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Research Article

Nicely graceful labellings of tadpoles

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Abstract: A vertex-labelling $f : V \to \{0, 1, \ldots, |E|\}$ for a finite undirected simple graph $G(V, E)$ is called graceful if f is injective and satisfies the additional property that $\{|f(u) - f(v)| : \text{for every edge} uv \in E\} = \{1, 2, \ldots, |E|\}$, where $|E|$ is the number of edges in G. Let M be a maximum matching in G and let f also satisfy the property that $f(u) + f(v) = W$ for every $uv \in M$, where W is a constant; then the labelling f is called nicely graceful. Furthermore, if M is a perfect matching in G, then f is said to be strongly graceful. In this paper, we investigate nicely and strongly graceful labellings of cycles and tadpoles that are obtained from a cycle by attaching a path to a vertex of the cycle. This leads to a complete characterisation of nicely graceful cycles and strongly graceful tadpoles.

Keywords: graph labelling, graceful labelling, strongly/nicely graceful, tadpole graphs.

AMS Subject classification: 05C78, 05C70

1. Introduction

A *simple graph* $G := G(V, E)$ consists of a non-empty set V of *vertices* and a possibly empty set E of 2-element subsets of V called *edges*. Throughout this paper, all graphs are simple, finite and connected. For convenience, for each $\{u, v\} \in E$, we write uv.

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When the context is clear, we simply denote $G(V, E)$ by G. Given two graphs $G(V, E)$, $G'(V', E')$, if $V' \subset V$ and $E' \subset E$, then G' is called a *subgraph* of G. A *matching* in G is a non-empty subset M of the edge set E such that any two edges in M are not adjacent in G. Vertices incident with an edge in M are called *matched* by M (or equivalently M matches them) and other vertices are called *unmatched*. The matching M is called *perfect* if all vertices are matched by M. In this case, the graph G is called with a *perfect matching*. A *maximum matching* in a graph is a matching of maximum cardinality. For general graph theory notions not defined here, the reader is referred to [5]. A set described as a list of elements of the form $\{x_1, x_2, \dots, x_k\}$ is meant to be empty if $k \leq 0$.

A cycle on n vertices will be denoted C_n and a path on n vertices and length $n-1$ will be denoted P_n. A *tadpole* $T_{[n,k]}$ (also called *dragon* or *kite* in the literature), is a graph obtained from a cycle C_n on n vertices by identifying a vertex of the cycle with a pendant vertex (or leaf) of a path of length $k \geq 1$. Equivalently, we will say that we *attach* a path with k vertices to the cycle by adding an edge connecting a vertex of the cycle with a leaf of the path. The tadpole $T_{[n,k]}$ has $n+k$ vertices and $n+k$ edges. Figures 4 and 4 give some examples of tadpoles. A tadpole $T_{[n,k]}$ has a perfect matching if and only if $n+k$ is even and a maximum matching of cardinality $(n+k-1)/2$, that leaves only one unmatched vertex, if $n+k$ is odd.

Let $|V| \leq |E|+1$ and let f be an *injective* function from V into $\{0, 1, \dots, |E|\}$. The *vertex-labelling* f induces an *edge-labelling* by allocating to the edge uv the label $|f(u) - f(v)|$. If this edge-labelling is injective, then f is called a *graceful labelling* for G, and the graph G is said to be *graceful* if such a labelling exists. The quantity $f(u) + f(v)$ will be called the *weight* of the edge uv relative to the labelling f.

Note that given the considered codomain $\{0, 1, \dots, |E|\}$ for f, the possible labels for the corresponding edge-labelling are from the set $\{1, 2, \dots, |E|\}$ and consequently, if f is graceful, then the induced edge-labelling is bijective. In general, the graceful labelling f itself is not bijective, unless $|V| = |E| + 1$ like, for instance, in trees. Note however that for any graceful labelling f, vertex-labels 0 and $|E|$ are necessarily used on two adjacent vertices in order to induce the edge-label $|E|$.

Graceful labellings of graphs were introduced in [9], mostly motivated by their link to graph decomposition. In this pioneer paper, it was conjectured that all trees are graceful; so far this conjecture, known as the *graceful tree conjecture*, remains open and has motivated a large amount of researches. Many papers in this area focus on various classes of trees and prove their gracefulness. This includes for instance paths and caterpillars (trees such that the removal of vertices of degree 1 leaves a path) [9] or lobsters with a perfect matching [8] (lobsters are trees such that the removal of vertices of degree 1 leaves a caterpillar) along with many other variants of trees (see [7] for details). It is straightforward to verify that stars are graceful.

Figure 1 presents an example of a graceful labelling of the path P_8 and two distinct graceful labellings for P_9, where integers in italic represent edge-labels. The first labelling of P_9 follows the same pattern as the one of P_8, alternating the minimum and maximum unused vertex-labels. The second labelling is derived from the labelling of P_8 adding a new vertex of label 8 adjacent to the previous leaf with label 0. Gracefulness of unicyclic graphs, that are obtained by adding an edge connecting two non-adjacent vertices of a tree, also raised a lot of interest across the research community (see, e.g., [3, 6, 7, 13]). In the present paper, we focus on tadpoles that are built from a path by adding an edge between a leaf and another vertex not yet adjacent to it.

The cycle C_n is graceful if and only if n is congruent to 0 or 3 modulo 4 [9]. Tadpoles $T_{[n,k]}$, with $n \geq 3$, and $k \geq 1$, are known to be graceful [13] and many other variants of *hairy cycles* obtained from a cycle by attaching non-crossing paths are proved to be graceful in [1]. In [2], it is shown that graphs obtained from a caterpillar by adding one edge are graceful, and in [13], it is conjectured that the non-graceful cycles C_n, that is when $n \equiv 1, 2 \pmod 4$, are the only examples of non-graceful unicyclic graphs.

Besides these researches on trees and unicyclic graphs, many other graph classes are known to be graceful. For more related results, we refer the reader to [7]. For practical and theoretical applications of graceful graphs, the reader is referred to [6, 10].

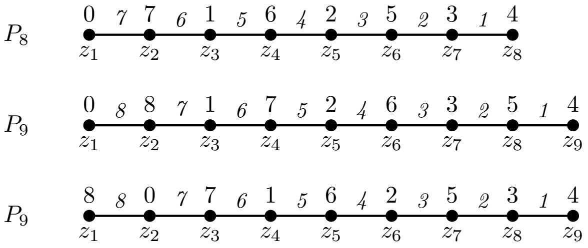

Figure 1. **Graceful labellings of P_8 and P_9.**

The notion of a *strongly graceful labelling* was introduced in [4]. Let G be a graceful graph with graceful labelling f and a perfect matching M. If in addition, we also have that $f(u) + f(v) = |E|$ for every $uv \in M$, then f is called a *strongly graceful labelling* of G, and the graph G is called *strongly graceful* if such a labelling and matching exist. Note that the graceful labelling of P_8 in Figure 1 is strongly graceful for the unique perfect matching. In Figure 1, we give an example of a strongly graceful labelling of the cycle on 12 vertices. The first motivation of strong gracefulness is the result that all trees are graceful if and only if all trees with a perfect matching are strongly graceful [4].

