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Research Article

Homogeneous symmetric functions and new generating functions for products of some numbers with bivariate Mersenne and bivariate Mersenne Lucas polynomials

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Abstract: The aim of this paper is to provide some operators for symmetric functions for the purpose of obtaining new generating functions for products of k-Fibonacci, k-Jacobsthal numbers, bivariate complex Fibonacci polynomials and Chebyshev polynomials with bivariate Mersenne and bivariate Mersenne Lucas polynomials.

Keywords: generating functions, bivariate Mersenne Lucas polynomials, k-Fibonacci numbers, k-Pell numbers, k-balancing numbers.

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1. Introduction

Generating functions are central in many fields as algebraic geometry, representation theory and combinatorics, they provide algebraic ways to encode complex information

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and allow us to extract enumeration results. Over several years, many studies have been conducted in the field of number sequences and polynomials and their properties and identities, a lot of these researches are based on getting the generating function [2, 4, 7–9, 11, 13, 14, 16, 18, 21].
Also many researchers were interested in k-Mersenne sequences and the sequences that are derived from their, in particular, In [24] K. Uslu and K.V. Deniz, studied k-Mersenne numbers. Also, in [12] M. Chelghem and A. Boussayoud defined and studied k-Mersenne Lucas numbers, and in [22] N. Saba and A. Boussayoud introduced the concept of bivariate Mersenne and bivariate Mersenne Lucas polynomials for the first time and they gave symmetric functions, explicit formula and d'Ocagne's identity of these polynomials, also they use the Binet's formula to obtain few well-known identities and furthermore, they investigated some summation formulas. That's why we saw their results as a good opportunity to present our work based on obtaining generating functions for some products of particular numbers and polynomials as k-Fibonacci and k-Jacobsthal numbers, bivariate complex Fibonacci polynomials and Chebyshev polynomials with this bivariate polynomials.

2. Definitions and properties

In this section, mentioning symmetric functions and some of their properties is important and useful. We also present some identities.
It is well-known that a symmetric function of an alphabet $A = \{a_1, a_2, a_3, \dots\}$ is a function of the letters which is invariant under the permutation of the letters of A. That is, a function $f(x_1, x_2, \dots, x_n)$ in n variables is symmetric if for all permutations of the index set $(1, 2, \dots, n)$, the following equality holds

$$f(x_1, x_2, \dots, x_n) = f(x_{s(1)}, x_{s(2)}, \dots, x_{s(n)}).$$

Definition 1. [6] Let k and n be two positive integers and let $\{a_1, a_2, \dots, a_n\}$ be the set of given variables. Then, the elementary symmetric function $e_k(a_1, a_2, \dots, a_n)$ is defined by

$$e_k^{(n)} = e_k(a_1, a_2, \dots, a_n) = \sum_{i_1+i_2+\dots+i_n=k} a_1^{i_1} a_2^{i_2} \dots a_n^{i_n}, \quad 0 \le k \le n, \tag{2.1}$$

with $i_1, i_2, \dots, i_n = 0 \vee 1$.

Proposition 1. *[6] The generating function of the elementary symmetric function is given by*

$$E(z) = \sum_{k\ge 0} e_k z^k = \prod_{i=1}^{n} (1 + a_i z).$$

Definition 2. [6] Let k and n be two positive integers and let $\{a_1, a_2, \ldots, a_n\}$ be the set of given variables. Then, the complete symmetric functions $h_k(a_1, a_2, \ldots, a_n)$ is defined by

$$h_k^{(n)} = h_k(a_1, a_2, \ldots, a_n) = \sum_{i_1+i_2+\ldots+i_n=k} a_1^{i_1} a_2^{i_2} \ldots a_n^{i_n}, \tag{2.2}$$

with $i_1, i_2, \ldots, i_n \geq 0$.

Proposition 2. *[6] The generating function of the completes symmetric function is given by*

$$H(z) = \sum_{k\geq 0} h_k z^k = \frac{1}{\prod_{i=1}^{n}(1-a_i z)}.$$

Definition 3. [1] Let n be a positive integer and $A = \{a_1, a_2\}$ be a set of given variables, the symmetric function S_n is defined by

$$S_n(A) = S_n(a_1 + a_2) = \frac{a_1^{n+1} - a_2^{n+1}}{a_1 - a_2},$$

with

$$S_0(a_1 + a_2) = 1, \quad S_1(a_1 + a_2) = a_1 + a_2, \quad S_2(a_1 + a_2) = a_1^2 + a_1 a_2 + a_2^2.$$

Remark 1. [1] We have $S_n(a_1 + a_2) = 0$, for $n < 0$.

Remark 2. Let $A = \{a_1, a_2\}$ be an alphabet. We have

$$S_n(a_1 + a_2) = h_n(a_1, a_2).$$

Definition 4. [10] For $n \in \mathbb{N}$, the Mersenne sequence, denoted by $(M_n)_{n\in\mathbb{N}}$, is defined recursively by

$$M_n = 3M_{n-1} - 2M_{n-2} \text{ For all } n \geqslant 2, \quad M_0 = 0, \quad M_1 = 1.$$

Definition 5. [24] For $n \in \mathbb{N}$ and $k \geq 1$, the k-Mersenne sequence, denoted by $(M_{k,n})_{n\in\mathbb{N}}$, is defined recursively by

$$M_{k,n} = 3kM_{k,n-1} - 2M_{k,n-2}, \text{ For all } n \geqslant 2, \quad M_{k,0} = 0, \quad M_{k,1} = 1.$$

The Binet's formula of k-Mersenne numbers is given by

$$M_{k,n} = \tfrac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2},$$

where $\lambda_1 = \frac{3k+\sqrt{9k^2-8}}{2}$ and $\lambda_2 = \frac{3k-\sqrt{9k^2-8}}{2}$ are the roots of the characteristic equation of the sequence $(M_{k,n})_{n\in\mathbb{N}}$.

Definition 6. [23] For $n \in \mathbb{N}$, the Mersenne Lucas sequence, denoted by $(m_n)_{n\in\mathbb{N}}$, is defined recursively by

$$m_n = 3m_{n-1} - 2m_{n-2} \text{ , For all } n \geqslant 2 \text{ , } m_0 = 2 \text{ , } m_1 = 3.$$

In [23], N. Saba and A. Boussayoud gave the following identities for Mersenne Lucas numbers.
The Binet's formula is given by

$$m_n = 2^n + 1.$$

The Negative extension

$$m_{-n} = \frac{m_n}{2^n}.$$

The generating function, the Catalan's identity and Cassini's identity are respectively given by

$$g(z) = \sum_{n=0}^{\infty} m_n z^n = \frac{2-3z}{1-3z+2z^2},$$

$$m_{n-r}m_{n+r} - m_n^2 = 2^{n-r}m_r^2 - 2^{n+2},$$

$$m_{n-1}m_{n+1} - m_n^2 = 2^{n-1}.$$

Also, the symmetric function as follows

$$m_n = 2S_n(\lambda_1 + [-\lambda_2]) - 3S_{n-1}(\lambda_1 + [-\lambda_2]),$$

with $\lambda_1 = 2$ and $\lambda_2 = 1$.

Definition 7. [12] For $n \in \mathbb{N}$, and $k \geq 1$ the k-Mersenne Lucas sequence, denoted by $(m_{k,n})_{n\in\mathbb{N}}$, is defined recursively by

$$m_{k,n} = 3km_{k,n-1} - 2m_{k,n-2}, \text{ For all } n \geqslant 2, \quad m_{k,0} = 2, \quad m_{k,1} = 3k.$$

Definition 8. [22] Let $n \geq 0$ be integer. The recurrence relation of bivariate Mersenne Lucas polynomials $(m_n(x,y))_{n\in\mathbb{N}}$ is given by

$$m_n(x,y) := \begin{cases} 2, \text{ if } n = 0 \\ 3y, \text{ if } n = 1 \\ 3ym_{n-1}(x,y) - 2xm_{n-2}(x,y), \text{ if } n \geq 2 \end{cases},$$

and it is assumed $x \neq 0$ and $9y^2 - 8x > 0$.

It is remarked that if $x = 1$ and $y = k$ (for $y \geq 1$) in the above recurrence relation, we obtain the sequence of k-Mersenne Lucas numbers, that is $m_n(1, k) = m_{k,n}$, when $m_n(1, 1) = m_n$, we recover the Mersenne Lucas numbers.
The first few terms of bivariate Mersenne Lucas polynomials are presented by $\left\{2, 3y, 9y^2 - 4x, 27y^3 - 18xy, 81y^4 - 72xy^2 + 8x^2, 243y^5 - 270xy^3 + 60x^2y, \dots\right\}$.
From the recurrence relation, the characteristic equation is extracted as follows

$$\lambda^2 - 3y\lambda + 2x = 0,$$

with characteristic roots

$$\lambda_1 = \frac{3y+\sqrt{9y^2-8x}}{2} \text{ and } \lambda_2 = \frac{3y-\sqrt{9y^2-8x}}{2}.$$

So that

$$\lambda_1 + \lambda_2 = 3y, \quad \lambda_1\lambda_2 = 2x \text{ and } \lambda_1 - \lambda_2 = \sqrt{9y^2 - 8x}.$$

The nth terms of the bivariate Mersenne Lucas polynomials are given in the next theorem.

