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Research Article

The crossing number of $K_{5,n}$ without one edge

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Abstract: It is conjectured that the crossing number of the complete bipartite graph $K_{m,n}$ without one edge e is equal to $\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor - \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$. In this paper, we establish the validity of this conjecture for $m = 5$ using combinatorial properties of cyclic permutations with proofs that can be generalized to all graphs $K_{m,n} \setminus e$ if m is at least six. Further, we give a conjecture concerning crossing numbers of $K_{m,n}$ without several edges incident with a common vertex.

Keywords: drawing, crossing number, near join product, cyclic permutation, complete bipartite graph.

AMS Subject classification: 05C10, 05C38

1. Introduction

Graph theory serves as a foundational framework for modeling and analyzing complex systems across various disciplines, ranging from computer science to social networks to biology. One fundamental aspect of graph theory is the concept of crossing numbers, which quantifies the minimum number of crossings required in a drawing of a graph [6]. The understanding and minimization of the crossing number of a graph have significant implications in numerous areas, encompassing both theoretical and practical domains. In computer science, minimizing the crossing number of a graph is essential for optimizing layout algorithms in circuit design, VLSI layout, and graph drawing applications. Moreover, in network design, reducing the crossing number

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enhances the efficiency and reliability of communication networks by minimizing signal interference and congestion. Furthermore, the crossing number has implications in spatial visualization and cartography, where minimizing crossings leads to clearer and more interpretable representations of geographic networks and transportation systems. In social network analysis, understanding the crossing number sheds light on the underlying structure and dynamics of social interactions, facilitating the identification of cohesive communities and influential nodes. Overall, reducing the number of crossings on graph edges can help in visualizing and understanding complex data, improving system performance, and optimizing graph algorithms [1, 19].

Let G be a simple graph (without loops or multiple edges). We use $V(G)$ and $E(G)$ to denote the vertex set and the edge set of G, respectively. The used graph terminology is taken from the book [25]. The *crossing number* of graph G, denoted $\mathrm{cr}(G)$, is defined as the minimum possible number of edge crossings over all drawings of G in the plane (for the definition of a *drawing* see Klešč [16]). A drawing with a minimum number of crossings (an optimal drawing) is always a *good* drawing, meaning that no edge crosses itself, no two edges cross more than once, no two edges incident with the same vertex cross, and no more than two edges cross at the same point. Let D be a good drawing of the graph G. We denote the number of crossings in D by $\mathrm{cr}_D(G)$. Let G_i and G_j be edge-disjoint subgraphs of G. We denote the number of crossings between edges of G_i and edges of G_j by $\mathrm{cr}_D(G_i, G_j)$, and the number of crossings among edges of G_i in D by $\mathrm{cr}_D(G_i)$.

For any three mutually edge-disjoint subgraphs G_i, G_j, and G_k of G by [16], the following equations hold:

$$\mathrm{cr}_D(G_i \cup G_j) = \mathrm{cr}_D(G_i) + \mathrm{cr}_D(G_j) + \mathrm{cr}_D(G_i, G_j)\,, \tag{1.1}$$

$$\mathrm{cr}_D(G_i \cup G_j, G_k) = \mathrm{cr}_D(G_i, G_k) + \mathrm{cr}_D(G_j, G_k)\,. \tag{1.2}$$

Determining the crossing number of the complete bipartite graph $K_{m,n}$ is one of the oldest crossing number open problems. Zarankiewicz [28] conjectured about the number of crossings of $K_{m,n}$ saying that the upper bound $\mathrm{cr}(K_{m,n}) \le Z(m)Z(n)$ holds with equality, where $Z(\alpha) = \lfloor \frac{\alpha}{2} \rfloor \lfloor \frac{\alpha-1}{2} \rfloor$ denotes Zarankiewicz's number. This conjecture

$$\mathrm{cr}(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \tag{1.3}$$

was proved for all positive integers m, n with respect to the restriction $\min\{m, n\} \le 6$ and $7 \le m \le 8$ with $n \le 10$ by Kleitman [14] and Woodal [26], respectively. Some useful statements and dependencies about Zarankiewicz's conjecture have already been stated [3, 5, 7, 23, 26, 27]. Much attention began to focus on the crossing number of $G \setminus e$ obtained by removing one edge e from some simple graph G. Zheng *et al.* [29] presented a new conjecture about $\mathrm{cr}(K_n \setminus e)$, which was independently confirmed by Chia and Lee [4] and Ouyang *et al.* [17] for any positive integer n at most twelve. He *et al.* [8] considered the complete bipartite graph $K_{m,n}$ minus one edge, denoted as $K_{m,n} \setminus e$, and predicted $\mathrm{cr}(K_{m,n} \setminus e)$ when $m = 3$ or $m = 4$ for the first time. According to symmetry, it doesn't matter which edge is removed. Moreover, due to isomorphism, it is clear that $\mathrm{cr}(K_{m,n} \setminus e) = \mathrm{cr}(K_{n,m} \setminus e)$ and so in what follows, we will assume that $m \le n$. The recursive inequality for the crossing numbers of $K_{m,n} \setminus e$ and $K_{m,n-1} \setminus e$ was established by Ouyang *et al.* [18] using a basic counting method. The crossing numbers of $K_{3,n} \setminus e$ and $K_{4,n} \setminus e$ obtained in this way confirm the following conjecture for any $m \le 4$ and $n \ge 1$.