We introduce a generalisation of strongly graceful labellings we call nicely graceful. Given a graph with a maximum matching M, a graceful labelling f such that, for every $uv \in M$, the sum $f(u)+f(v) = W$ for some constant W is called *nicely graceful*. The constant W is then called the *weight* of the labelling relative to M. A graph with such a labelling for a given maximum matching is called a *nicely graceful graph of weight W*. We then refer to it as *nice gracefulness*. A graph is strongly graceful if and only if it has a perfect matching and is nicely graceful of weight $|E|$. The notion originally appeared for the proof of Theorem 2, but reveals to be interesting by itself.

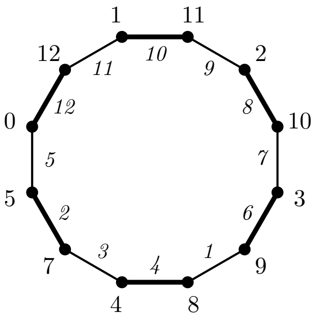

Figure 2. **A strongly graceful labelling of C_{12}; edges of the matching are outlined in bold.**

In this paper, we focus on strongly and nicely graceful properties of cycles and tadpoles. In Section 2, we establish some remarks that are the main ingredients of the proofs in the following sections. In Section 3, we characterise strongly and nicely graceful cycles for all possible weights. In Section 4, we characterise strongly graceful tadpoles and give some sufficient conditions for the existence of nicely graceful labellings of tadpoles. We present our conclusions in Section 5. The Table 1 sums-up the main results proved in the paper.

n	k	**Weight** $\lvert E\rvert$ Yes / No	**Weight** $\lvert E\rvert \pm 1$ Yes / No	**Weight** $\lvert E\rvert \pm 2$ Yes / No
Cycle C_n (see Th 1)				
$n \equiv 0 \pmod 4$ $n > 4$	N/A	Yes	N/A	N/A
$n = 4$		No		
$n \equiv 1 \pmod 4$		N/A (not graceful)		
$n \equiv 2 \pmod 4$				
$n \equiv 3 \pmod 4$		Yes	Yes	Only C_3
Tadpole $T_{[n,k]}$				
$n \equiv 0 \pmod 4$ $n > 4$	$k \equiv 0 \pmod 2$	Yes (Th 2)	N/A	N/A
	$k \equiv 1 \pmod 2$	Yes (Th 2)	Yes (Th 2)	No (Rmk 5)
$n = 4$	$k \equiv 0 \pmod 2$	No (Prop 6)	N/A	N/A
	$k \equiv 1 \pmod 2$	Yes (Th 2)	No (Prop 6)	Yes (Th 2)
$n \equiv 1 \pmod 4$	$k \equiv 0 \pmod 2$ $k \geq (n-1)/2$	Yes (Th 2)	No (Prop 4)	Only if $k = (n-1)/2$ (Th 3)
	$k \equiv 0 \pmod 2$ $k < (n-1)/2$	?		
	$k \equiv 1 \pmod 2$	No (Prop 3)	N/A	N/A
$n \equiv 1 \pmod 8$	$k = 2$	No (Prop 5)	No (see $n \equiv 1 \pmod 4$)	No (Th 3)
$n \equiv 2 \pmod 4$	$k \equiv 0 \pmod 2$	No (Prop 3)	N/A	N/A
	$k \equiv 1 \pmod 2$ $k \leq n/2 - 2$	Yes (Th 2)	Yes (Th 2)	No (Rmk 5)
	$k \equiv 1 \pmod 2$ $k > n/2 - 2$	?	?	
$n \equiv 3 \pmod 4$	$k \equiv 0 \pmod 2$	Yes (Th 2)	Yes (Th 2)	Only if $k \in \{(n-3)/2, (n+1)/2\}$ (Th 3)
	$k \equiv 1 \pmod 2$	Yes (Th 2)	N/A	N/A

Table 1. Nicely graceful cases covered in the paper. The symbol "?" corresponds to the cases we still do not know. When there is a perfect matching, nicely graceful for the weight $|E|$ corresponds to strongly graceful.

2. Some useful remarks

In this section, we present some key concepts and a few remarks we will use in the following sections.

Remark 1. The canonical labelling of a path.
Let $P_{n+1} = z_1 z_2 \dots z_{n+1}$ be a path on $n+1$ vertices. If n is odd then P_{n+1} contains a unique perfect matching $\{z_{2i+1}z_{2i+2} : i = 0, 1, \dots, (n-1)/2\}$ and the canonical strongly graceful labelling f of P_{n+1} (or canonical labelling for short) is defined as follows:

$$\begin{aligned} f(z_{2i+1}) &= i, & i &= 0, 1, \dots, \tfrac{n-1}{2}, \\ f(z_{2i+2}) &= n - i, & i &= 0, 1, \dots, \tfrac{n-1}{2}. \end{aligned} \tag{2.1}$$

We can similarly define a canonical labelling of an odd path P_{n+1} for n even such that:

$$\begin{aligned} f(z_{2i+1}) &= i, & i &= 0, 1, \dots, \tfrac{n}{2}, \\ f(z_{2i+2}) &= n - i, & i &= 0, 1, \dots, \tfrac{n}{2} - 1. \end{aligned} \tag{2.2}$$

In both cases, the edge-labelling induced by the canonical labelling is decreasing from n to 1. Note as well that, in the case of an odd path, the matching $\{z_{2i+1}z_{2i+2} : i = 0, 1, \dots, n/2 - 1\}$ is maximum and z_{n+1} is the only unmatched vertex. For every edge in the matching, the sum of the labels of its endpoints is constant and equal to n, similar to the strongly graceful case. The matching $\{z_{2i}z_{2i+1} : i = 1, 2, \dots, n/2\}$ is also maximum and z_1 is the only unmatched vertex. For every edge in this matching, the sum of the labels of its endpoints is also constant and equal to $n+1$. This is the idea of a nicely graceful labelling of a given weight. In the above examples, the weight is n and $n+1$, respectively.

Figure 1 represents the canonical labellings of P_8 and of P_9 as well as a second graceful labelling for P_9. We notice that these two graceful labellings of P_9 are nicely graceful relative to the two possible maximum matchings (of cardinality 4) $M_1 = \{z_1z_2, z_3z_4, z_5z_6, z_7z_8\}$ and $M_2 = \{z_2z_3, z_4z_5, z_6z_7, z_8z_9\}$. For the canonical labelling of P_9, the corresponding weight is 8 for M_1 and 9 for M_2; the corresponding weights are 8 and 7 for the second labelling of P_9.

Let f be a labelling of $G(V, E)$ from V to $\{0, 1, \dots, |E|\}$. The labelling f' from V to $\{0, 1, \dots, |E|\}$ defined such that $f'(v) = |E| - f(v)$ for every vertex $v \in V$ is called the *complementary labelling* of f. Note that the complementary of the complementary is the labelling itself. For instance, the two labellings of P_9 in Figure 1 are complementary.