Theorem 1. *[23] The Binet's formula for bivariate Mersenne Lucas polynomials is given by*

$$m_n(x, y) = \lambda_1^n + \lambda_2^n.$$

The negative extension of bivariate Mersenne Lucas polynomials is given by $(m_{-n}(x, y))_{n\in\mathbb{N}}$ as

$$m_{-n}(x, y) = \frac{m_n(x,y)}{(2x)^n}.$$

And they gave the generating function and symmetric function for bivariate Mersenne Lucas polynomials in the following theorems.

Theorem 2. *[23] For $n \in \mathbb{N}$, the generating function of bivariate Mersenne Lucas polynomials is given by*

$$g(z) = \sum_{n=0}^{\infty} m_n(x, y)z^n = \frac{2-3yz}{1-3yz+2xz^2}.$$

Theorem 3. *[22] For $n \in \mathbb{N}$, the symmetric functions of bivariate Mersenne and bivariate Mersenne Lucas polynomials are given by*

1. $M_n(x, y) = S_{n-1}(\lambda_1 + [-\lambda_2])$,
2. $m_n(x, y) = 2S_n(\lambda_1 + [-\lambda_2]) - 3yS_{n-1}(\lambda_1 + [-\lambda_2])$,

with $\lambda_1 = \frac{3y+\sqrt{9y^2-8x}}{2}$ *and* $\lambda_2 = \frac{3y-\sqrt{9y^2-8x}}{2}$.

The k-Fibonacci numbers $(F_{k,n})_{n\in\mathbb{N}}$ were introduced by S. Falcon and A. Plaza in [15] . They are defined recurrently by

$$F_{k,n} = kF_{k,n-1} + F_{k,n-2} \text{ for } k \geq 1 \text{ and } n \geq 2,$$

with $F_{k,0} = 1$ and $F_{k,1} = k$.
The bivariate complex Fibonacci polynomials $(F_n(x, y))_{n\in\mathbb{N}}$ were defined in [3] as follows

$$F_{n+1}(x, y) = ixF_n(x, y) + yF_{n-1}(x, y) \text{ for } n \geq 1,$$

with $F_0(x, y) = 0$ and $F_1(x, y) = 1$.
The Binet's and the explicit formulas of bivariate complex Fibonacci polynomials are respectively given by

$$F_n(x, y) = \frac{\lambda_1^n(x,y) - \lambda_2^n(x,y)}{\lambda_1(x,y) - \lambda_2(x,y)},$$

and

$$F_n(x, y) = \sum_{j=0}^{[n-1/2]} \binom{n-j-1}{1} (ix)^{n-2j-1} y^j.$$

Note that $\lambda_1(x, y)$, $\lambda_2(x, y)$ are the roots of the characteristic equation $\lambda^2 - ix\lambda - y = 0$.
Similarly, we define the k-Jacobsthal numbers by mean of the recurrence relation.
For any positive integer k, the k-Jacobsthal sequence say $(J_{k,n})_{n\in\mathbb{N}}$ is defined recurrently as follows [17, 19, 20].

$$J_{k,n+1} = kJ_{k,n} + 2J_{k,n-1}, \text{ for } n \geq 1,$$

with $J_{k,0} = 0, J_{k,1} = 1$.
The Binet's formula of k-Jacobsthal numbers is given by

$$J_{k,n} = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2},$$

where $\lambda_1 = \dfrac{k + \sqrt{8 + k^2}}{2}$ and $\lambda_2 = \dfrac{k - \sqrt{8 + k^2}}{2}$, are the roots of the characteristic equation

$$\lambda^2 - k\lambda - 2 = 0 \text{ and } \lambda_1 > \lambda_2.$$

Definition 9. [6] The symmetrizing operator $\delta^k_{a_1a_2}$ is defined by

$$\delta^k_{a_1a_2} f(a_1) = \frac{a_1^k f(a_1) - a_2^k f(a_2)}{a_1 - a_2}, \quad \forall k \in \mathbb{N}. \tag{2.3}$$

Remark 3. If $f(a_i) = a_i$, in formula (2.3), we obtain

$$\delta^k_{a_1a_2} f(a_1) = \frac{a_1^{k+1} - a_2^{k+1}}{a_1 - a_2}.$$

Remark 4. Let $A = \{a_1, a_2\}$ be an alphabet. We have

$$\delta^k_{a_1a_2}(a_1) = S_k(a_1 + a_2).$$

3. Main results

In this section, we introduce an operator to derive new generating functions of products of some well-known numbers and polynomials.

The following theorem is the basis for all applications and results that follow.

Theorem 4. *Let A, B and C be three alphabets, respectively, $\{a_1, a_2, a_3\}$, $\{b_1, b_2\}$ and $\{c_1, c_2\}$. Then we have*

$$\sum_{n=0}^{\infty} S_n(A) S_{n+k-1}(B) S_{n+k-1}(C) z^n = \frac{b_1^k b_2^k \begin{pmatrix} \left(\sum\limits_{n=0}^{\infty} S_n(-A) b_2^n c_1^n z^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-A) b_1^n c_1^n z^n\right) \sum\limits_{n=0}^{\infty} S_n(-A) S_{n-k-1}(B) c_2^{n+k} z^n \\ -\left(\sum\limits_{n=0}^{\infty} S_n(-A) b_2^n c_2^n z^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-A) b_1^n c_2^n z^n\right) \sum\limits_{n=0}^{\infty} S_n(-A) S_{n-k-1}(B) c_1^{n+k} z^n \end{pmatrix}}{(c_1 - c_2) \prod\limits_{a \in A} (1 - a b_1 c_1 z) \prod\limits_{a \in A} (1 - a b_2 c_1 z) \prod\limits_{a \in A} (1 - a b_1 c_2 z) \prod\limits_{a \in A} (1 - a b_2 c_2 z)}, \tag{3.1}$$

for all $k \in \mathbb{N}_0$.

Proof. By applying the operator $\delta_{c_1 c_2}^k \delta_{b_1 b_2}^k$ to the series $f(b_1 c_1 z) = \sum\limits_{n=0}^{\infty} S_n(A) b_1^n c_1^n z^n$, we have

$$\begin{aligned}
\delta_{c_1 c_2}^k \delta_{b_1 b_2}^k \left(f(b_1 c_1 z)\right) &= \delta_{c_1 c_2}^k \delta_{b_1 b_2}^k \left(\sum_{n=0}^{\infty} S_n(A) b_1^n c_1^n z^n\right) \\
&= \delta_{c_1 c_2}^k \left(\frac{b_1^k \sum\limits_{n=0}^{\infty} S_n(A) b_1^n c_1^n z^n - b_2^k \sum\limits_{n=0}^{\infty} S_n(A) b_2^n c_1^n z^n}{b_1 - b_2}\right) \\
&= \delta_{c_1 c_2}^k \left(\sum_{n=0}^{\infty} S_n(A) \frac{b_1^{n+1} - b_2^{n+k}}{b_1 - b_2} c_1^n z^n\right) \\
&= \delta_{c_1 c_2}^k \left(\sum_{n=0}^{\infty} S_n(A) S_{n+k-1}(B) c_1^n z^n\right) \\
&= \frac{c_1^k \sum\limits_{n=0}^{\infty} S_n(A) S_{n+k-1}(B) c_1^n z^n - c_2^k \sum\limits_{n=0}^{\infty} S_n(A) S_{n+k-1}(B) c_2^n z^n}{c_1 - c_2}
\end{aligned}$$

$$= \frac{\sum_{n=0}^{\infty} S_n(A)S_{n+k-1}(B)c_1^{n+k}z^n - \sum_{n=0}^{\infty} S_n(A)S_{n+k-1}(B)c_2^{n+k}z^n}{c_1 - c_2}$$

$$= \sum_{n=0}^{\infty} S_n(A)S_{n+k-1}(B)\left(\frac{c_1^{n+k} - c_2^{n+k}}{c_1 - c_2}\right)z^n$$

$$= \sum_{n=0}^{\inf} S_n(A)S_{n+k-1}(B)S_{n+k-1}(C)z^n.$$

On the other hand, since

$$f(b_1c_1z) = \frac{1}{\prod_{a\in A}(1 - ab_1c_1z)},$$

$$\delta_{c_1c_2}^k\delta_{b_1b_2}^k\left(\frac{1}{\prod_{a\in A}(1 - ab_1c_1z)}\right) = \delta_{c_1c_2}^k\left(\frac{\frac{b_1^k}{\prod_{a\in A}(1 - ab_1c_1z)} - \frac{b_2^k}{\prod_{a\in A}(1 - ab_2c_1z)}}{b_1 - b_2}\right)$$

$$= \delta_{c_1c_2}^k\left(\frac{b_1^k\prod_{a\in A}(1 - ab_2c_1z) - b_2^k\prod_{a\in A}(1 - ab_1c_1z)}{(b_1 - b_2)\prod_{a\in A}(1 - ab_1c_1z)\prod_{a\in A}(1 - ab_2c_1z)}\right).$$

