$

where $P_1 = 2^{n_2} - 1$ and $P_2 = 2^{n_2-1}$. The equality holds in the above if and only if both $\mathcal{G}$ and $\mathcal{H}$ are regular.

5. Conclusion

In this article, the relation between the forgotten topological index of the hypergraph and some other vertex-degree-based topological index has been obtained. We have given the bounds for the forgotten topological index of uniform hypergraphs, minimally connected hypergraphs, and some families of minimally connected hypergraphs. In addition, the expression for the F-index of some binary operations on hypergraphs has been obtained.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

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