Remark 2. The following statements hold.

- If f is graceful, then f' is graceful as well.
- If f is strongly graceful for a matching M, then so does f'.

- If f is nicely graceful for a matching M and the weight W, then f' is nicely graceful for M and the weight is $2|E| - W$.

Proof. The labelling f' is injective since the function $x \mapsto |E| - x$ is bijective from $\{0, 1, \ldots, |E|\}$ to $\{0, 1, \ldots, |E|\}$. Moreover, for any two vertices u, v in V we get

$$|f'(u) - f'(v)| = |(|E| - f(u)) - (|E| - f(v))| = |f(u) - f(v)|$$

and

$$f'(u) + f'(v) = (|E| - f(u)) + (|E| - f(v)) = 2|E| - f(u) - f(v).$$

□

Observe that the edges are assigned the same labels under f and f'. If f is nicely graceful for the weight $|E|$, then for any edge uv in the matching, $f'(u) = f(v)$ and $f'(v) = f(u)$ (see, e.g., the two labellings of P_9 in Figure 1).

Remark 3. Let f be a nicely graceful labelling of weight W for a maximum matching M of the graph $G(V, E)$. Then:

$$2|M| - 1 \leq W \leq 2|E| + 1 - 2|M|.$$

Proof. The maximum label of a matched vertex is at least $2|M| - 1$ and consequently the weight W is at least $2|M| - 1$. Assume by contradiction that a nicely graceful labelling f has a weight greater than $2|E| + 1 - 2|M|$, then the complementary labelling has a weight less than $2|M| - 1$, which contradicts the first inequality. □

In particular, if f is nicely graceful while $G(V, E)$ has a maximum matching M satisfying $|E| = 2|M| + 1$, then the only possible weights are between $|E| - 2$ and $|E| + 2$. The example of a star $S(V, E)$, with a central vertex linked to $|V| - 1$ leaves, illustrates that all possible values from $2|M| - 1$ to $2|E| + 1 - 2|M|$ are possible for the weight of nicely graceful labellings. A maximum matching M in the star is of cardinality 1 and consequently any graceful labelling is nicely graceful. A labelling from V to $\{0, 1, \ldots, |E|\}$ is graceful if and only if the central vertex has label 0 or $|E|$. In the former case, the weight ranges from $1 = 2|M| - 1$ to $|E|$, depending on the chosen maximum matching, while in the latter case it ranges from $|E|$ to $2|E| - 1 = 2|E| + 1 - 2|M|$.

Remark 4. If f is nicely graceful while G has a perfect matching, then the only possible weight is $|E|$ and consequently, f is strongly graceful.

Proof. Here, we just use the fact that there is a vertex of label $|E|$ to guarantee an edge of label $|E|$. Therefore $k \geq |E|$ and by using the complementary labelling, we get $k \leq |E|$. □

The following remark outlines that, in connected unicyclic graphs with a single unmatched vertex relative to a maximum matching, a nicely graceful labelling of weight $|E|-2$ can only exist in very specific cases. We will look deeper into this case in Theorem 3.

Remark 5. Let $G(V,E)$ be a graph with a maximum matching M satisfying $|E|=|V|=1+2|M|$, $|M|\geq 2$ and with a nicely graceful labelling of weight $|E|-2$. Denote u the unique unmatched vertex. It is of label $|E|$ and since all other vertices are matched, they have a label at most $|E|-2$. Therefore, the edges of label $|E|$ and $|E|-1$ are incident to u, say edge uv with v of label 0 and uw with w of label 1. Since $|E|=|V|=1+2|M|$, the matched vertices use all labels from 0 to $|E|-2$, while no vertex has the label $|E|-1$. So, v is adjacent to a vertex v' of label $|E|-2$ and w is adjacent to a vertex w' of label $|E|-3$. The edges vv' and ww' are in the matching and the vertices u,v,w,v',w' are all different; indeed, since $|E|\geq 5$, their labels, $|E|,0,1,|E|-2$ and $|E|-3$, are all different. Then, the only possible edges of label $|E|-3$ are $v'w$ (creating a cycle on 4 vertices $wuvv'$), vw' (creating a cycle on 4 vertices $vuww'$), or uz with z of label 3. Note that, if $|E|=5$, then $z=v'$ and if $|E|=6$, then $z=w'$, in all other cases, z,u,v,w,v',w' are all different. To sum-up, either the graph G has a tadpole $T_{[4,1]}$ as a subgraph or u has three distinct neighbours of label 0, 1 and 3.

The following Remark 6 plays a crucial role in what follows and is at the origin of many of the nicely graceful labellings we obtain. The remark outlines a labelling of a path obtained by merging a canonical labelling for a part of the path with the complementary of the canonical labelling for the second part. Then, we briefly explain how to use it to build nicely graceful labellings for some classes of tadpoles. We just give here the main idea and we provide extended details in the proofs of the results using this approach, in Theorem 1 and Theorem 2.

Remark 6. Consider a path $z_1z_2\dots z_{q+1}$ with $q+1$ vertices and q edges and consider the canonical labelling f of weight W for the considered matching. If $q+1$ is even, then f is strongly graceful with $W=q$. If $q+1$ is odd, then f is nicely graceful of weight $W=q$ for the maximum matching that matches all vertices but z_{q+1} or $W=q+1$ for the maximum matching that matches all vertices but z_1.

Let us select an edge z_iz_{i+1}, not in the matching and of label $\lambda\neq 1$; then we consider one of the two following transformations. Either we replace $f(y)$ with $W-f(y)$ for $z_{i+1},z_{i+2},\dots,z_{q+1}$ (in other words, we swap labels of the the end vertices of edges in the matching from z_{i+1}) or we do the same for $z_1,z_2,\dots,z_i$. The resulting vertex-labelling is still bijective while the resulting edge-labelling is obtained by replacing the label λ of the edge z_iz_{i+1} with 1 while all other edge-labels are preserved. Consequently, the new edge-labelling has two occurrences of the label 1 while no occurrence of the label λ. We will use these transformations for "saving" the label λ for another edge. Suppose for instance that, after applying one of these transformations, we delete the vertex z_{q+1} from the previous construction (to avoid two edge-labels 1), and either add an edge between the vertex z_1 of label 0 and the unique vertex of label λ or between z_q and a vertex of label $f(z_q)+\lambda$ or $f(z_q)-\lambda$. In both cases, we create a tadpole or cycle with a nicely graceful labelling of weight W. The transformations are constrained by the fact that the label λ is obtained for an edge not in the matching and for this reason, nicely graceful labelling can be obtained using this technique only for some tadpoles. The nicely graceful labellings we obtain for cycles in Theorem 1 are inspired by this technique and we use it directly for some cases in Theorem 2, as detailed in Table 2.

A tadpole $T_{[n,k]}$ can be obtained, from a path $z_1 z_2 \ldots z_{n+k}$, either by adding the edge $z_1 z_n$ (we call it the *out-labelling* case) or the edge $z_{k+1} z_{n+k}$ (we call it the *in-labelling* case). In the former case, the path $z_1 z_2 \ldots z_{n+k}$ visits all vertices of the cycle first and then leaves the cycle to the attached path. In the latter case, it first visits all vertices of the attached path before entering the cycle.