Using the fact that

$$\sum_{n=0}^{\infty} S_n(-A)a_1^nb_1^nz^n = \prod_{a\in A}(1 - ab_1c_1z),$$

then

$$\delta_{c_1c_2}^k\delta_{b_1b_2}^k\left(\frac{1}{\prod_{a\in A}(1 - ab_1c_1z)}\right) = \delta_{c_1c_2}^k\left(\frac{\sum_{n=0}^{\infty} S_n(-A)b_1^kb_2^nc_1z^n - \sum_{n=0}^{\infty} S_n(-A)b_2^kb_1^nc_1^nz^n}{(b_1 - b_2)\prod_{a\in A}(1 - ab_1c_1z)\prod_{a\in A}(1 - ab_2c_1z)}\right)$$

$$= \delta_{c_1c_2}^{k}\left(\frac{-b_1^kb_2^k\sum_{n=0}^{\infty}S_n(-A)\left(\frac{b_1^{n-k}-b_2^{n-k}}{b_1-b_2}\right)c_1^nz^n}{\prod_{a\in A}(1-ab_1c_1z)\prod_{a\in A}(1-ab_2c_1z)}\right)$$

$$= \delta_{c_1c_2}^{k}\left(\frac{-b_1^kb_2^k\sum_{n=0}^{\infty}S_n(-A)S_{n-k-1}(B)c_1^nz^n}{\prod_{a\in A}(1-ab_1c_1z)\prod_{a\in A}(1-ab_2c_1z)}\right)$$

$$= \frac{1}{c_1-c_2}\left(\frac{-c_1^kb_1^kb_2^k\sum_{n=0}^{\infty}S_n(-A)S_{n-k-1}(B)c_1^nz^n}{\prod_{a\in A}(1-ab_1c_1z)\prod_{a\in A}(1-ab_2c_1z)}+\frac{c_2^kb_1^kb_2^k\sum_{n=0}^{\infty}S_n(-A)S_{n-k-1}(B)c_2^nz^n}{\prod_{a\in A}(1-ab_1c_2z)\prod_{a\in A}(1-ab_2c_2z)}\right)$$

$$= \frac{b_1^kb_2^k\left(\begin{array}{l}\left(\sum_{n=0}^{\infty}S_n(-A)b_2^nc_1^nz^n\right)\left(\sum_{n=0}^{\infty}S_n(-A)b_1^nc_1^nz^n\right)\sum_{n=0}^{\infty}S_n(-A)S_{n-k-1}(B)c_2^{n+k}z^n\\ -\left(\sum_{n=0}^{\infty}S_n(-A)b_2^nc_2^nz^n\right)\left(\sum_{n=0}^{\infty}S_n(-A)b_1^nc_2^nz^n\right)\sum_{n=0}^{\infty}S_n(-A)S_{n-k-1}(B)c_1^{n+k}z^n\end{array}\right)}{(c_1-c_2)\prod_{a\in A}(1-ab_1c_1z)\prod_{a\in A}(1-ab_2c_1z)\prod_{a\in A}(1-ab_1c_2z)\prod_{a\in A}(1-ab_2c_2z)}.$$

This completes the proof. □

In this theorem we used an operator in order to derive new generating functions for different products of numbers and polynomials.
If $k = 0, 1, 2$ and $a_3 = 0$ in theorem (4), we deduce the following results.

Lemma 1. *[5] Let A, B and C be three alphabets, respectively, $\{a_1, a_2\}$, $\{b_1, b_2\}$ and $\{c_1, c_2\}$. Then we have*

$$\sum_{n=0}^{\infty}S_n(A)S_{n-1}(B)S_{n-1}(C)z^n = \frac{N_1}{D}, \quad n\in\mathbb{N}, \tag{3.2}$$

with

$$\begin{aligned}N_1 &= (a_1+a_2)z - a_1a_2(b_1+b_2)(c_1+c_2)z^2 + b_1b_2c_1c_2(a_1+a_2)(2a_1a_2-(a_1+a_2)^2)z^3\\ &\quad + a_1a_2b_1b_2c_1c_2(b_1+b_2)(c_1+c_2)(a_1+a_2)^2z^4 - b_1b_2c_1c_2a_1^2a_2^2(a_1+a_2)(b_1b_2(c_1+c_2)^2+\\ &\quad c_1c_2(b_1+b_2)^2 - c_1c_2b_1b_2)z^5 + a_1^3a_2^3b_1^2b_2^2c_1^2c_2^2(b_1+b_2)(c_1+c_2)z^6.\end{aligned}$$

$$D = 1 - (a_1 + a_2)(b_1 + b_2)(c_1 + c_2)z + (b_1b_2(a_1 + a_2)^2(c_1 + c_2)^2 + ((b_1 + b_2)^2 - 2b_1b_2)$$
$$((a_1 + a_2)^2c_1c_2 - 2a_1a_2c_1c_2 + a_1a_2(c_1 + c_2)^2))z^2 - (a_1 + a_2)(b_1 + b_2)(c_1 + c_2)$$
$$(b_1b_2c_1c_2(a_1 + a_2)^2 + b_1b_2a_1a_2(c_1 + c_2)^2 + a_1a_2c_1c_2(b_1 + b_2)^2 - 5a_1a_2c_1c_2b_1b_2)z^3$$
$$+ (a_1^2a_2^2c_1^2c_2^2(b_1 + b_2)^4 + c_1^2c_2^2b_1^2b_2^2(a_1 + a_2)^4 + a_1^2a_2^2b_1^2b_2^2(c_1 + c_2)^4 - a_1a_2b_1b_2c_1c_2$$
$$(4b_1b_2c_1c_2(a_1 + a_2)^2 + 4a_1a_2c_1c_2(b_1 + b_2)^2 + 4a_1a_2b_1b_2(c_1 + c_2)^2 - (a_1 + a_2)^2(b_1 + b_2)^2$$
$$(c_1 + c_2)^2) + 6a_1^2a_2^2b_1^2b_2^2c_1^2c_2^2)z^4 - a_1a_2b_1b_2c_1c_2(a_1 + a_2)(b_1 + b_2)(c_1 + c_2)(b_1b_2c_1c_2(a_1 + a_2)^2$$
$$+ a_1a_2c_1c_2(b_1 + b_2)^2 + a_1a_2b_1b_2(c_1 + c_2)^2 - 5a_1a_2b_1b_2c_1c_2)z^5 + (a_1^2a_2^2b_1^3b_2^3c_1^2c_2^2(a_1 + a_2)^2(c_1 + c_2)^2$$
$$+ a_1^2a_2^2b_1^2b_2^2c_1^2c_2^2((b_1 + b_2)^2 - 2b_1b_2)((a_1 + a_2)^2c_1c_2 - 2a_1a_2c_1c_2 + a_1a_2(c_1 + c_2)^2))z^6$$
$$- a_1^3a_2^3b_1^3b_2^3c_1^3c_2^3(a_1 + a_2)(b_1 + b_2)(c_1 + c_2)z^7 + a_1^4a_2^4b_1^4b_2^4c_1^4c_2^4z^8.$$

Lemma 2. *[5] Let A, B and C be three alphabets, respectively, $\{a_1, a_2\}$, $\{b_1, b_2\}$ and $\{c_1, c_2\}$. Then we have*

$$\sum_{n=0}^{\infty} S_n(A)S_{n+1}(B)S_{n+1}(C)z^n = \frac{N_2}{D}, \quad n \in \mathbb{N}_0, \tag{3.3}$$

with

$$N_2 = (c_1 + c_2)(b_1 + b_2) - (a_1 + a_2)(c_1c_2(b_1 + b_2)^2 + b_1b_2(c_1 + c_2)^2 - c_1c_2b_1b_2)z + c_1c_2b_1b_2(a_1 + a_1)^2$$
$$(b_1 + b_1)(c_1 + c_1)z^2 - c_1^2c_2^2b_1^2b_2^2(a_1 + a_1)((a_1 + a_1)^2 - 2a_1a_2)z^3 - a_1^2a_2^2b_1^2b_2^2c_1^2c_2^2(b_1 + b_2)$$
$$(c_1 + c_2)z^4 + a_1^2a_2^2b_1^3b_2^3c_1^3c_2^3(a_1 + a_2)z^5.$$

From the previous lemma, we deduce the following relationship

$$\sum_{n=0}^{\infty} S_{n-1}(A)S_n(B)S_n(C)z^n = \frac{N_3}{D}, \quad n \in \mathbb{N}_0, \tag{3.4}$$

with

$$N_3 = (c_1 + c_2)(b_1 + b_2)z - (a_1 + a_2)(c_1c_2(b_1 + b_2)^2 + b_1b_2(c_1 + c_2)^2 - c_1c_2b_1b_2)z^2 + c_1c_2b_1b_2(a_1$$
$$+ a_1)^2(b_1 + b_1)(c_1 + c_1)z^3 - c_1^2c_2^2b_1^2b_2^2(a_1 + a_1)((a_1 + a_1)^2 - 2a_1a_2)z^4 - a_1^2a_2^2b_1^2b_2^2c_1^2c_2^2$$
$$(b_1 + b_2)(c_1 + c_2)z^5 + a_1^2a_2^2b_1^3b_2^3c_1^3c_2^3(a_1 + a_2)z^6.$$