Conjecture 1 ([4],[18]). For $m, n \ge 1$,

$$\mathrm{cr}(K_{m,n} \setminus e) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor - \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor.$$

Conjecture 1 can be formulated as to whether $\mathrm{cr}(K_{5,n} \setminus e) \le 4\lfloor \frac{n-1}{2} \rfloor \lfloor \frac{n-2}{2} \rfloor + 2\lfloor \frac{n-1}{2} \rfloor$ holds with equality when $m = 5$. It is worth noting that $(n-1)(n-2)$ is equal to $4\lfloor \frac{n-1}{2} \rfloor \lfloor \frac{n-2}{2} \rfloor + 2\lfloor \frac{n-1}{2} \rfloor$ and we will refer to this in some parts of the paper. Chia and Lee [4] only considered the special case of $n = 5$, e.g., $\mathrm{cr}(K_{5,5} \setminus e) = 12$. The crossing number of $K_{5,n+1} \setminus e$ has recently been found by Huang and Wang [12] thanks to the definition of a complete graph on n vertices with edge-labeling associated with some good drawing ϕ such as that $\mathrm{cr}_\phi(K_{5,n+1} \setminus e) < n(n-1)$. The proof of their main theorem is based on the idea of assuming odd partitions of a complete bipartite graph with the help of Lemma 1.

Lemma 1 ([15]). *Let ϕ and ϕ' be two good drawings of the complete bipartite graph $K_{m,n}$. If both m and n are odd, then $\mathrm{cr}_\phi(K_{m,n}) \equiv \mathrm{cr}_{\phi'}(K_{m,n}) \pmod 2$.*

Due to the very strong assumption of an odd number of vertices on both partitions, it will not be possible to use such an idea for the graphs $K_{6,n} \setminus e$. Although the main goal of the paper is to give a conjecture concerning crossing numbers of $K_{m,n}$ without several edges incident with one vertex, we also offer a simple proving alternative for $\mathrm{cr}(K_{5,n} \setminus e)$ described in Corollary 2 because a similar idea of the proof should be used for all graphs $K_{m,n} \setminus e$ if m is at least six. For this purpose, we introduce for the first time the concept of a „near" join product between two graphs, which can also be linked to combinatorial properties of cyclic permutations. Section 4 is devoted to a new conjecture (4.1) concerning crossing numbers of $K_{m,n} \setminus \bigcup_{i=1}^{k} e_i$ for k different edges $e_1, e_2, \dots, e_k$ incident with just one vertex. Of course, its correctness is confirmed by using several isomorphisms for a lot of possible cases. In the proofs of the paper, we will often use the term "region" also in nonplanar subdrawings. In this case, crossings are considered to be vertices of the "map".

2. Cyclic Permutations and Configurations

Let $G^* = (V(G^*), E(G^*))$ be the graph on six vertices isomorphic to the complete bipartite graph $K_{1,4}$ and one isolated vertex, and let also $V(G^*) = \{v_0, v_1, v_2, v_3, v_4, v_5\}$.

In the rest of the paper, let v_0 and v_5 be the vertex notation of the vertex of degree four and the isolated vertex, respectively. We consider ,,near" join product of G^* with the discrete graph D_n on n vertices denoted by G_n. The graph G_n consists of one copy of the graph G^* and n vertices $t_1, t_2, \dots, t_n$, where each vertex t_i, $i = 1, 2, \dots, n$, is adjacent to every vertex of G^* except for the vertex v_0. In the case of $i = 0$, we define $G_0 = G^*$. It is easy to see that the graph G_n is isomorphic to the graph $K_{5,n+1} \setminus e$ obtained by removing one edge e from the complete bipartite graph $K_{5,n+1}$. For $n \geq 1$, the first and second partition of $K_{5,n+1}$ consists of the vertices v_1, v_2, v_3, v_4, v_5 and $v_0, t_1, t_2, \dots, t_n$, respectively. A removed edge e is the edge between the vertices v_0 and v_5.

Let T^i, $1 \leq i \leq n$, denote the subgraph induced by the five edges incident with the vertex t_i. Thus, $T^1 \cup \dots \cup T^n$ is isomorphic to the complete bipartite graph $K_{5,n}$ and

$$G_n = G^* \cup \left(\bigcup_{i=1}^{n} T^i \right) \cong G^* \cup K_{5,n}. \tag{2.1}$$

We consider a good drawing D of the graph G_n. The *rotation* $\mathrm{rot}_D(t_i)$ of a vertex t_i in the drawing D as the cyclic permutation that records the (cyclic) counter-clockwise order in which the edges leave t_i have been defined by Hernández-Vélez *et al.* [9]. We use the notation (12345) if the counter-clockwise order the edges incident with the vertex t_i is t_iv_1, t_iv_2, t_iv_3, t_iv_4, and t_iv_5. We have to emphasize that a rotation is a cyclic permutation. We will separate all subgraphs T^i, $i = 1, \dots, n$, of the graph G_n into three mutually-disjoint subsets depending on how many times the considered T^i crosses the edges of G^* in D. For $i = 1, \dots, n$, let $R_D = \{T^i : \mathrm{cr}_D(G^*, T^i) = 0\}$ and $S_D = \{T^i : \mathrm{cr}_D(G^*, T^i) = 1\}$. Every other subgraph T^i crosses the edges of G^* at least twice in D. Moreover, let F^i denote the subgraph $G^* \cup T^i$ for $T^i \in R_D$, where $i \in \{1, \dots, n\}$. Thus, for a given subdrawing of G^* in D, any subgraph F^i is exactly represented by $\mathrm{rot}_D(t_i)$.