$n \geq 3$	k	**Weight** $n+k$ Out / In	**Weight** $n+k+1$ Out / In
$n \equiv 0 \pmod 4$	$k \equiv 1 \pmod 2$	Use an in-labelling *swap in the cycle*	
$n \equiv 1 \pmod 4$	$k \equiv 0 \pmod 2$, $k \geq (n-1)/2$	Use an out-labelling *swap in the path*	
$n \equiv 2 \pmod 4$	$k \equiv 1 \pmod 2$, $k \leq n/2 - 4$	Use an out-labelling *swap in the cycle*	Use an out-labelling *swap in the cycle*
	$k \equiv 1 \pmod 2$, $k = n/2 - 2$		

Table 2. Cases in Theorem 2, where we use Remark 6 to build a nicely graceful labelling of the considered tadpole.

Remark 7. Note that, for any two integers a and b, the quantities $a+b$ and $a-b$ have the same parity and consequently, for any nicely graceful labelling, the weights and the labels of edges in the matching have the same parity.

3. Gracefulness of cycles

In this part, we outline some properties of cycles regarding nice and strong gracefulness. It is proved in [12] (see Theorem 2 in the paper) that C_n is strongly graceful if and only if $n \equiv 0 \pmod 4$ and $n > 4$. Note that the fact that C_4 is not strongly graceful was not mentioned in that paper; this is now rectified. Here, we extend the result showing that graceful cycles with more than 4 vertices are actually nicely graceful for possible weights between $n-1$ and $n+1$.

If n is even, then the cycle has a perfect matching, and the nicely graceful labellings are actually strongly graceful according to Remark 4. If n is odd, then a maximum matching is of cardinality $(n-1)/2$ and consequently, the possible weights are from $n-2$ to $n+2$ as stated by Remark 3. However, Remark 5 tells us that the possible weights in the case of a cycle with more than 3 vertices are between $n-1$ and $n+1$.

Theorem 1. *The cycle C_n is nicely graceful if and only if $n \equiv 0 \pmod 4$, $n > 4$ or $n \equiv 3 \pmod 4$. Moreover, if $n \equiv 3 \pmod 4, n \geq 7$, then there are nicely graceful labellings of weights $n-1$, n and $n+1$ and if $n = 3$, then there are nicely graceful labellings of weights $1, 2, 3, 4$ and 5.*

Proof. The necessary condition comes from the fact that C_4 is not strongly graceful as noticed in Proposition 1 and that $C_n, n \equiv 1, 2 \pmod 4$ are not graceful as already mentioned.

For each case, we propose a maximum matching M and a related nicely graceful labelling for the considered weight. It is worth noting that the result holds for any maximum matching using a simple reordering of vertices. Let us first consider independently the case $n = 3$. If G is a cycle C_3 with vertices a, b and c, then the vertex labelling f with $f(a) = 0$, $f(b) = 1$ and $f(c) = 3$ is nicely graceful of weight 1 for the maximum matching $\{ab\}$, 3 for the maximum matching $\{ac\}$ and 4 for the maximum matching $\{bc\}$. The labelling g with $g(a) = 0$, $g(b) = 2$ and $g(c) = 3$ is nicely graceful of weight 2 for the matching $\{ab\}$ and 5 for the maximum matching $\{bc\}$.

For the other cases, we define an orientation of the cycle and denote the vertices $c_1, c_2, \ldots, c_n$ following this orientation. The idea is the same as in Remark 6, where the cycle C_n can be seen as a path P_{n+1} where the two leaves are identified. We describe completely how to obtain the labelling on the cycle since it gives an easy and direct method to build it.

Case 1. $n \equiv 0 \pmod 4$.

This case was shown in [12]. However, we propose a different proof and the same method will apply to the other cases with different initial conditions.

Since n is even, C_n has a perfect matching and the only possible weight is n. Then, to guarantee the weight n, all possible combinations for labels of the end vertices of the edges in the matching are $(n, 0), (n-1, 1), \ldots, (n/2+1, n/2-1)$ for the corresponding edge-labels $n, n-2, \ldots, 2$. We have here exactly $n/2$ combinations, all used and consequently, all vertex-labels but $n/2$ are used. We choose the perfect matching $M = \{c_{2i+1}c_{2i+2} : i = 0, 1, \ldots, n/2-1\}$. We choose the labels $n, n-2, \ldots, 2$ for the edges in the matching taken in this order (all even labels in decreasing order — see Remark 7) and all odd labels will be assigned to edges not in the matching. We just need to choose between $(n-i, i)$ or $(i, n-i)$ for each edge in the matching $c_{2i+1}c_{2i+2}$. The idea is to start with, say, $(i, n-i)$ for $i = 0, 1, \ldots, j$ and swap to $(n-i, i)$ for $i = j+1, j+2, \ldots, n/2-1$ for an index j that will be determined later. At this stage, the labels of c_1, c_2, c_{n-1}, c_n are determined: $f(c_1) = 0$, $f(c_2) = n$, $f(c_{n-1}) = n/2+1$ and $f(c_n) = n/2-1$. The label induced for the edge c_nc_1 is $n/2-1$, which is odd since $n \equiv 0 \pmod 4$. We determine j in order to have the edge-labels $n, n-1, \ldots, n/2, 1, n/2-2, \ldots, 2, n/2-1$ when following the cycle from c_1 to c_n. This is guaranteed by choosing $j = n/4$. So, formally, the corresponding labelling is defined by:

$$\begin{aligned}
f(c_{2i+1}) &= i, && i = 0, 1, \ldots, \tfrac{n}{4},\\
f(c_{2i+2}) &= n-i, && i = 0, 1, \ldots, \tfrac{n}{4},\\
f(c_{2i+1}) &= n-i, && i = \tfrac{n}{4}+1, \tfrac{n}{4}+2, \ldots, \tfrac{n}{2}-1,\\
f(c_{2i+2}) &= i, && i = \tfrac{n}{4}+1, \tfrac{n}{4}+2, \ldots, \tfrac{n}{2}-1.
\end{aligned}$$

Note that $n \geq 8$ guarantees that $n/4+1 \leq n/2-1$.

Case 2. $n \equiv 3 \pmod 4$ and the weight is n.