Lemma 3. *[5] Let A, B and C be three alphabets, respectively, $\{a_1, a_2\}$, $\{b_1, b_2\}$ and $\{c_1, c_2\}$. Then we have*

$$\sum_{n=0}^{\infty} S_n(A)S_n(B)S_n(C)z^n = \frac{N_4}{D}, \quad n \in \mathbb{N}_0, \tag{3.5}$$

with

$$N_4 = 1-(a_1a_2c_1c_2(b_1+b_2)^2+a_1a_2b_1b_2(c_1+c_2)^2+b_1b_2c_1c_2(a_1+a_2)^2-3a_1a_2c_1c_2b_1b_2)z^2 + 2a_1a_2b_1b_2c_1c_2(a_1+a_2)(b_1+b_2)(c_1+c_2)z^3-(b_1b_2a_1^2a_2^2c_1^2c_2^2(b_1+b_2)^2+c_1c_2a_1^2a_2^2b_1^2b_2^2 (c_1+c_2)^2+a_1a_2b_1^2b_2^2c_1^2c_2^2(a_1+a_2)^2-3a_1^2a_2^2b_1^2b_2^2c_1^2c_2^2)z^4+a_1^3a_2^3b_1^3b_2^3c_1^3c_2^3z^6.$$

The following relationship is deduced from the previous lemma

$$\sum_{n=0}^{\infty} S_{n-1}(A)S_{n-1}(B)S_{n-1}(C)z^n = \frac{N_5}{D}, \quad n \in \mathbb{N}_0, \tag{3.6}$$

with

$$N_5 = z-(a_1a_2c_1c_2(b_1+b_2)^2+a_1a_2b_1b_2(c_1+c_2)^2+b_1b_2c_1c_2(a_1+a_2)^2-3a_1a_2c_1c_2b_1b_2)z^3+ 2a_1a_2b_1b_2c_1c_2(a_1+a_2)(b_1+b_2)(c_1+c_2)z^4-(b_1b_2a_1^2a_2^2c_1^2c_2^2(b_1+b_2)^2+c_1c_2a_1^2a_2^2b_1^2b_2^2 (c_1+c_2)^2+a_1a_2b_1^2b_2^2c_1^2c_2^2(a_1+a_2)^2-3a_1^2a_2^2b_1^2b_2^2c_1^2c_2^2)z^5+a_1^3a_2^3b_1^3b_2^3c_1^3c_2^3z^7.$$

4. The novel generating functions of several products

This case consists of two related parts

Firstly

For getting a new generating function, involving the product of squares of k-Fibonacci numbers with bivariate Mersenne and bivariate Mersenne Lucas polynomials, we replace a_2 by $[-a_2]$, b_2 by $[-b_2]$ and c_2 by $[-c_2]$ and we put $a_1-a_2=3y$, $b_1-b_2=k$, $c_1-c_2=k$, $a_1a_2=-2x$, $b_1b_2=1$ and $c_1c_2=1$ in (3.4) and (3.5) we obtain

$$\sum_{n=0}^{\infty} S_{n-1}(a_1+[-a_2])S_n(b_1+[-b_2])S_n(c_1+[-c_2]) = \frac{N_6}{D_1}, \tag{4.1}$$

and

$$\sum_{n=0}^{\infty} S_n(a_1+[-a_2])S_n(b_1+[-b_2])S_n(c_1+[-c_2]) = \frac{N_7}{D_1}, \tag{4.2}$$

with

$N_6 = 12x^2yz^6 - 4k^2x^2z^5 - 3y(-4x+9y^2)z^4 + 9k^2y^2z^3 - 3y(-2k^2-1)z^2 + k^2z.$

$D_1 = 16x^4z^8 - 24k^2x^3yz^7 + (-36k^2x^2y^2 + 4x^2(k^2+2)(2k^2x+4x-9y^2))z^6 - 6k^2xy(-4k^2x-10x+9y^2)z^5 + (8k^4x^2+24x^2-2x(-9k^4y^2-16k^2x+36y^2)+81y^4)z^4 - 3k^2y(-4k^2x-10x+9y^2)z^3 + (-9k^2y^2+(k^2+2)(2k^2x+4x-9y^2))z^2 - 3k^2yz + 1.$

$N_7 = 8x^3z^6 - (-8k^2x^2-12x^2+18xy^2)z^4 + 12k^2xyz^3 - (-4k^2x-6x+9y^2)z^2 + 1.$

From the previous results, we conclude the following theorems.

Theorem 5. *For $n \in \mathbb{N}$, the new generating function for the product of squares of k-Fibonacci numbers with bivariate Mersenne polynomials is given by*

$$\sum_{n=0}^{\infty} F_{k,n}^2 M_n(x,y) z^n = \frac{12x^2yz^6 - 4k^2x^2z^5 - 3y(-4x+9y^2)z^4 + 9k^2y^2z^3 - 3y(-2k^2-1)z^2 + k^2z}{D_1}.$$

Corollary 1. *For $n \in \mathbb{N}$, the new generating function for the product of squares of Fibonacci numbers with bivariate Mersenne polynomials is given by*

$$\sum_{n=0}^{\infty} F_n^2 M_n(x,y) z^n = \frac{12x^2yz^6 - 4x^2z^5 - 3y(-4x+9y^2)z^4 + 9y^2z^3 + 9yz^2 + z}{D_2},$$

with
$D_2 = 16x^4z^8 - 24x^3yz^7 + \left(-36x^2y^2 + 12x^2\left(6x - 9y^2\right)\right)z^6 - 6xy\left(-14x + 9y^2\right)z^5 + \left(32x^2 - 2x\left(-16x + 27y^2\right) + 81y^4\right)z^4 - 3y\left(-14x + 9y^2\right)z^3 + \left(18x - 36y^2\right)z^2 - 3yz + 1.$

Theorem 6. *For $n \in \mathbb{N}$, the new generating function of the product of squares of k-Fibonacci numbers and bivariate Mersenne Lucas polynomials is given by*

$$\sum_{n=0}^{\infty} F_{k,n}^2 m_n(x,y) z^n = \frac{N_{F_{k,n}^2 m_n(x,y)}}{D_1},$$

with

$$\begin{aligned} N_{F_{k,n}^2 m_n(x,y)} &= \left(16x^3 - 36x^2y^2\right)z^6 + 12k^2x^2yz^5 + \left(16k^2x^2 + 24x^2 - 72xy^2 + 81y^4\right)z^4 \\ &+ \left(24k^2xy - 27k^2y^3\right)z^3 + \left(8k^2x - 18k^2y^2 + 12x - 27y^2\right)z^2 - 3k^2yz + 2. \end{aligned}$$

Proof. We have $m_n(x,y) = 2S_n(a_1 + [-a_2]) - 3yS_{n-1}(a_1 + [-a_2])$, (see [22]).
then

$$\begin{aligned} \sum_{n=0}^{\infty} F_{k,n}^2 m_n(x,y) z^n &= \sum_{n=0}^{\infty} ((2S_n(a_1 + [-a_2]) - 3yS_{n-1}(a_1 + [-a_2]))(S_n(b_1 + [-b_2])S_n(c_1 + [-c_2])))z^n \\ &= 2\sum_{n=0}^{\infty} S_n(a_1 + [-a_2])S_n(b_1 + [-b_2])S_n(c_1 + [-c_2])z^n \\ &- 3y\sum_{n=0}^{\infty} S_{n-1}(a_1 + [-a_2])S_n(b_1 + [-b_2])S_n(c_1 + [-c_2])z^n \\ &= 2\frac{1 + ((4k^2-6)x - 9y^2)z^2 - 12k^2xyz^3 + ((-8k^2+12)x^2 + 18xy^2)z^4 - 8x^3z^6}{D_1} \\ &- 3y\frac{k^2z + (-6k^2+3)yz^2 + 9k^2y^2z^3 + (-12xy - 27y^3)z^4 - 4k^2x^2z^5 + 12x^2yz^6}{D_1}, \end{aligned}$$

so

$$\sum_{n=0}^{\infty} F_{k,n}^2 m_n(x,y) z^n = \frac{N_{F_{k,n}^2 m_n(x,y)}}{D_1},$$

then

$$
\begin{aligned}
N_{F^2_{k,n}m_n(x,y)} &= \left(16x^3-36x^2y^2\right)z^6+12k^2x^2yz^5+\left(16k^2x^2+24x^2-72xy^2+81y^4\right)z^4\\
&+ \left(24k^2xy-27k^2y^3\right)z^3+\left(8k^2x-18k^2y^2+12x-27y^2\right)z^2-3k^2yz+2.
\end{aligned}
$$

As required. □

Corollary 2. *For $n \in \mathbb{N}$, the new generating function of the product of squares of Fibonacci numbers and bivariate Mersenne Lucas polynomials is given by*

$$
\begin{aligned}
\sum_{n=0}^{\infty} F_n^2 m_n(x,y)z^n &= \frac{\left(16x^3-36x^2y^2\right)z^6+12x^2yz^5+\left(40x^2-72xy^2+81y^4\right)z^4+\left(24xy-27y^3\right)z^3}{D_2}\\
&+ \frac{\left(20x-45y^2\right)z^2-3yz+2}{D_2}.
\end{aligned}
$$

In this step we use the main theorem and appropriate lemmas to derive new generating functions of squares of k-Fibonacci numbers with bivariate Mersenne and bivariate Mersenne Lucas polynomials.