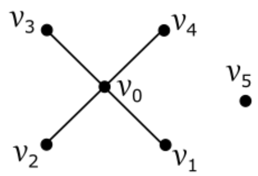

Figure 1. **One possible drawing of the graph G^*.**

Since there is only one possible drawing of the graph G^*, without loss of generality, we can choose the vertex notation of G^* in such a way as shown in Figure 1. Our aim shall be to list all possible rotations $\mathrm{rot}_D(t_i)$ for $i \in \{1, \dots, n\}$ which can appear in D if $T^i \in R_D$. Since there is only one subdrawing of $F^i \setminus v_5$ represented by the rotation (1234), there are four possibilities how to obtain the subdrawing of F^i

depending on which region the edge t_iv_5 is placed in. Let $\mathcal{M} = \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4\}$ be the set of these four configurations. In the rest of the paper, we represent a cyclic permutation by the permutation with 1 in the first position. Thus, the configurations $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, and $\mathcal{A}_4$ are represented by the cyclic permutations (12534), (12345), (12354), and (15234), respectively. For $p \in \{1, 2, 3, 4\}$, we say that a subdrawing of F^i has the configuration $\mathcal{A}_p$ denoted by $\mathrm{conf}(F^i) = \mathcal{A}_p$. For our purposes, it does not matter which of the regions is unbounded, and so we can assume that the drawings are as shown in Figure 2. Of course, in a fixed drawing D of the graph G_n, some configurations from $\mathcal{M} = \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4\}$ need not appear. We denote by $\mathcal{M}_D$ the set of all configurations for the drawing D belonging to $\mathcal{M}$.

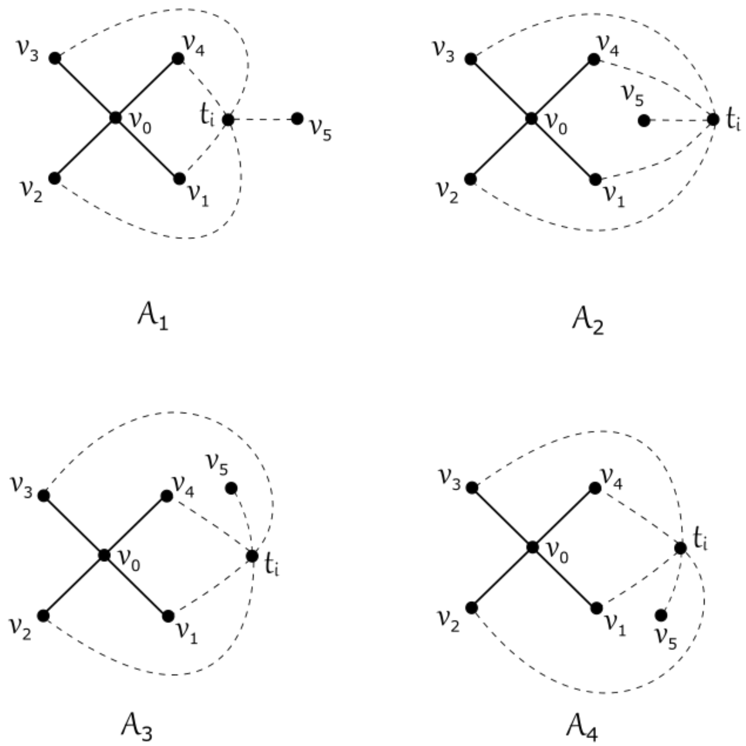

Figure 2. **Four possible subdrawings of F^i for $T^i \in R_D$.**

Let $\mathcal{X}$, $\mathcal{Y}$ be two configurations from $\mathcal{M}_D$. We shortly denote by $\mathrm{cr}_D(\mathcal{X}, \mathcal{Y})$ the number of crossings in D between T^i and T^j for such that F^i, F^j have configurations $\mathcal{X}$, $\mathcal{Y}$, respectively. Finally, let $\mathrm{cr}(\mathcal{X}, \mathcal{Y}) = \min\{\mathrm{cr}_D(\mathcal{X}, \mathcal{Y})\}$ over all good drawings D of the graph G_n with $\mathcal{X}, \mathcal{Y} \in \mathcal{M}_D$. Our aim is to establish $\mathrm{cr}(\mathcal{X}, \mathcal{Y})$ for all pairs $\mathcal{X}, \mathcal{Y} \in \mathcal{M}$. Now, the configurations $\mathcal{A}_1$ and $\mathcal{A}_2$ are represented by the cyclic permutations (12534) and (12345), respectively. Since the minimum number of interchanges of adjacent elements of (12534) required to produce the cyclic permutation (12345) is two, any subgraph T^j with the configuration $\mathcal{A}_2$ of F^j crosses the edges of T^i with the configuration $\mathcal{A}_1$ of F^i at least twice, that is, $\mathrm{cr}(\mathcal{A}_1, \mathcal{A}_2) \geq Z(5) - 2 = \lfloor \frac{5}{2} \rfloor \lfloor \frac{4}{2} \rfloor - 2 = 2$, where $Z(\alpha) = \lfloor \frac{\alpha}{2} \rfloor \lfloor \frac{\alpha-1}{2} \rfloor$ is Zarankiewicz's number and both considered permutations are of the length five. Details have been worked out by Woodall [26]. The same reason gives $\mathrm{cr}(\mathcal{A}_1, \mathcal{A}_3) \geq 3$, $\mathrm{cr}(\mathcal{A}_1, \mathcal{A}_4) \geq 3$, $\mathrm{cr}(\mathcal{A}_2, \mathcal{A}_3) \geq 3$, $\mathrm{cr}(\mathcal{A}_2, \mathcal{A}_4) \geq 3$, $\mathrm{cr}(\mathcal{A}_3, \mathcal{A}_4) \geq 2$, and $\mathrm{cr}(\mathcal{A}_p, \mathcal{A}_p) \geq 4$ for any $p = 1, \dots, 4$. The resulting lower bounds for the number of crossings of two configurations from $\mathcal{M}$ are summarized in the symmetric Table 1.