This time, since n is odd, a maximum matching in C_n is of cardinality $(n-1)/2$

with only one unmatched vertex. Let us choose as maximum matching $M = \{c_{2i+1}c_{2i+2} : i = 0, 1, \ldots, (n-3)/2\}$ such that c_n is unmatched. This time, the possible combinations for labels of the end vertices of the edges in the matching are $(n, 0), (n-1, 1), \ldots, (n-i, i), \ldots, ((n+1)/2, (n-1)/2)$. Since n is odd, the corresponding edge-labels are odd (Remark 7) and consist of all possible odd numbers from n down to 1. Contrary to the previous case, the number of pairs is one more than the cardinality of the matching and consequently, all pairs but one correspond to labels of the end vertices of edges in the matching. We choose to exclude the last pair, which means that labels of edges in the matching are odd numbers from n down to 3. The label 1 as well as the even labels will be allocated to edges not in the matching and the only two possible labels for c_n are $(n+1)/2$ and $(n-1)/2$. As previously, We define the labels of matched vertices by choosing the labels $(i, n-i)$ for the end vertices (c_{2i+1}, c_{2i+2}) of the ith edge in the matching, for $i = 0, 1, \ldots, j$, and swap to $(n-i, i)$ for $i = j+1, j+2, \ldots, (n-3)/2$. In particular, we have $f(c_1) = 0$, $f(c_2) = n$, $f(c_{n-1}) = (n-3)/2$, $f(c_{n-2}) = (n+3)/2$. The vertex c_n is adjacent to vertices of labels 0 and $(n-3)/2$, which is even since $n \equiv 3 \pmod 4$. So, we select the even label $(n+1)/2$ for c_n so that the two edges incident to c_n have the even labels 2 and $(n+1)/2$. The index j is determined to avoid a second edge with label $(n+1)/2$; this imposes $j = (n-3)/4$. It is positive since $n \geq 4$. So, the corresponding labelling is:

$$\begin{aligned}
f(c_{2i+1}) &= i, && i = 0, 1, \ldots, \tfrac{n-3}{4},\\
f(c_{2i+2}) &= n-i, && i = 0, 1, \ldots, \tfrac{n-3}{4},\\
f(c_{2i+1}) &= n-i, && i = \tfrac{n+1}{4}, \tfrac{n+5}{4}, \ldots, \tfrac{n-3}{2},\\
f(c_{2i+2}) &= i, && i = \tfrac{n+1}{4}, \tfrac{n+5}{4}, \ldots, \tfrac{n-3}{2},\\
f(c_n) &= \tfrac{n+1}{2},
\end{aligned}$$

that induces the edge-labels $n, n-1, \ldots, (n+3)/2, 1, (n-1)/2, (n-3)/2, \ldots, 2$ and $(n+1)/2$. Since $n > 4$ and $n \equiv 3 \pmod 4$, we have $n \geq 7$ and consequently $(n+1)/4 \leq (n-3)/2$.

Let us finally determine, with the same method, a nicely graceful labelling of weight $n-1$. A labelling of weight $n+1$ will then be deduced using the complementary labelling (see Remark 2).

Case 3. $n \equiv 3 \pmod 4$ and the weight is $n-1$.

We choose the same maximum matching as previously and edges in the matching will now have even labels. The possible combinations for labels of the end vertices of edges in the matching are $(n-1, 0), (n-2, 1), \ldots, (n-i-1, i), \ldots, ((n+1)/2, (n-3)/2)$ corresponding to the even edge-labels $n-1, n-3, \ldots, 2$. The only possibilities for the last vertex c_n are $f(c_n) = (n-1)/2$ or $f(c_n) = n$. We choose the latter one that induces the edge-labels n and $(n+3)/2$ for the two edges $c_1c_n, c_{n-1}c_n$ incident to c_n. We choose j in order to avoid a second edge of label $(n+3)/2$. This leads to $j = (n-7)/4$ (since $n \geq 4$ and $n \equiv 3 \pmod 4$ we have $n \geq 7$). The corresponding labelling is:

$$\begin{aligned} f(c_{2i+1}) &= i, & i &= 0, \ldots, \tfrac{n-7}{4}, \\ f(c_{2i+2}) &= n-i-1, & i &= 0, 1, \ldots, \tfrac{n-7}{4}, \\ f(c_{2i+1}) &= n-i-1, & i &= \tfrac{n-3}{4}, \tfrac{n+1}{4}, \ldots, \tfrac{n-3}{2}, \\ f(c_{2i+2}) &= i, & i &= \tfrac{n-3}{4}, \tfrac{n+1}{4}, \ldots, \tfrac{n-3}{2}, \\ f(c_n) &= n, \end{aligned}$$

that induces the edge-labels $n-1, n-2, \ldots, (n+5)/2, 1, (n+1)/2, (n-1)/2, \ldots, 2,$ $(n+3)/2$, n. As in the previous case we have $n \geq 7$ and consequently, $(n-3)/4 < (n-3)/2$ and $(n-7)/4 \geq 0$. □

The previous results outline the cycle C_4 as a particular case. The following result shows how C_4 may constraint the existence of nicely graceful labellings for some maximum matchings.

Proposition 1. *Let G be a graceful graph with a maximum matching M, and that contains a subgraph C_4 sharing two edges with M, then there is no nicely graceful labelling for this matching.*

Proof. Let M be a maximum matching for $G(V,E)$ and let ab and cd be two different edges of M. We show the result by contrapositive. Assume that f is a nicely graceful labelling for M. Then we have $f(a)+f(b) = f(c)+f(d)$ and consequently, $|f(a)-f(c)| = |f(b)-f(d)|$ and $|f(a)-f(d)| = |f(b)-f(c)|$. Since f is graceful, we cannot have ac and bd both in E and similarly, we cannot have ad and bc both in E. Consequently, a, b, c, d cannot be the vertices of a C_4 that is a subgraph of G. □

An immediate consequence is that C_4 itself is not strongly graceful. Note however that the graph formed by a cycle C_4 with vertices a, b, c, d and a fifth vertex e linked to a (called Tadpole $T_{[4,1]}$ in Section 4) is nicely graceful for the weight 5 and maximum matching $\{bc, ae\}$ but not for the matching $\{ab, cd\}$ according to the previous result. We discuss later (see Theorem 2) the case of tadpoles with a complete characterisation of when they are strongly graceful. In [11] Solairaju and Malathi define the class of graphs obtained by connecting n copies of C_4 as follows. Denote by $v_{i,1}, v_{i,2}, v_{i,3}, v_{i,4}$ the four consecutive vertices of the ith copy of C_4 and join by an edge $v_{i,1}$ with $v_{i+1,1}$ and $v_{i,3}$ with $v_{i+1,3}$ for $i = 1, 2, \ldots, n-1$. They show that these graphs are graceful. But, based on Proposition 1, these graphs can not be strongly graceful since a perfect matching necessarily shares two edges with each copy of C_4.

A graph is called *unicyclic* if it contains exactly one cycle. In this paper we are only dealing with connected unicyclic graphs. Such graphs have as many edges as vertices. Consider a graph $G(V,E)$ and a matching $M \subset E$. We introduce the graph $G_M = (V_M, E_M)$ obtained from G by contracting (see [5]) each edge in M into a single vertex and replacing parallel edges that may arise with a single edge. More formally, $V_M = M \cup U_M$, where U_M is the set of vertices that are unmatched by M and:

For every $u, v \in U_M$, $uv \in E_M \Leftrightarrow uv \in E$.
For every $u \in U_M$, $m = ab \in M$, $um \in E_M \Leftrightarrow E \cap \{ua, ub\} \neq \emptyset$.
For every $m, m' \in M$, $m = ab$, $m' = a'b'$, $mm' \in E \Leftrightarrow E \cap \{aa', ab', ba', bb'\} \neq \emptyset$.