Secondly

Obtaining a new generating functions of the products of squares of k-Jacobsthal numbers with bivariate Mersenne and bivariate Mersenne Lucas polynomials requires the following replacements a_2 by $[-a_2]$, b_2 by $[-b_2]$ and c_2 by $[-c_2]$ we make the following substitutions $a_1-a_2=3y$, $b_1-b_2=k$, $c_1-c_2=k$, $a_1a_2=-2x$, $b_1b_2=2$ and $c_1c_2=2$ in (3.6) and (3.2), we obtain the following

$$
\sum_{n=0}^{\infty} S_{n-1}(a_1+[-a_2])S_{n-1}(b_1+[-b_2])S_{n-1}(c_1+[-c_2]) = \frac{N_8}{D_3}, \tag{4.3}
$$

and

$$
\sum_{n=0}^{\infty} S_{n}(a_1+[-a_2])S_{n-1}(b_1+[-b_2])S_{n-1}(c_1+[-c_2]) = \frac{N_9}{D_3}, \tag{4.4}
$$

where

$N_8 = 512x^3z^7 - \left(-64k^2x^2-192x^2+288xy^2\right)z^5 + 48k^2xyz^4 - \left(-8k^2x-24x+36y^2\right)z^3+z.$

$N_9 = 128k^2x^3z^6-48x^2y\left(-4k^2-4\right)z^5+72k^2xy^2z^4+12y\left(4x-9y^2\right)z^3-2k^2xz^2+3yz.$

$D_3 = 4096x^4z^8-1536k^2x^3yz^7+\left(-1152k^2x^2y^2+64x^2\left(k^2+4\right)\left(2k^2x+8x-18y^2\right)\right)z^6-24k^2xy\left(-8k^2x-40x+36y^2\right)z^5+\left(32k^4x^2+384x^2-8x\left(-9k^4y^2-32k^2x+144y^2\right)+1296y^4\right)z^4-3k^2y\left(-8k^2x-40x+36y^2\right)z^3 + \left(-18k^2y^2+\left(k^2+4\right)\left(2k^2x+8x-18y^2\right)\right)z^2 - 3k^2yz+1.$

From our previous results, we obtain the following theorems.

Theorem 7. *For $n \in \mathbb{N}$, the new generating function of the product of squares of k-Jacobsthal numbers with bivariate Mersenne polynomials is given by*

$$\sum_{n=0}^{\infty} J_{k,n}^2 M_n(x,y) z^n = \frac{N_{J_{k,n}^2 M_n(x,y)}}{D_3},$$

and we have

$$N_{J_{k,n}^2 M_n(x,y)} = 512x^3z^7 - \left(-64k^2x^2 - 192x^2 + 288xy^2\right) z^5 + 48k^2xyz^4 - \left(-8k^2x - 24x + 36y^2\right) z^3 + z.$$

Corollary 3. *For $n \in \mathbb{N}$, the new generating function of the product of squares of Jacobsthal numbers with bivariate Mersenne polynomials is given by*

$$\sum_{n=0}^{\infty} J_n^2 M_n(x,y) z^n = \frac{512x^3z^7 + \left(256x^2 - 288xy^2\right) z^5 + 48xyz^4 + \left(32x - 36y^2\right) z^3 + z}{D_4},$$

and we have
$D_4 = 4096x^4z^8 - 1536x^3yz^7 + \left(3200x^3 - 6912x^2y^2\right) z^6 + \left(1152x^2y - 864xy^3\right) z^5 + \left(672x^2 - 1080xy^2 + 1296y^4\right) z^4 + \left(144xy - 108y^3\right) z^3 + \left(50x - 108y^2\right) z^2 - 3yz + 1.$

Theorem 8. *For $n \in \mathbb{N}$, the new generating function of the product of squares of the product of k-Jacobsthal numbers and bivariate Mersenne Lucas polynomials is given by*

$$\sum_{n=0}^{\infty} J_{k,n}^2 m_n(x,y) z^n = \frac{N_{J_{k,n}^2 m_n(x,y)}}{D_3},$$

where

$$\begin{aligned} N_{J_{k,n}^2 m_n(x,y)} &= -1536x^3yz^7 + 256k^2x^3z^6 + \left(192k^2x^2y - 192x^2y + 864xy^3\right) z^5 \\ &+ \left(-24k^2xy + 24xy - 108y^3\right) z^3 - 4k^2xz^2 + 3yz. \end{aligned}$$

Proof. With the knowledge that $m_n(x,y) = 2S_n(\lambda_1 + [-\lambda_2]) - 3yS_{n-1}(\lambda_1 + [-\lambda_2])$ (see [22]),
then, we obtain the following

$$\begin{aligned} \sum_{n=0}^{\infty} J_{k,n}^2 m_n(x,y) z^n &= \sum_{n=0}^{\infty} (2S_n(a_1 + [-a_2]) - 3yS_{n-1}(a_1 + [-a_2]))(S_{n-1}(b_1 + [-b_2])S_{n-1}(c_1 + [-c_2]))z^n \\ &= 2\sum_{n=0}^{\infty} S_n(a_1 + [-a_2])S_{n-1}(b_1 + [-b_2])S_{n-1}(c_1 + [-c_2])z^n \\ &- 3y\sum_{n=0}^{\infty} S_{n-1}(a_1 + [-a_2])S_{n-1}(b_1 + [-b_2])S_{n-1}(c_1 + [-c_2])z^n \end{aligned}$$

$$
\begin{aligned}
&= 2\left(\frac{128k^2x^3z^6 - 48x^2y\left(-4k^2-4\right)z^5 + 72k^2xy^2z^4 + 12y\left(4x-9y^2\right)z^3 - 2k^2xz^2 + 3yz}{D_3}\right) \\
&- 3y\left(\frac{512x^3z^7 - \left(-64k^2x^2 - 192x^2 + 288xy^2\right)z^5 + 48k^2xyz^4}{D_3}\right) \\
&+ 3y\left(\frac{\left(-8k^2x - 24x + 36y^2\right)z^3 + z}{D_3}\right),
\end{aligned}
$$

so

$$\sum_{n=0}^{\infty} J_{k,n}^2 m_n(x,y) z^n = \frac{N_{J_{k,n}^2 m_n(x,y)}}{D_3},$$

then

$$
\begin{aligned}
N_{J_{k,n}^2 m_n(x,y)} &= -1536x^3yz^7 + 256k^2x^3z^6 + \left(192k^2x^2y - 192x^2y + 864xy^3\right)z^5 \\
&+ \left(-24k^2xy + 24xy - 108y^3\right)z^3 - 4k^2xz^2 + 3yz.
\end{aligned}
$$

This completes the proof. □

Corollary 4. *For $n \in \mathbb{N}$, the new generating function of the product of squares of Jacobsthal numbers with bivariate Mersenne Lucas polynomials is given by*

$$\sum_{n=0}^{\infty} J_n^2 m_n(x,y) z^n = \frac{-1536x^3yz^7 + 256x^3z^6 + 864xy^3z^5 - 108y^3z^3 - 4xz^2 + 3yz}{D_4},$$

As a consequence of this step, we derive new generating functions of squares of k-Jacobsthal numbers with bivariate Mersenne and bivariate Mersenne Lucas polynomials.

5. On the generating functions of some bivariate polynomials

Now, we will do the following substitutions with the aim of the achievement of new generating functions of the product of bivariate complex Fibonacci polynomials with bivariate Mersenne and bivariate Mersenne Lucas polynomials, then we replace a_2 by $[-a_2]$, b_2 by $[-b_2]$ and c_2 by $[-c_2]$ and we put $a_1 - a_2 = is$, $a_1a_2 = t$, $b_1 - b_2 = ir$, $b_1b_2 = h$, $c_1 - c_2 = 3y$, $c_1c_2 = -2x$, in (3.2) and (3.6), we obtain the following

$$\sum_{n=0}^{\infty} S_{n-1}(a_1 + [-a_2])S_{n-1}(b_1 + [-b_2])S_{n-1}(c_1 + [-c_2]) = \frac{N_{10}}{D_5}, \tag{5.1}$$

and

$$\sum_{n=0}^{\infty} S_n(a_1 + [-a_2])S_{n-1}(b_1 + [-b_2])S_{n-1}(c_1 + [-c_2]) = \frac{N_{11}}{D_5}, \tag{5.2}$$

where

$N_{10} = 8h^3t^3x^3z^7 + \left(-4h^2s^2tx^2 + 12h^2t^2x^2 - 18h^2t^2xy^2 - 4hr^2t^2x^2\right)z^5 - 12hrstxyz^4 + \left(-2hs^2x + 6htx - 9hty^2 - 2r^2tx\right)z^3 + z.$