$-$	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$
$\mathcal{A}_1$	4	2	3	3
$\mathcal{A}_2$	2	4	3	3
$\mathcal{A}_3$	3	3	4	2
$\mathcal{A}_4$	3	3	2	4

Table 1. The minimum number of crossings between T^i and T^j with configurations from $\mathcal{M}$ of F^i and F^j, respectively.

3. The crossing number of G_n

Two vertices t_i and t_j of G_n are *antipodal* in a drawing of G_n if the subgraphs T^i and T^j do not cross each other. A drawing is *antipode-free* if it has no antipodal vertices. In the proof of the main theorem, the following three statements related to some restricted subdrawings of the graph G_n will be needful.

$\mathrm{conf}(F^i)$	$\mathrm{rot}_D(t_j)$
$\mathcal{A}_1$	(13452), (12435)
$\mathcal{A}_2$	(15342), (12543)
$\mathcal{A}_3$	(15324), (14523)
$\mathcal{A}_4$	(14235), (13254)

Table 2. The corresponding rotations for $T^i \in R_D$ with $\mathrm{conf}(F^i) = \mathcal{A}_p$ and $T^j \in S_D$ such that $\mathrm{cr}_D(T^i, T^j) = 1$.

Lemma 2. *For $n \geq 2$, let D be a good drawing of G_n with the vertex notation of the graph G^* in such a way as shown in Figure 1. Let for $T^i \in R_D$, the corresponding subgraph F^i has the configuration $\mathcal{A}_p \in \mathcal{M}_D$ for some $p \in \{1, \dots, 4\}$. If there is a subgraph $T^j \in S_D$ with $\mathrm{cr}_D(T^i, T^j) = 1$, then all possible $\mathrm{rot}_D(t_j)$ are given in Table 2.*

Proof. Let us assume the configuration $\mathcal{A}_1$ of F^i, i.e., $\mathrm{rot}_D(t_i) = (12534)$. The unique subdrawing $D(F^i)$ of the subgraph F^i contains four regions with the vertex t_i on their boundaries. If there is a subgraph $T^j \in S_D$ with $\mathrm{cr}_D(T^i, T^j) = 1$, then the vertex t_j must be placed in the same region as the vertex v_5. That means that the edge v_0v_2 or v_0v_3 of G^* must be crossed by the edge t_jv_1 or t_jv_4, respectively. This forces either $\mathrm{rot}_D(t_j) = (13452)$ or $\mathrm{rot}_D(t_j) = (12435)$. For the remainder configurations $\mathcal{A}_2$, $\mathcal{A}_3$, and $\mathcal{A}_4$ of F^i, using the same arguments, one can easy to verify mentioned rotations of t_j in Table 2. □

In the following, let us define two mutually-disjoint subsets $S_D(\mathcal{A}_1, \mathcal{A}_2)$ and $S_D(\mathcal{A}_3, \mathcal{A}_4)$ of S_D based on the corresponding cyclic permutations described in Table 2. More precisely,

$$S_D(\mathcal{A}_1, \mathcal{A}_2) = \{T^j \in S_D : \mathrm{rot}_D(t_j) \in \{(13452), (12435), (15342), (12543)\}\}$$

and

$$S_D(\mathcal{A}_3, \mathcal{A}_4) = \{T^j \in S_D : \mathrm{rot}_D(t_j) \in \{(15324), (14523), (14235), (13254)\}\}.$$

For some $p \in \{1, 3\}$, the set $S_D(\mathcal{A}_p, \mathcal{A}_{p+1})$ can be nonempty even if $\mathcal{A}_p, \mathcal{A}_{p+1} \notin \mathcal{M}_D$. We remark that if T^i does not cross the edges of G^*, then $\mathrm{rot}_D(t_i)$ must contain the elements 1, 2, 3, and 4 in such a way that the omission of the element 5 induces the cyclic sub-permutation (1234). Further, due to symmetries of four mentioned configurations, let us define the function $\pi : \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$ with

$$\pi(1) = 4, \pi(2) = 1, \pi(3) = 2, \pi(4) = 3.$$

Let $\Pi : \mathcal{M} \to \mathcal{M}$ be the function obtained by applying π on corresponding cyclic permutations of configurations in $\mathcal{M}$, respectively. Thus, we have

$$\Pi(\mathcal{A}_1) = \mathcal{A}_4,\ \Pi(\mathcal{A}_4) = \mathcal{A}_2,\ \Pi(\mathcal{A}_2) = \mathcal{A}_3,\ \Pi(\mathcal{A}_3) = \mathcal{A}_1. \tag{3.1}$$

The subsets $S_D(\mathcal{A}_1, \mathcal{A}_2)$ and $S_D(\mathcal{A}_3, \mathcal{A}_4)$ defined above are complementary to each other according to

$$\Pi \circ \Pi(\mathcal{A}_1) = \mathcal{A}_2,\ \Pi \circ \Pi(\mathcal{A}_2) = \mathcal{A}_1,\ \Pi \circ \Pi(\mathcal{A}_3) = \mathcal{A}_4,\ \Pi \circ \Pi(\mathcal{A}_4) = \mathcal{A}_3.$$

Moreover, the configurations $\mathrm{conf}(F^i)$ and the rotations $\mathrm{rot}_D(t_j)$ in rows of Table 2 can be obtain by successive applying of Π and π, respectively.