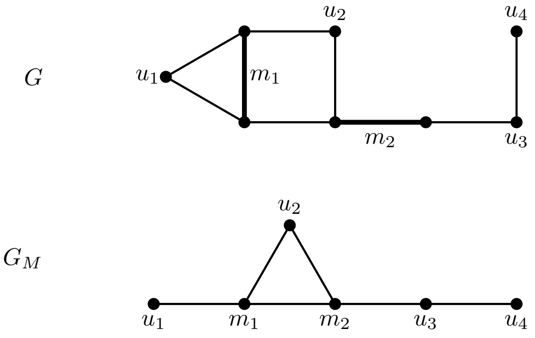

Figure 3. **An example of the graph G_M obtained from the graph G and the matching $M = \{m_1, m_2\}$. We have $U_M = \{u_1, u_2, u_3, u_4\}$.**

Figure 3 gives an example of a graph G_M for some graph G and matching M. The following result will be helpful on various occasions:

Proposition 2. *Let $G(V, E)$ be such that $|V| = |E|$ and M be a maximum matching in G satisfying $|M| \geq (|V| - 1)/2$. If there is a strongly graceful labelling for M in G or a nicely graceful labelling of weight $|E| - 1$ or $|E| + 1$, then G_M is bipartite.*

Proof. In both considered cases, all edges in M have an even label and since there are exactly $|M|$ even numbers from 1 to $|E|$, all edges not in M have an odd label. So, we can partition $M = M^e \cup M^o$ where the end vertices of edges in M^e have an even label while the end vertices of edges in M^o have an odd label. Any edge in G_M between two elements of M corresponds, in G, to at least one edge in $E \setminus M$ linking two end vertices of edges in the matching with labels of different parity. Similarly, $U_M = U_M^e \cup U_M^o$ depending of the parity of the related vertex-label. Under our hypotheses, there is at most one unmatched vertex ($|U_M^o| + |U_M^e| \leq 1$). Any edge, in G, between an unmatched vertex and a matched one links vertices with labels of different parity. Hence, edges in G_M are between $M^e \cup U_M^e$ and $M^o \cup U_M^o$, which concludes the proof. □

Remark 8. Let G be a graph with a matching M and a cycle C_p on p vertices. We denote $E(C_p) \subset E$ the edge set of C_p. The cycle C_p induces, in G_M, a cycle with $p - |E(C_p) \cap M|$ vertices.

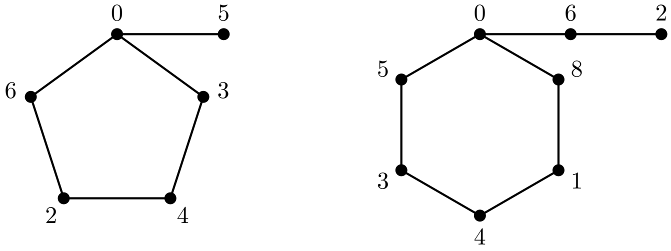

Figure 4. **Graceful labellings of $T_{[5,1]}$ and $T_{[6,2]}$.**

This leads to the following immediate corollary.

Corollary 1. *Assume that the conditions of Proposition 2 hold and there is a cycle C_p in G. Then, p and $|E(C_p) \cap M|$ have the same parity.*

Proof. Under the considered hypotheses G_M is bipartite; using Remark 8, p and $|E(C_p) \cap M|$ have the same parity to ensure that the corresponding cycle in G_M is even. □

In Section 4, this corollary is used to identify non-nicely graceful cases of tadpoles.

4. Gracefulness of tadpoles

We remind that, for any two integers $n \geq 3$ and $k \geq 0$, a tadpole $T_{[n,k]}$ is a graph formed by a cycle C_n on n vertices and one vertex of the cycle is identified with a leaf of a path of length k (so, with $k+1$ vertices). If $k = 0$, then the tadpole is just a cycle and if $k \geq 1$, then the vertex identified with a leaf of the path is of degree 3 (we will denote it t) and the other leaf of the path is of degree 1 (we will denote it s). All other vertices are of degree 2. A tadpole $T_{[n,k]}$ has $n+k$ vertices and $n+k$ edges and, as already mentioned; it has a maximum matching of cardinality $\lfloor (n+k)/2 \rfloor$.
Recall that all tadpoles $T_{[n,k]}$ with $k \geq 1$ are graceful [9]. In what follows we give some sufficient conditions for tadpoles not to be nicely graceful as well as sufficient conditions for tadpoles to be nicely graceful. In all, it gives necessary and sufficient conditions for tadpoles to be strongly graceful while some cases remain open regarding nice gracefulness.
In Figure 4, we depict two examples of graceful labellings of tadpoles, one with the cycle length congruent to 1 modulo 4 and one with the cycle length congruent to 2 modulo 4. In both cases, there is a perfect matching but the proposed labellings are not strongly graceful. Proposition 3 below shows that, for these two cases, there is no strongly graceful labelling.

Proposition 3. *Tadpoles $T_{[n,k]}$ with $n+k$ even and $n \equiv 1,2 \pmod 4$ are not strongly graceful.*

Proof. Suppose first $n \equiv 2 \pmod 4$ and k is even. The related tadpole has a perfect matching and any perfect matching shares $n/2$ edges with the cycle. Since $n/2$ is odd, Corollary 1 ensures that no strongly graceful labelling can exist.
Suppose now that $n \equiv 1 \pmod 4$ and k is odd. The tadpole still have a perfect matching and any perfect matching shares $(n-1)/2$ edges with the cycle. Since $(n-1)/2$ is even, Corollary 1 ensures that no strongly graceful labelling can exist. ☐

Similarly, we establish other criteria of non-existence of nicely graceful labelling:

Proposition 4. *Consider a tadpole $T_{[n,k]}$ with $n+k$ odd.*

1. *If $n \equiv 2 \pmod 4$, then there is no nicely graceful labelling of weight $n+k-1$ or $n+k+1$ for a matching that matches all vertices of the cycle.*
2. *If $n \equiv 0 \pmod 4$, then there is no nicely graceful labelling of weight $n+k-1$ or $n+k+1$ for a matching with exactly one unmatched vertex on the cycle.*
3. *If $n \equiv 1 \pmod 4$, then there is no nicely graceful labelling of weight $n+k-1$ or $n+k+1$.*

Proof. Suppose first that $n \equiv 2 \pmod 4$ and k is odd. Assume there is a maximum matching such that all vertices in the cycle are matched. Since the cycle is even, and only one vertex in the cycle is of degree more than 2, $n/2$ edges of the matching are actually edges of the cycle. Since $n/2$ is odd, Corollary 1 ensures that no nicely graceful labelling can exist for the weight $n+k-1$ or $n+k+1$.
Suppose now $n \equiv 0 \pmod 4$ and k is odd and consider a maximum matching such that one vertex in the cycle is unmatched. Then all other vertices in the cycle are matched and consequently, the matching shares $(n-2)/2$ edges with the cycle. Here, $(n-2)/2$ is odd and Corollary 1 ensures that no nicely graceful labelling can exist for the weight $n+k-1$ or $n+k+1$.
Suppose finally that $n \equiv 1 \pmod 4$ and k is even. The cycle necessarily shares $(n-1)/2$ edges with the maximum matching but $(n-1)/2$ is even while the cycle is odd. Corollary 1 ensures that no nicely graceful labelling can exist for the weight $n+k-1$ or $n+k+1$. ☐

Proposition 5. *If $n \equiv 1 \pmod 8$, then $T_{[n,2]}$ has no nicely graceful labelling of weight $n+2$.*

Proof. Suppose by contradiction that there is a nicely graceful labelling f of weight $n+2$. Let M be a maximum matching; note first that M shares $(n-1)/2$ edges with the cycle. We denote the cycle $c_1c_2\dots c_n$, where edges $c_{2i-1}c_{2i}$, $i = 1,2,\dots,(n-1)/2$ are in M. The last vertex c_n is either unmatched or there is a vertex x outside the cycle such that $c_nx \in M$. In all, $|M| = (n+1)/2$.