$N_{11} = -12h^2irt^3x^2yz^6 + \left(4h^2ist^2x^2 - 18h^2ist^2xy^2 - 4hir^2st^2x^2\right)z^5 - 6hirs^2txyz^4 + \left(-2his^3x + 4histx\right)z^3 + 3irtyz^2 + isz.$

$$\begin{aligned} D_5 &= 16h^4t^4x^4z^8 + 24h^3rst^3x^3yz^7 \\ &+ (-16h^3s^2t^2x^3 + 36h^3s^2t^2x^2y^2 + 32h^3t^3x^3 - 72h^3t^3x^2y^2 + 8h^2r^2s^2t^2x^3 - 16h^2r^2t^3x^3 \\ &+ 36h^2r^2t^3x^2y^2)z^6 + \left(12h^2rs^3tx^2y - 60h^2rst^2x^2y + 54h^2rst^2xy^3 + 12hr^3st^2x^2y\right)z^5 \\ &+ \left(4h^2s^4x^2 - 16h^2s^2tx^2 + 24h^2t^2x^2 - 72h^2t^2xy^2 + 81h^2t^2y^4 + 18hr^2s^2txy^2 - 16hr^2t^2x^2 + 4r^4t^2x^2\right)z^4 \\ &+ \left(6hrs^3xy - 30hrstxy + 27hrsty^3 + 6r^3stxy\right)z^3 \\ &+ \left(-4hs^2x + 9hs^2y^2 + 8htx - 18hty^2 + 2r^2s^2x - 4r^2tx + 9r^2ty^2\right)z^2 + 3rsyz + 1. \end{aligned}$$

From these results, we conclude the following theorems

Theorem 9. *For $n \in \mathbb{N}$, the new generating function of the product of bivariate Mersenne polynomials with bivariate complex Fibonacci polynomials is given by*

$$\sum_{n=0}^{\infty} F(s,t)F(r,h)M_n(x,y)z^n = \frac{N_{F(s,t)F(r,h)M_n(x,y)}}{D_5},$$

and we have

$N_{F(s,t)F(r,h)M_n(x,y)} = 8h^3t^3x^3z^7 + \left(-4h^2s^2tx^2 + 12h^2t^2x^2 - 18h^2t^2xy^2 - 4hr^2t^2x^2\right)z^5 - 12hrstxyz^4 + \left(-2hs^2x + 6htx - 9hty^2 - 2r^2tx\right)z^3 + z.$

Theorem 10. *For $n \in \mathbb{N}$, the new generating function of the product of bivariate Mersenne Lucas polynomials with bivariate complex Fibonacci polynomials is given by*

$$\sum_{n=0}^{\infty} F(s,t)F(r,h)m_n(x,y)z^n = \frac{N_{F(s,t)F(r,h)m_n(x,y)}}{D_5},$$

where

$$\begin{aligned} N_{F(s,t)F(r,h)m_n(x,y)} &= -24h^3t^3x^3yz^7 - 24h^2irt^3x^2yz^6 + (12h^2s^2tx^2y + 8h^2ist^2x^2 - 36h^2ist^2xy^2 \\ &- 36h^2t^2x^2y + 54h^2t^2xy^3 - 8hir^2st^2x^2 + 12hr^2t^2x^2y)z^5 + (-12hirs^2txy \\ &+ 36hrstxy^2)z^4 + \left(-4his^3x + 6hs^2xy + 8histx - 18htxy + 27hty^3 + 6r^2txy\right)z^3 \\ &+ 6irtyz^2 + (2is - 3y)\,z. \end{aligned}$$

Proof. As we mentioned previously $m_n(x,y) = 2S_n\left(a_1 + [-a_2]\right) - 3yS_{n-1}\left(a_1 + [-a_2]\right)$, (see [22]).

$$\begin{aligned}\sum_{n=0}^{\infty}F(s,t)F(r,h)m_n(x,y)z^n &= \sum_{n=0}^{\infty}(2S_n(a_1+[-a_2])-3yS_{n-1}(a_1+[-a_2]))(S_{n-1}(b_1+[-b_2])S_{n-1}(c_1+[-c_2]))z^n\\ &= 2\sum_{n=0}^{\infty}S_n(a_1+[-a_2])S_{n-1}(b_1+[-b_2])S_{n-1}(c_1+[-c_2])z^n\\ &- 3y\sum_{n=0}^{\infty}S_{n-1}(a_1+[-a_2])S_{n-1}(b_1+[-b_2])S_{n-1}(c_1+[-c_2]))z^n\\ &= 2(\frac{-12h^2irt^3x^2yz^6+\left(4h^2ist^2x^2-18h^2ist^2xy^2-4hir^2st^2x^2\right)z^5-6hirs^2txyz^4}{D_5}\\ &+ \frac{\left(-2his^3x+4histx\right)z^3+3irtyz^2+isz}{D_5})\\ &- 3y(\frac{8h^3t^3x^3z^7+\left(-4h^2s^2tx^2+12h^2t^2x^2-18h^2t^2xy^2-4hr^2t^2x^2\right)z^5-12hrstxyz^4}{D_5}\\ &+ \frac{\left(-2hs^2x+6htx-9hty^2-2r^2tx\right)z^3+z}{D_5}),\end{aligned}$$

so

$$\sum_{n=0}^{\infty}F(s,t)F(r,h)m_n(x,y)z^n=\frac{N_{7_{F(s,t)F(r,h)m_n(x,y)}}}{D_5},$$

then

$$\begin{aligned}N_{F(s,t)F(r,h)m_n(x,y)} &= -24h^3t^3x^3yz^7-24h^2irt^3x^2yz^6+(12h^2s^2tx^2y+8h^2ist^2x^2-36h^2ist^2xy^2\\ &- 36h^2t^2x^2y+54h^2t^2xy^3-8hir^2st^2x^2+12hr^2t^2x^2y)z^5+(-12hirs^2txy\\ &+ 36hrstxy^2)z^4+\left(-4his^3x+6hs^2xy+8histx-18htxy+27hty^3+6r^2txy\right)z^3\\ &+ 6irtyz^2+(2is-3y)\,z.\end{aligned}$$

This completes the proof. □

By this theorems we derive new generating functions of squares of bivariate complex Fibonacci polynomials with bivariate Mersenne and bivariate Mersenne Lucas polynomials.

6. Ordinary generating functions of the products of bivariate Mersenne and bivariate Mersenne Lucas polynomials with Chebyshev polynomials

Finally by making the following substitutions, we will derive new generating functions of the product of Chebyshev polynomials of the first and second kind with bivariate Mersenne polynomials and with bivariate Mersenne Lucas polynomials, then we replace a_1 by $[2a_1]$, a_2 by $[-2a_2]$, b_1 by $[2b_1]$, b_2 by $[-2b_2]$ and c_2 by $[-c_2]$.

For the first kindreplacing $a_1-a_2=t$, $b_1-b_2=s$, $a_1a_2=-\frac{1}{4}$, $b_1b_2=-\frac{1}{4}$, $c_1-c_2=3y$ and $c_1c_2=-2x$ in (3.2), (3.4), (3.5) and (3.6), we obtain

$$\sum_{n=0}^{\infty} S_{n-1}(2a_1+[-2a_2])S_n(2b_1+[-2b_2])S_n(c_1+[-c_2]) = \frac{T_1}{D_6}, \tag{6.1}$$

$$\sum_{n=0}^{\infty} S_n(2a_1+[-2a_2])S_n(2b_1+[-2b_2])S_n(c_1+[-c_2]) = \frac{T_2}{D_6}, \tag{6.2}$$

$$\sum_{n=0}^{\infty} S_n(2a_1+[-2a_2])S_{n-1}(2b_1+[-2b_2])S_{n-1}(c_1+[-c_2]) = \frac{T_3}{D_6}, \tag{6.3}$$

and

$$\sum_{n=0}^{\infty} S_{n-1}(2a_1+[-2a_2])S_{n-1}(2b_1+[-2b_2])S_{n-1}(c_1+[-c_2]) = \frac{T_4}{D_6}, \tag{6.4}$$

where

$T_1 = 16tx^3z^6 - 24sx^2yz^5 + \left(-32t^3x^2+16tx^2\right)z^4 + 48st^2xyz^3 + \left(-16s^2tx+4tx-18ty^2\right)z^2+6syz.$
$T_2 = 8x^3z^6 + \left(-16s^2x^2-16t^2x^2+12x^2-18xy^2\right)z^4 + 48stxyz^3 + \left(-8s^2x-8t^2x+6x-9y^2\right)z^2+1.$
$T_3 = 24sx^2yz^6 + \left(-32s^2tx^2+8tx^2-36txy^2\right)z^5 + 48st^2xyz^4 + \left(-16t^3x+8tx\right)z^3 - 6syz^2+2tz.$
$T_4 = 8x^3z^7 + \left(-16s^2x^2-16t^2x^2+12x^2-18xy^2\right)z^5 + 48stxyz^4 + \left(-8s^2x-8t^2x+6x-9y^2\right)z^3+z.$

$$\begin{aligned} D_6 &= 16x^4z^8 - 96stx^3yz^7 + \left(128s^2t^2x^3 - 64s^2x^3 + 144s^2x^2y^2 - 64t^2x^3 + 144t^2x^2y^2 + 32x^3 - 72x^2y^2\right)z^6 \\ &+ \left(-192s^3tx^2y - 192st^3x^2y + 240stx^2y - 216stxy^3\right)z^5 + (64s^4x^2 + 288s^2t^2xy^2 \\ &- 64s^2x^2 + 64t^4x^2 - 64t^2x^2 + 24x^2 - 72xy^2 + 81y^4)z^4 + \left(-96s^3txy - 96st^3xy + 120stxy - 108sty^3\right)z^3 \\ &+ \left(32s^2t^2x - 16s^2x + 36s^2y^2 - 16t^2x + 36t^2y^2 + 8x - 18y^2\right)z^2 - 12styz + 1. \end{aligned}$$

From our previous results, we obtain the following theorems.