Lemma 3. *For $n \geq 2$, let D be a good and antipode-free drawing of G_n with the subdrawing of G^* induced by D given in Figure 1. Let there be some subgraph $T^i \in R_D$ which is crossed just once by other subgraph $T^j \in S_D(\mathcal{A}_{1+p}, \mathcal{A}_{2+p})$ for some $p \in \{0, 2\}$. Then*

$$\mathrm{cr}_D(T^i \cup T^j, T^k) \geq 5 \quad \text{if} \quad T^k \in R_D \cup S_D \setminus S_D(\mathcal{A}_{3-p}, \mathcal{A}_{4-p})$$

for every T^k, $k \neq i, j$.

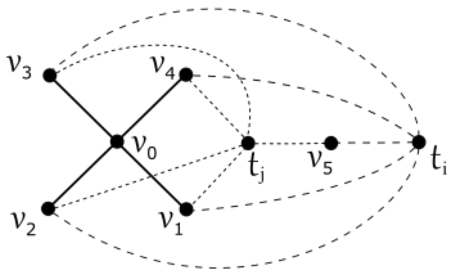

Figure 3. One possible subdrawing of $G^* \cup T^i \cup T^j$ with $\mathrm{rot}_D(t_i) = (12345)$ and $\mathrm{rot}_D(t_j) = (15342)$, and $\mathrm{cr}_D(T^i, T^j) = 1$.

Proof. Let D be a good and antipode-free drawing of G_n with the subdrawing $D(G^*)$ shown in Figure 1, and let $T^i \in R_D$ be a subgraph with the configuration $\mathcal{A}_2 \in \mathcal{M}_D$ of F^i, i.e., $\mathrm{rot}_D(t_i) = (12345)$. Let also $T^j \in S_D(\mathcal{A}_1, \mathcal{A}_2)$ be a subgraph satisfying $\mathrm{cr}_D(T^i, T^j) = 1$ with $\mathrm{rot}_D(t_j) = (15342)$ (the proof can proceed in a similar way as for a T^j with $\mathrm{rot}_D(t_j) = (12543)$).
In Figure 3, there is only one possible subdrawing of $G^* \cup T^i \cup T^j$ containing ten different regions. For a $T^k \in R_D \cup S_D$ with $k \neq i, j$, let us first consider that t_k is located in the region of $D(G^* \cup T^i \cup T^j)$ with two vertices v_1 and v_2 of G^* on its boundary. It is easy to see that $\mathrm{cr}_D(T^i \cup T^j, T^k) < 5$ only in the case when each of the edges t_kv_3, t_kv_4 and t_kv_5 of T^k forces just one crossing on edges of the subgraph $T^i \cup T^j$. The edge t_kv_4 must cross one edge of G^*, so T^k cannot be from R_D. As D is antipode-free, t_kv_3, t_kv_4 and t_kv_5 crosses t_iv_2, t_jv_2 and t_iv_1, respectively. So, $\mathrm{rot}_D(t_k) = (14235)$ and $T^k \in S_D(\mathcal{A}_3, \mathcal{A}_4)$.
Now, let the vertex t_k be placed in the region of $D(G^* \cup T^i \cup T^j)$ with three vertices v_0, v_2 and v_3 of G^* on its boundary. The edges t_kv_1 and t_kv_5 enforce at least one and two crossings on $T^i \cup T^j$, respectively. Assuming $\mathrm{cr}_D(T^i \cup T^j, T^k) < 5$, the edge t_kv_4 must cross v_0v_3 of G^*, and the vertices t_k and t_j contradict an antipodality-free of D. Finally, if t_k is placed in one of the other region of $D(G^* \cup T^i \cup T^j)$ then it is not difficult to verify that $\mathrm{cr}_D(T^i \cup T^j, T^k) \geq 5$ in all discussed cases.
As the same idea can be applied for the remainder configurations $\mathcal{A}_1$, $\mathcal{A}_3$, and $\mathcal{A}_4$ of F^i using the transformation Π, the proof of Lemma 3 is done. □

Corollary 1. *For $n \geq 2$, let D be a good and antipode-free drawing of G_n satisfying $|R_D| \geq 1$ and $2|R_D| + |S_D| \geq 2\lceil \frac{n}{2} \rceil + 1$ with the vertex notation of the graph G^* in such a way as shown in Figure 1. If $\mathrm{cr}(G_{n-2}) = (n-2)(n-3)$ and some subgraph $T^i \in R_D$ is crossed just once by a $T^j \in S_D$, then there are at least $n(n-1)$ crossings in D.*

Proof. For easier reading, let $r = |R_D|$ and $s = |S_D|$. By the assumption of Corollary 1, $r \geq 1$ and $2r + s \geq 2\lceil \frac{n}{2} \rceil + 1$. Consequently, let us denote $s_1 = |S_D(\mathcal{A}_1, \mathcal{A}_2)|$ and $s_2 = |S_D(\mathcal{A}_3, \mathcal{A}_4)|$. Since at least one of the sets $S_D(\mathcal{A}_1, \mathcal{A}_2)$ and $S_D(\mathcal{A}_3, \mathcal{A}_4)$ must be nonempty in D, without lost of generality due to their symmetry by (3.1), let us suppose that $s_1 \geq s_2$. Let us also consider a subgraph $T^j \in S_D(\mathcal{A}_1, \mathcal{A}_2)$ and some $T^i \in R_D$ with $\mathrm{cr}_D(T^i, T^j) = 1$. By (1.3), we obtain $\mathrm{cr}_D(K_{5,3}) \geq 4$. This implies that any other subgraph T^k, $k \neq i, j$, must cross $T^i \cup T^j$ at least three times in D. Hence, by fixing the subgraph $T^i \cup T^j$ and using Lemma 3, we have