As $|E| = n+2$ is odd, all edges in M have an odd label. Since there are $(n+3)/2$ possible odd labels from 1 to $n+2$, exactly one edge not in the matching has an odd label and all other edges not in the matching have an even label. For any graceful labelling, the sum of labels of edges in the cycle is even (see [9], Lemma 1); since $(n-1)/2 \equiv 0 \pmod 2$ and at most one edge not in the matching has an odd label, all edges in the cycle but not in the matching have an even label. So, the labels of edges not in the matching and in the cycle are $\{2i : i = 1, 2, \ldots, (n+1)/2\}$. We denote $\lambda_i = |f(c_{2i+1}) - f(c_{2i})|$, $i = 1, 2, \ldots, (n-1)/2$ and $\lambda_0 = |f(c_1) - f(c_n)|$ the labels of edges in the cycle but not in the matching. For any $j = 0, 1, \ldots, (n-5)/4$, $c_{4j+1}c_{4j+2}$ and $c_{4j+3}c_{4j+4}$ are edges in the matching and consequently, $\lambda_{2j+1} = |f(c_{4j+4}) - f(c_{4j+1})|$. Since

$$f(c_1) - f(c_n) + \sum_{j=0}^{\frac{n-5}{4}} [f(c_{4j+4}) - f(c_{4j+1}) + f(c_{4j+5}) - f(c_{4j+4})] = 0,$$

there are numbers $\varepsilon_i \in \{-1, 1\}$ such that

$$\sum_{i=0}^{\frac{n-1}{2}} \varepsilon_i \lambda_i = 0$$

or equivalently

$$\sum_{i=0}^{\frac{n-1}{2}} \varepsilon_i \frac{\lambda_i}{2} = 0.$$

But $\{\lambda_i/2 : i = 0, 1, \ldots, (n-1)/2\} = \{1, 2, \ldots, (n+1)/2\}$ with an odd number of odd indexes. Consequently, the quantity $\sum_{i=0}^{(n-1)/2} \varepsilon_i \lambda_i/2$ is odd and thus cannot be 0, a contradiction. This completes the proof. □

We get a special case when $n = 4$. An immediate consequence of Proposition 1 is that there is no nicely graceful labelling for a matching with two edges in the C_4. So, $T_{[4,2p]}$ is not strongly graceful since any perfect matching will share two edges with the C_4. Moreover, due to Proposition 4, for $T_{[4,2p+1]}$ and the weight $4+2p$ or $6+2p$, there is no nicely graceful labelling for a matching sharing only one edge with the C_4. The following proposition sums up the negative results for a tadpole with a C_4.

Proposition 6. *Let* $p \in \mathbb{N}$, $T_{[4,2p+1]}$ *is not nicely graceful for the weight* $4+2p$ *or* $6+2p$ *and* $T_{[4,2p]}$ *is not strongly graceful.*

In the next theorem, we identify some nicely graceful tadpoles that can be obtained as direct applications of Remark 6 (see Table 2). We completely detail the method for the last three cases and show different arguments we find interesting for the first cases.

Theorem 2. *The following conditions are sufficient for tadpoles* $T_{[n,k]}$ $(n \geq 3)$ *to be nicely graceful:*

1. $n \equiv 3 \pmod 4$ *for the weights* $n+k$ *and* $n+k\pm 1$ *if* k *is odd;*
2. $n \equiv 0 \pmod 4$, $n > 4$ *for the weights* $n+k$ *and* $n+k\pm 1$ *if* k *is odd;*
3. $n = 4$, $k \equiv 1 \pmod 2$ *for the weights* $n+k$ *and* $n+k\pm 2$*;*
4. $n \equiv 1 \pmod 4$, $k \equiv 0 \pmod 2$, $k \geq (n-1)/2$, *for the weight* $n+k$*;*
5. $n \equiv 2 \pmod 4$, $k \equiv 1 \pmod 2$, $k \leq n/2-2$, *for the weights* $n+k$ *and* $n+k\pm 1$.

Proof. The case $k=0$ corresponds to Theorem 1. Consider G, a tadpole $T_{[n,k]}$ obtained from the cycle $C_n = c_1c_2\dots c_n$ by attaching a path $x_1x_2\dots x_k$ to a vertex denoted t on the cycle and using either the edge tx_1 or tx_k. We define below a labelling f in the different cases that all can be seen as an application of Remark 6. We will use it directly for Case 2 (k odd) and for Cases 4 and 5. For Cases 1 and 2 with k even, we prefer showing how to build a nicely graceful labelling of the tadpole G from a nicely graceful labelling of the cycle. All these ideas might be transferable to other classes of graphs.

Case 1. $n \equiv 3 \pmod 4$.
We choose $t = c_n$ and we assume that tx_k is an edge of G, and consequently x_1 is of degree 1. We consider a maximum matching M_n of the cycle such that t is unmatched. We consider two cases whether k is even or odd. If k is even, then we consider the possible weights, $n+k$ and $n+k+1$ since the case $n+k-1$ will be deduced using the complementary labelling. So, we denote the weight $n+k+\varepsilon$ with $\varepsilon \in \{0,1\}$.
Using Theorem 1, for every $\varepsilon \in \{0,1\}$, there is a nicely graceful labelling g_ε of the cycle C_n of weight $n+\varepsilon$ with respect to the matching M_n. Given our choice of M_n and $t = c_n$, the labellings proposed in the proof of Theorem 1 plus, for $\varepsilon = 1$, Remark 6, imply $g_0(t) = (n+1)/2$ and $g_1(t) = 0$ (recall that in Theorem 1, the proposed labelling had the weight $n-1$) and this holds as well for $n=3$. We then consider the labelling $f_\varepsilon = g_\varepsilon + k/2$ obtained from g_ε by adding the constant $k/2$ for each vertex of the cycle. This labelling uses vertex-labels between $k/2$ and $n+k/2$ and the related edge-labels are $\{1,2,\dots,n\}$. In addition, for each edge in M_n, the induced weight is $n+k+\varepsilon$. We complete this labelling for the whole graph G as follows:

$$\begin{aligned} f_\varepsilon(x_{2i+1}) &= i, & i &= 0,1,\dots,\tfrac{k}{2}-1,\\ f_\varepsilon(x_{2i+2}) &= n+k-i, & i &= 0,1,\dots,\tfrac{k}{2}-1. \end{aligned}$$