Theorem 11. *For $n \in \mathbb{N}$, the new generating function of the product of Chebyshev polynomials of the first kind with bivariate Mersenne polynomials is given by*

$$\sum_{n=0}^{\infty} T_n(t)T_n(s)M_n(x,y)z^n = \frac{N_{T_n(t)T_n(s)M_n(x,y)}}{D_6},$$

and we have

$$\begin{aligned} N_{T_n(t)T_n(s)M_n(x,y)} &= (8stx^3z^7 + \left(-48stx^2y + 12x^2y\right)z^6 + (-16s^3tx^2 + 64s^2t^2x^2 - 8s^2x^2 \\ &+ 36s^2xy^2 - 16st^3x^2 - 4stx^2 - 18stxy^2 - 8t^2x^2 + 36t^2xy^2)z^5 \\ &+ (-48s^3txy + 48s^2t^2xy - 48st^3xy + 12xy - 27y^3)z^4 \\ &+ (16s^4x - 8s^3tx - 8s^2x - 8st^3x + 6stx + 27sty^2 + 16t^4x - 8t^2x)z^3 \\ &+ (-12s^2y + 12sty - 12t^2y + 3y)z^2 + \left(-2s^2 + 5st - 2t^2\right)z. \end{aligned}$$

Proof. We have the expressions

$$M_n(x,y) = S_{n-1}(c_1 + [-c_2]) \text{ (see [22])},$$
$$T_n(t) = S_n(2a_1 + [-2a_2]) - tS_{n-1}(2a_1 + [-2a_2]), \text{ (see [6])},$$

then $\sum_{n=0}^{\infty} T_n(s)T_n(t)M_n(x,y)z^n$ is equal to

$$\begin{aligned}
&= \sum_{n=0}^{\infty}(S_n(2a_1 + [-2a_2]) - tS_{n-1}(2a_1 + [-2a_2]))(S_n(2b_1 + [-2b_2]) \\
&- tS_{n-1}(2b_1 + [-2b_2]))S_{n-1}(c_1 + [-c_2])z^n \\
&= \sum_{n=0}^{\infty}([S_n(2a_1 + [-2a_2])S_n(2b_1 + [-2b_2])]S_{n-1}(c_1 + [-c_2]) \\
&- tS_n(2a_1 + [-2a_2])S_{n-1}(2b_1 + [-2b_2])S_{n-1}(c_1 + [-c_2]) \\
&- sS_{n-1}(2a_1 + [-2a_2])S_n(2b_1 + [-2b_2])S_{n-1}(c_1 + [-c_2]) \\
&+ stS_{n-1}(2a_1 + [-2a_2])S_{n-1}(2b_1 + [-2b_2])S_{n-1}(c_1 + [-c_2]))z^n \\
&= \frac{16tx^3z^6 - 24sx^2yz^5 + \left(-32t^3x^2 + 16tx^2\right)z^4 + 48st^2xyz^3 + \left(-16s^2tx + 4tx - 18ty^2\right)z^2 + 6syz}{D_6} \\
&- s(\frac{24sx^2yz^6 + \left(-32s^2tx^2 + 8tx^2 - 36txy^2\right)z^5 + 48st^2xyz^4 + \left(-16t^3x + 8tx\right)z^3 - 6syz^2 + 2tz}{D_6}) \\
&- t(\frac{24tx^2yz^6 + \left(-32t^2sx^2 + 8sx^2 - 36sxy^2\right)z^5 + 48ts^2xyz^4 + \left(-16s^3x + 8sx\right)z^3 - 6tyz^2 + 2sz}{D_6}) \\
&+ st(\frac{8x^3z^7 + \left(-16s^2x^2 - 16t^2x^2 + 12x^2 - 18xy^2\right)z^5 + 48stxyz^4 + \left(-8s^2x - 8t^2x + 6x - 9y^2\right)z^3 + z}{D_6}) \\
&= \frac{N_{T_n(t)T_n(s)M_n(x,y)}}{D_6},
\end{aligned}$$

where

$$\begin{aligned}
N_{T_n(t)T_n(s)M_n(x,y)} &= (8stx^3z^7 + \left(-48stx^2y + 12x^2y\right)z^6 + (-16s^3tx^2 + 64s^2t^2x^2 \\
&- 8s^2x^2 + 36s^2xy^2 - 16st^3x^2 - 4stx^2 - 18stxy^2 - 8t^2x^2 + 36t^2xy^2)z^5 \\
&+ (-48s^3txy + 48s^2t^2xy - 48st^3xy + 12xy - 27y^3)z^4 + (16s^4x \\
&- 8s^3tx - 8s^2x - 8st^3x + 6stx + 27sty^2 + 16t^4x - 8t^2x)z^3 \\
&+ (-12s^2y + 12sty - 12t^2y + 3y)z^2 + \left(-2s^2 + 5st - 2t^2\right)z.
\end{aligned}$$

□

Theorem 12. *For $n \in \mathbb{N}$, the new generating function for the product of Chebyshev polynomials of the first kind with bivariate Mersenne Lucas polynomials is given by*

$$\sum_{n=0}^{\infty} T_n(t)T_n(s)m_n(x,y)z^n = \frac{N_{T_n(t)T_n(s)m_n(x,y)}}{D_6},$$

where

$$
\begin{aligned}
N_{T_n(t)T_n(s)m_n(x,y)} &= (-24stx^3yz^7 + (64s^2t^2x^3 - 32s^2x^3 + 72s^2x^2y^2 - 32t^2x^3 + 72t^2x^2y^2 \\
&+ 16x^3 - 36x^2y^2)z^6 + \left(-144s^3tx^2y - 144st^3x^2y + 180stx^2y - 162stxy^3\right) z^5 \\
&+ \left(64s^4x^2 + 288s^2t^2xy^2 - 64s^2x^2 + 64t^4x^2 - 64t^2x^2 + 24x^2 - 72xy^2 + 81y^4\right) z^4 \\
&+ \left(-120s^3txy - 120st^3xy + 150stxy - 135sty^3\right) z^3 + (48s^2t^2x - 24s^2x \\
&+ 54s^2y^2 - 24t^2x + 54t^2y^2 + 12x - 27y^2)z^2 - 21styz + 2).
\end{aligned}
$$

Proof. We have the expressions

$$
\begin{aligned}
m_n(x,y) &= 2S_n(c_1 + [-c_2]) - 3yS_{n-1}(c_1 + [-c_2]), \text{ (see [22])}, \\
T_n(t) &= S_n(2a_1 + [-2a_2]) - tS_{n-1}(2a_1 + [-2a_2]) \text{ , (see [6])},
\end{aligned}
$$