$$\mathrm{cr}_D(G_n) = \mathrm{cr}_D(G_{n-2}) + \mathrm{cr}_D(K_{5,n-2}, T^i \cup T^j) + \mathrm{cr}_D(G^*, T^i \cup T^j) + \mathrm{cr}_D(T^i \cup T^j) \geq$$

$$\geq (n-2)(n-3) + 5(r-1) + 5(s - s_2 - 1) + 3s_2 + 3(n - r - s) + 1 + 1 =$$

$$= (n-2)(n-3) + 3n + 2r + s + s - 2s_2 - 8 \geq (n-2)(n-3) + 3n + n + 2 - 8 \geq n(n-1),$$

where the inequality $2r + s + s - 2s_2 \geq n + 2$ can be used. For n odd and at least three, we have $2r + s \geq 2\lceil \frac{n}{2} \rceil + 1 \geq n + 2$. In the case when n is even, the equality $2r + s = n + 1$ forces an odd value of s but $s \geq 2s_2 + 1$ must hold due to $s - s_1 - s_2 \geq 0$ and $s_1 \geq s_2$. □

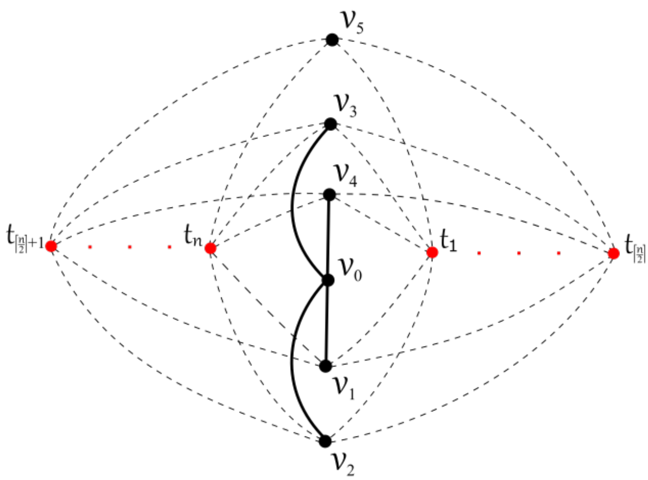

Figure 4. **The good drawing of G_n with $n(n-1)$ crossings.**

Theorem 2. $\mathrm{cr}(G_n) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor = n(n-1)$ *for* $n \geq 0$.

Proof. In Figure 4, the edges of $K_{5,n}$ cross each other $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ times, each subgraph T^i for $i \in \{1, \ldots, \lceil \frac{n}{2} \rceil\}$ on the right side does not cross the edges of the graph

G^* and each subgraph T^i for $i \in \{\lceil \frac{n}{2} \rceil + 1, \dots, n\}$ on the left side crosses edges of G^* exactly twice. Thus $\mathrm{cr}(G_n) \leq 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor = n(n-1)$. We prove the reverse inequality by induction on n. Both graphs G_0 and G_1 are planar, hence $\mathrm{cr}(G_0) = \mathrm{cr}(G_1) = 0$. The graph G_2 contains $K_{3,4}$ as a subgraph, and therefore $\mathrm{cr}(G_2) \geq 2$ by (1.3). So, $\mathrm{cr}(G_2) = 2$. Suppose now that, for some $n \geq 3$, there is an optimal drawing D of G_n with

$$\mathrm{cr}_D(G_n) < 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor = n(n-1) \tag{3.2}$$

and

$$\mathrm{cr}(G_m) = m(m-1) \quad \text{for any non-negative integer } m < n. \tag{3.3}$$

Without loss of generality, we can choose the vertex notation of $D(G^*)$ in such a way as shown in Figure 1. Next we claim that the considered drawing D must be antipode-free. For a contradiction suppose that $\mathrm{cr}_D(T^i, T^j) = 0$ for two different subgraphs T^i and T^j. Both subgraphs T^i and T^j cannot be from the set R_D at the same time due to positive values in Table 1. We can assume that $\mathrm{cr}_D(G^*, T^i \cup T^j) \geq 2$, otherwise if exactly one, say T^i, is from R_D then one can easily verify that $\mathrm{cr}_D(G^*, T^j) \geq 2$ using possible subdrawings in Figure 2. Next by (1.3), $\mathrm{cr}(K_{3,5}) = 4$. This implies that any other T^k, $k \neq i, j$, crosses $T^i \cup T^j$ at least four times. So, for the number of crossings in D we have

$$\mathrm{cr}_D(G_n) = \mathrm{cr}_D(G_{n-2}) + \mathrm{cr}_D(T^i \cup T^j) + \mathrm{cr}_D(K_{5,n-2}, T^i \cup T^j) +$$

$$+ \mathrm{cr}_D(G^*, T^i \cup T^j) \geq (n-2)(n-3) + 4(n-2) + 2 = n(n-1).$$

This contradiction with the assumption (3.2) confirms that the considered drawing D must be antipode-free. Moreover, if $r = |R_D|$ and $s = |S_D|$, the assumption (3.2) together with $\mathrm{cr}(K_{5,n}) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ again by (1.3) imply that in D there is at least one subgraph T^i which does not cross the edges of G^*. More precisely:

$$\mathrm{cr}_D(G^*) + 0r + 1s + 2(n - r - s) \leq \mathrm{cr}_D(G^*) + \mathrm{cr}_D(G^*, K_{5,n}) < 2\left\lfloor \frac{n}{2} \right\rfloor,$$

i.e.,

$$s + 2(n - r - s) < 2\left\lfloor \frac{n}{2} \right\rfloor. \tag{3.4}$$

This enforces that $r \geq 1$, and $2r + s \geq 2\lceil \frac{n}{2} \rceil + 1$. Until the end of the proof let us assume that i, j, k, l are mutually different. If there exist $T^i \in R_D$ and $T^l \in S_D$ such that $\mathrm{cr}_D(T^i, T^l) = 1$, then Corollary 1 contradicts the assumption (3.2). In the following, we can consider that $\mathrm{cr}_D(T^i, T^l) \geq 2$ holds for every $T^i \in R_D$ and $T^l \in S_D$.
Case 1: Let $\{\mathcal{A}_p, \mathcal{A}_{p+1}\} \subseteq \mathcal{M}_D$ for some $p \in \{1, 3\}$. Without lost of generality, let us suppose that $\{\mathcal{A}_1, \mathcal{A}_2\} \subseteq \mathcal{M}_D$. We discuss two possibilities over congruence n modulo 2.

1. Let n be odd, and let us also consider two different subgraphs T^i, $T^j \in R_D$ such that F^i and F^j have configurations $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. Then, $\mathrm{cr}_D(T^i \cup T^j, T^k) \geq 6$ holds for any $T^k \in R_D$ by summing the values in the corresponding two rows of Table 1. Moreover, it is obvious that the condition $\mathrm{cr}_D(G^* \cup T^i \cup T^j, T^k) \geq 5$ is fulfilling for any $T^k \in S_D$ from the observation described above before this case. If T^k crosses the edges of G^* at least twice, then $\mathrm{cr}_D(G^* \cup T^i \cup T^j, T^k) \geq 4$ because the drawing D is antipode-free. Thus, by fixing the subgraph $G^* \cup T^i \cup T^j$ using (1.3), we have

$$\mathrm{cr}_D(G_n) = \mathrm{cr}_D(K_{5,n-2}) + \mathrm{cr}_D(K_{5,n-2}, G^* \cup T^i \cup T^j) + \mathrm{cr}_D(G^* \cup T^i \cup T^j) \geq$$

$$\geq 4\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 6(r-2) + 5s + 4(n-r-s) + 2 = 4\frac{n-3}{2}\frac{n-3}{2} + 4n +$$
$$+(2r+s) - 10 \geq (n-3)(n-3) + 4n + (n+2) - 10 \geq n(n-1).$$

2. Let n be even. In the rest of the proof and thanks to Π, let the number of subgraphs with the associated configuration $\mathcal{A}_1$ be at least as much as the number of subgraphs with the configuration $\mathcal{A}_2$. If we consider a subgraph $T^i \in R_D$ with the configuration $\mathcal{A}_1$ of F^i, then

$$\sum_{T^j \in R_D} \mathrm{cr}_D(T^i, T^j) \geq 3(r-2) + 2 = 3(r-1) - 1.$$

So, by fixing the subgraph $G^* \cup T^i$ thanks to (1.3), we obtain

$$\mathrm{cr}_D(K_{5,n-1}) + \mathrm{cr}_D(K_{5,n-1}, G^* \cup T^i) + \mathrm{cr}_D(G^* \cup T^i) \geq 4\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor +$$

$$+3(r-1) - 1 + 3(n-r) + 0 = (n-2)(n-2) + 3n - 4 \geq n(n-1).$$

Case 2: $\{\mathcal{A}_p, \mathcal{A}_{p+1}\} \not\subseteq \mathcal{M}_D$ for any $p = 1, 3$. We can assume that $T^i \in R_D$. Then $\mathrm{cr}_D(T^i, T^j) \geq 3$ trivially holds for any $T^j \in R_D$. Thus, by fixing the subgraph $G^* \cup T^i$, we have

$$\mathrm{cr}_D(G_n) \geq 4\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 3(n-1) + 0 \geq n(n-1).$$

Thus, it was shown in all mentioned cases that there is no optimal drawing D of the graph G_n with less than $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor$ crossings. This completes the proof of the main Theorem 2. □

The previous Theorem 2 together with an isomorphism between the graphs G_n and $K_{5,n+1} \setminus e$ give us the next corollary.

Corollary 2. $\mathrm{cr}(K_{5,n} \setminus e) = 4\lfloor \frac{n-1}{2} \rfloor \lfloor \frac{n-2}{2} \rfloor + 2\lfloor \frac{n-1}{2} \rfloor = (n-1)(n-2)$ *for* $n \geq 1$.

4. The crossing numbers of $K_{m,n}$ without several edges

As mentioned in the Introduction, all edges of the complete bipartite graph $K_{m,n}$ are equivalent to each other, and therefore, it doesn't matter which edge e is removed in the case of $\mathrm{cr}(K_{m,n} \setminus e)$. The situation rapidly changes if we want to remove several different edges of $K_{m,n}$ because the removed edges may be in different adjacency relationships. This section is devoted to some natural generalization of removing a single edge to the case of removing several edges incident with a common vertex.
Let $K_{m,n}$ be the complete bipartite graph on $m+n$ vertices with partitions $V_1 \cup V_2 = V(K_{m,n})$ containing an edge between every pair of vertices from V_1 and V_2 of sizes m and n, respectively. In the rest of the paper, let $I = \{0, 1, \dots, m\}$ and $m \le n$. For arbitrary $k \in I$, let $e_1, e_2, \dots, e_k$ be different edges incident with just one vertex of the vertex set V_2. Now, we can postulate that

$$\mathrm{cr}\left(K_{m,n} \setminus \bigcup_{i=1}^{k} e_i\right) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor - \left\lfloor \frac{n-1}{2} \right\rfloor \sum_{i=1}^{k} \left\lfloor \frac{m-i}{2} \right\rfloor \tag{4.1}$$

for all integers $m, n \ge 1$.