then

$$
\begin{aligned}
\sum_{n=0}^{\infty} T_n(s)T_n(t)m_n(x,y)z^n &= \sum_{n=0}^{\infty}(S_n(2a_1 + [-2a_2]) - tS_{n-1}(2a_1 + [-2a_2])(S_n(2b_1 + [-2b_2]) \\
&- tS_{n-1}(2b_1 + [-2b_2])).(2S_n(c_1 + [-c_2]) - 3yS_{n-1}(c_1 + [-c_2]))z^n \\
&= \sum_{n=0}^{\infty} S_n(2a_1 + [-2a_2])S_n(2b_1 + [-2b_2]) - tS_n(2a_1 + [-2a_2])S_{n-1}(2b_1 + [-2b_2]) \\
&- tS_{n-1}(2a_1 + [-2a_2])S_n(b_1 + [-b_2]) + t^2S_{n-1}(2a_1 + [-2a_2])S_{n-1}(2b_1 + [-2b_2]) \\
&. \ [2S_n(c_1 + [-c_2]) - 3yS_{n-1}(c_1 + [-c_2])]z^n \\
&= \sum_{n=0}^{\infty} 2(S_n(2a_1 + [-2a_2])S_n(2b_1 + [-2b_2]))S_n(c_1 + [-c_2]) \\
&- 3y(S_n(2a_1 + [-2a_2])S_n(2b_1 + [-2b_2]))S_{n-1}(c_1 + [-c_2]) \\
&- 2s(S_n(2a_1 + [-2a_2])S_{n-1}(2b_1 + [-2b_2])S_n(c_1 + [-c_2]) \\
&+ 3ys(S_n(2a_1 + [-2a_2])S_{n-1}(2b_1 + [-2b_2]))S_{n-1}(c_1 + [-c_2])) \\
&- 2tS_{n-1}(2a_1 + [-2a_2])S_n(2b_1 + [-2b_2])S_n(c_1 + [-c_2]) \\
&+ 3yt(S_{n-1}(2a_1 + [-2a_2])S_n(2b_1 + [-2b_2]))S_{n-1}(c_1 + [-c_2]) \\
&+ 2tsS_{n-1}(2a_1 + [-2a_2])S_{n-1}(2b_1 + [-2b_2])S_n(c_1 + [-c_2]) \\
&- 3yst(S_{n-1}(2a_1 + [-2a_2])S_{n-1}(2b_1 + [-2b_2]))S_{n-1}(c_1 + [-c_2])z^n \\
&= 2\frac{(8x^3z^6 + \left(-16s^2x^2 - 16t^2x^2 + 12x^2 - 18xy^2\right) z^4 + 48stxyz^3 + \left(-8s^2x - 8t^2x + 6x - 9y^2\right) z^2 + 1)}{D_6} \\
&- 3y\frac{(12x^2yz^6 - 16stx^2z^5 + \left(12xy - 27y^3\right) z^4 + 36sty^2z^3 + \left(-12s^2y - 12t^2y + 3y\right) z^2 + 4stz)}{D_6} \\
&- 2s\frac{(16sx^3z^6 - 24tx^2yz^5 + \left(-32s^3x^2 + 16sx^2\right) z^4 + 48s^2txyz^3 + \left(-16st^2x + 4sx - 18sy^2\right) z^2 + 6tyz)}{D_6} \\
&+ 3ys\frac{(24sx^2yz^6 + \left(-32s^2tx^2 + 8tx^2 - 36txy^2\right) z^5 + 48st^2xyz^4 + \left(-16t^3x + 8tx\right) z^3 - 6syz^2 + 2tz)}{D_6} \\
&- 2t\frac{(16tx^3z^6 - 24sx^2yz^5 + \left(-32t^3x^2 + 16tx^2\right) z^4 + 48st^2xyz^3 + \left(-16s^2tx + 4tx - 18ty^2\right) z^2 + 6syz)}{D_6} \\
&+ 3yt\frac{(24tx^2yz^6 + \left(-32st^2x^2 + 8sx^2 - 36sxy^2\right) z^5 + 48s^2txyz^4 + \left(-16s^3x + 8sx\right) z^3 - 6tyz^2 + 2sz)}{D_6} \\
&+ 2ts\frac{(32stx^3z^6 + \left(-48s^2x^2y - 48t^2x^2y + 12x^2y\right) z^5 + 72stxy^2z^4 + \left(12xy - 27y^3\right) z^3 - 8stxz^2 + 3yz)}{D_6} \\
&- 3yts\frac{(8x^3z^7 + \left(-16s^2x^2 - 16t^2x^2 + 12x^2 - 18xy^2\right) z^5 + 48stxyz^4 + \left(-8s^2x - 8t^2x + 6x - 9y^2\right) z^3 + z)}{D_6} \\
&= \frac{N_{T_n(t)T_n(s)m_n(x,y)}}{D_6},
\end{aligned}
$$

with

$$\begin{aligned}N_{T_n(t)T_n(s)m_n(x,y)} &= (-24stx^3yz^7 + (64s^2t^2x^3 - 32s^2x^3 + 72s^2x^2y^2 - 32t^2x^3 + 72t^2x^2y^2 + 16x^3 \\ &- 36x^2y^2)z^6 + \left(-144s^3tx^2y - 144st^3x^2y + 180stx^2y - 162stxy^3\right) z^5 \\ &+ \left(64s^4x^2 + 288s^2t^2xy^2 - 64s^2x^2 + 64t^4x^2 - 64t^2x^2 + 24x^2 - 72xy^2 + 81y^4\right) z^4 \\ &+ \left(-120s^3txy - 120st^3xy + 150stxy - 135sty^3\right) z^3 + (48s^2t^2x - 24s^2x \\ &+ 54s^2y^2 - 24t^2x + 54t^2y^2 + 12x - 27y^2)z^2 - 21styz + 2).\end{aligned}$$

This completes our proof. □

For the second kind Replacing $a_1 - a_2 = t$, $b_1 - b_2 = s$, $a_1a_2 = -\frac{1}{4}$, $b_1b_2 = -\frac{1}{4}$, $c_1 - c_2 = 3y$ and $c_1c_2 = -2x$ in (3.4) and (3.5), then we obtain

$$\sum_{n=0}^{\infty} S_n(2a_1 + [-2a_2])S_n(2b_1 + [-2b_2])S_n(c_1 + [-c_2]) = \frac{U_1}{D_6}, \tag{6.5}$$

and

$$\sum_{n=0}^{\infty} S_{n-1}(2a_1 + [-2a_2])S_n(2b_1 + [-2b_2])S_n(c_1 + [-c_2]) = \frac{U_2}{D_6}, \tag{6.6}$$

where
$U_1 = 8x^3z^6 + \left(-16s^2x^2 - 16t^2x^2 + 12x^2 - 18xy^2\right) z^4 + 48stxyz^3 + \left(-8s^2x - 8t^2x + 6x - 9y^2\right) z^2 + 1.$
$U_2 = 12x^2yz^6 - 16stx^2z^5 + \left(12xy - 27y^3\right) z^4 + 36sty^2z^3 + \left(-12s^2y - 12t^2y + 3y\right) z^2 + 4stz.$

Therefore, we state the following theorems.

Theorem 13. *For $n \in \mathbb{N}$, the new generating function for the product of Chebyshev polynomials of the second kind with bivariate Mersenne polynomials is given by*

$$\begin{aligned}\sum_{n=0}^{\infty} U_n(t)U_n(s)M_n(x,y)z^n &= \frac{12x^2yz^6 - 16stx^2z^5 + \left(12xy - 27y^3\right) z^4 + 36sty^2z^3}{D_6} \\ &+ \frac{\left(-12s^2y - 12t^2y + 3y\right) z^2 + 4stz}{D_6}.\end{aligned}$$

Theorem 14. *For $n \in \mathbb{N}$, the new generating function of the product of Chebyshev polynomials of the second kind with bivariate Mersenne Lucas polynomials is given by*

$$\sum_{n=0}^{\infty} U_n(t)U_n(s)m_n(x,y)z^n = \frac{N_{U_n(t)U_n(s)m_n(x,y)}}{D_6},$$

and we have

$$\begin{aligned} N_{U_n(t)U_n(s)m_n(x,y)} &= (16x^3-36x^2y^2)z^6+48stx^2yz^5+\left(-32s^2x^2-32t^2x^2+24x^2-72xy^2+81y^4\right)z^4 \\ &+ \left(96stxy-108sty^3\right)z^3+\left(-16s^2x+36s^2y^2-16t^2x+36t^2y^2+12x-27y^2\right)z^2 \\ &- 12styz+2. \end{aligned}$$

Proof. As we have

$$m_n(x,y)=2S_n(c_1+[-c_2])-3yS_{n-1}(c_1+[-c_2]),\ (\text{see [22]}),$$

and

$$U_n(t)=S_n(2a_1+[-2a_2]),\ (\text{see [6]}),$$

then

$$\begin{aligned} \sum_{n=0}^{\infty}U_n(t)U_n(s)m_n(x,y)z^n &= \sum_{n=0}^{\infty}S_n(2a_1+[-2a_2])S_n(2b_1+[-2b_2])[2S_n(c_1+[-c_2])-3yS_{n-1}(c_1+[-c_2])]z^n \\ &= 2\sum_{n=0}^{\infty}S_n(2a_1+[-2a_2])S_n(2b_1+[-2b_2])S_n(c_1+[-c_2])z^n \\ &- 3y\sum_{n=0}^{\infty}S_n(2a_1+[-2a_2])S_n(2b_1+[-2b_2])S_{n-1}(c_1+[-c_2])z^n \\ &= 2\frac{8x^3z^6+\left(-16s^2x^2-16t^2x^2+12x^2-18xy^2\right)z^4+48stxyz^3}{D_6} \\ &+ \frac{+\left(-8s^2x-8t^2x+6x-9y^2\right)z^2+1}{D_6} \\ &- 3y\frac{12x^2yz^6-16stx^2z^5+\left(12xy-27y^3\right)z^4+36sty^2z^3}{D_6} \\ &+ \frac{\left(-12s^2y-12t^2y+3y\right)z^2+4stz}{D_6} \\ &= \frac{N_{U_n(t)U_n(s)m_n(x,y)}}{D_6}, \end{aligned}$$

where

$$\begin{aligned} N_{U_n(t)U_n(s)m_n(x,y)} &= (16x^3-36x^2y^2)z^6+48stx^2yz^5+\left(-32s^2x^2-32t^2x^2+24x^2-72xy^2+81y^4\right)z^4 \\ &+ \left(96stxy-108sty^3\right)z^3+\left(-16s^2x+36s^2y^2-16t^2x+36t^2y^2+12x-27y^2\right)z^2 \\ &- 12styz+2. \end{aligned}$$

Thus our proof is complete. □

Here, we use our main theorem to derive new generating functions of the product of Chebyshev polynomials of the first and second kind with bivariate Mersenne polynomials and with bivariate Mersenne Lucas polynomials.

7. Conclusion

In our work, we have introduced new generating functions for several products of special numbers and polynomials as k-Fibonacci and k-Jacobsthal numbers, bivariate complex Fibonacci polynomials and Chebyshev polynomials with bivariate Mersenne and bivariate Mersenne Lucas polynomials. This opens up several avenues for further research and application in areas.

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