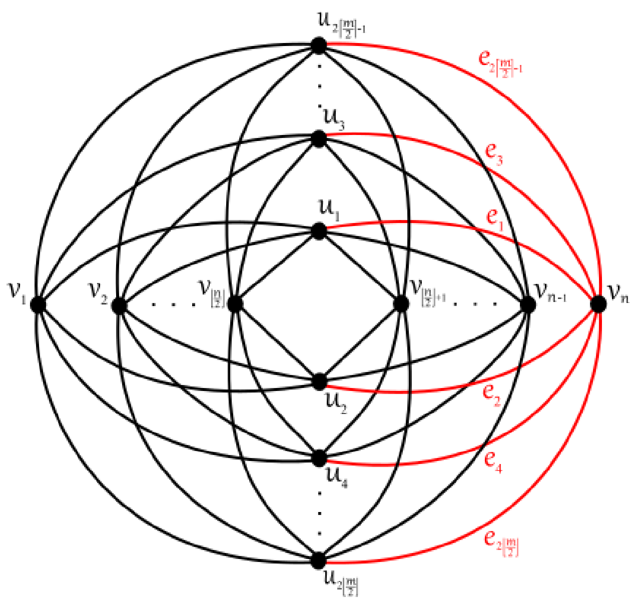

Figure 5. **The good drawing of $K_{m,n}$ with $\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ crossings.**

For $k = 0$, we obtain the Zarankiewicz's conjecture mentioned in the Introduction. If $k \ge 1$, then the upper bound for the conjecture (4.1) can be reached by removing the edges $e_1, \dots, e_k$ from the drawing in Figure 5 because each edge $e_i = u_i v_n$ is crossed exactly $\lfloor \frac{n-1}{2} \rfloor \lfloor \frac{m-i}{2} \rfloor$ times. Our conjecture (4.1) was established as Conjecture 1 in the case of $k = 1$, and Corollary 2 confirms the validity of this conjecture for $m = 5$. For $k = m$, we have $\sum_{i=1}^{m} \lfloor \frac{m-i}{2} \rfloor = \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor$, and so

$$\left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor - \left\lfloor \frac{n-1}{2} \right\rfloor \sum_{i=1}^{m} \left\lfloor \frac{m-i}{2} \right\rfloor = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor.$$

It is important to note that an isolated vertex does not affect a crossing number of the graph in which it is contained, and therefore, $\mathrm{cr}(K_{m,n} \setminus \bigcup_{i=1}^{m} e_i) = \mathrm{cr}(K_{m,n-1})$. Both cases $k = m - 1$ and $k = m - 2$ imply the same value on the right side of (4.1) due to

$$\sum_{i=1}^{m} \left\lfloor \frac{m-i}{2} \right\rfloor = \sum_{i=1}^{m-1} \left\lfloor \frac{m-i}{2} \right\rfloor = \sum_{i=1}^{m-2} \left\lfloor \frac{m-i}{2} \right\rfloor = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor.$$

On the other hand, $K_{m,n} \setminus \bigcup_{i=1}^{m} e_i$ is a subgraph of $K_{m,n} \setminus \bigcup_{i=1}^{m-1} e_i$ that is a subgraph of $K_{m,n} \setminus \bigcup_{i=1}^{m-2} e_i$, and therefore,

$$\mathrm{cr}(K_{m,n} \setminus \bigcup_{i=1}^{m-2} e_i) \geq \mathrm{cr}(K_{m,n} \setminus \bigcup_{i=1}^{m-1} e_i) \geq \mathrm{cr}(K_{m,n} \setminus \bigcup_{i=1}^{m} e_i).$$

Both reverse inequalities can be verified thanks to the considered subdrawings obtained from Figure 5. By Ouyang *et al.* [18], the conjecture (4.1) was already solved in the case of $m = 4$ with $k = 1$. All remaining cases for m at most four also confirm the validity of (4.1) based on the arguments above if $k = 0, m-2, m-1, m$. At the moment we do not even know the crossing number of $K_{m,n} \setminus \bigcup_{i=1}^{k} e_i$ for $m = 5$ and $k = 2$, i.e., whether $\mathrm{cr}(K_{5,n} \setminus \{e_1, e_2\}) \leq 4\lfloor \frac{n-1}{2} \rfloor \lfloor \frac{n-2}{2} \rfloor + \lfloor \frac{n-1}{2} \rfloor$ holds with equality.

5. Conclusions

We expect that the mentioned idea of the near join product between two graphs with similar forms of discussions can be used to estimate unknown values of the crossing numbers of other families of graphs without one edge. Above all, conjectures about the crossing numbers of $K_n \setminus e$ and $K_{m,n} \setminus e$ are established, but not yet for multipartite graphs without one edge. In the case of multipartite graphs, not all edges are equivalent and it is all the more difficult to determine suitable estimates for all alternatives. The crossing numbers of $K_{1,4,n} \setminus e$ and $K_{2,3,n} \setminus e$ are well-known by Su [24], and he also stated a question considering the exact values of the crossing numbers of $K_{1,5,n} \setminus e$, $K_{2,4,n} \setminus e$, and $K_{3,3,n} \setminus e$. Recently, a partial answer to his question has been offered for the last mentioned graph $K_{3,3,n} \setminus e$ by Staš [21]. Partial results for $K_{1,2,2,n} \setminus e$, $K_{1,1,3,n} \setminus e$, $K_{1,1,4,n} \setminus e$, $K_{1,1,1,2,n} \setminus e$, and $K_{1,1,1,1,1,n} \setminus e$ were already provided by Asano [2], Huang and Zhao [13], Ho [10, 11], Staš [20], and Staš and Timková [22], respectively.

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Statements and Declarations

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