Corresponding edge-labels are $\{n+1, n+2, \dots, n+k\}$ while the used vertex-labels are in $\{1,2,\dots,k/2-1\} \cup \{n+k/2+1, n+k/2+2, \dots, n+k\}$.
The matching M_n can be extended, either with $M_1 = \{tx_k, x_{2i+1}x_{2i} : i = 1,2,\dots, k/2-1\}$ or with $M_0 = \{x_{2i}x_{2i-1} : i = 1,2,\dots,k/2\}$, to make a maximum matching

of G. In the former case, x_1 is unmatched by the resulting maximum matching and the weight of edges in M_1 is $n+k+1$ while in the latter case the vertex t will be unmatched and edges in M_0 have the weight $n+k$. So, in all cases, f_ε is nicely graceful for the weight $n+k+\varepsilon$ and the matching $M_n \cup M_\varepsilon$. The complementary labelling of f_1 is nicely graceful of weight $n+k-1$ for the matching $M_n \cup M_1$.

Suppose now that k is odd, which makes $k+n$ even. For the perfect matching we choose $M_n \cup \{x_{2i+1}x_{2i+2} : i = 0,1,\ldots,(k-3)/2\} \cup \{x_k t\}$.

As in the previous case, we consider a nicely graceful labelling g_1 of the cycle C_n of weight $n+1$ for the matching M_n and we define $f = g_1 + (k-1)/2$.

We complete f as follows:

$$\begin{aligned} f(x_{2i+1}) &= i, & i &= 0,1,\ldots,\tfrac{k-1}{2},\\ f(x_{2i+2}) &= n+k-i, & i &= 0,1,\ldots,\tfrac{k-1}{2}-1. \end{aligned}$$

Thus, f is a strongly graceful labelling for the whole tadpole $T_{[n,k]}$ (the weight is $n+k$).

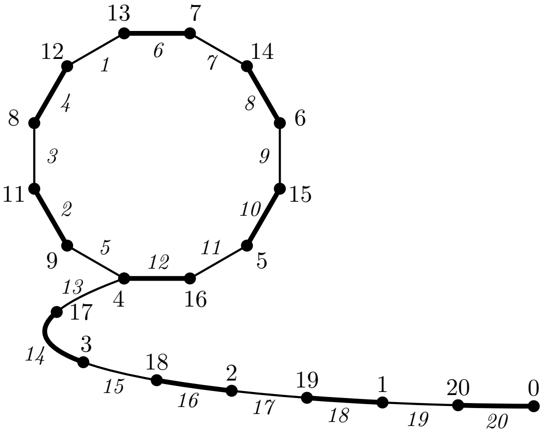

Figure 5. **The tadpole $T_{[12,8]}$ with a strongly graceful labelling (in italic are edge-labels and edges in the matching are outlined in bold).**

Case 2. $n \equiv 0 \pmod 4, n > 4$.

As previously, we assume that tx_k is an edge of G. Let M_n be a perfect matching of the cycle. Assume first that k is even and consider the perfect matching of G defined as: $M = M_n \cup \{x_{2i+1}x_{2i+2} : i = 0,1,\ldots,k/2-1\}$. As previously, we consider a strongly graceful labelling f of the cycle for this matching, ensuring that t is labelled 0, then we add $k/2$ to each vertex-label and we complete this labelling for the whole graph as follows:

$$\begin{aligned} f(x_{2i+1}) &= i, & i &= 0,1,\ldots,\tfrac{k}{2}-1,\\ f(x_{2i+2}) &= n+k-i, & i &= 0,1,\ldots,\tfrac{k}{2}-1. \end{aligned}$$

This defines a strongly graceful labelling for the tadpole $T_{[n,k]}$ and the perfect matching M (see Figure 4), which concludes the case $n \equiv 0 \pmod 4$, $n > 4$ and k even. Note in addition that the vertex of degree 1, $x_1 = s$, is labelled 0. As a consequence, if we add one vertex only connected to it, to create a tadpole $T_{[n,k+1]}$; then the previous labelling can be completed by labelling the new vertex with $n+k+1$ to create a nicely graceful labelling of the tadpole $T_{[n,k+1]}$ for the maximum matching M. This concludes the proof for $n \equiv 0 \pmod 4$, $n > 4$ and k odd for the weight $n+k-1$ and also for the weight $n+k+1$ using the complementary labelling. Note that the previous method does not work for $n = 4$ because C_4 is not strongly graceful.

We now focus on $n \equiv 0 \pmod 4), n > 4$, k odd for the weight $n+k$ to conclude Case 2. We cannot directly apply the method used for the case $n \equiv 3 \pmod 4$ because a nicely (thus strongly) graceful labelling of the cycle C_n, $n \equiv 0 \pmod 4$, $n > 4$ has necessarily the weight n. Then, if k is odd, we cannot reach the weight $n+k$ by adding the same constant to each vertex-label on the cycle. So, we need a special proof for Case 2 with $n \equiv 0 \pmod 4$, k odd and the weight $n+k$.

We do a direct proof obtained with a modified version of Remark 6. As previously, the cycle is denoted $c_1c_2\dots c_n$ and we choose $t = c_1$ and tx_k is an edge of G (so, $s = x_1$). We consider the maximum matching $M = \{x_{2i+1}x_{2i+2} : i = 0, 1, \dots, (k-3)/2\} \cup \{x_kc_1\} \cup \{c_{2i}c_{2i+1} : i = 1, 2, \dots, (n-2)/2\}$ such that c_n is the only unmatched vertex. The idea is to use the transformation in Remark 6 on the path $P = x_1x_2\dots c_1c_2\dots c_nx$, where x is an additional vertex. P has $n+k+1$ vertices and $M \cup \{c_nx\}$ is a perfect matching of P. We start with the canonical labelling of P, of weight $n+k$, and apply Remark 6 for $\lambda = n/2$. Note that $n/2$ is even and thus, it corresponds to the label of an edge not in the matching. After the transformation, the vertex x has the label $(n+k-1)/2$ and c_n the label $(n+k+1)/2$. Considering the path $P' = x_1x_2\dots c_1c_2\dots c_n$, obtained from P by removing x, we get the following vertex-labelling f of P':

$$\begin{aligned}
f(x_{2i+1}) &= i, && i = 0, 1, \dots, \tfrac{k-1}{2},\\
f(x_{2i+2}) &= n+k-i, && i = 0, 1, \dots, \tfrac{k-3}{2},\\
f(c_1) &= n + \tfrac{k+1}{2}, &&\\
f(c_n) &= \tfrac{n+k+1}{2}, &&\\
f(c_{2i}) &= \tfrac{k-1}{2} + i, && i = 1, 2, \dots, j,\\
f(c_{2i+1}) &= n + \tfrac{k+1}{2} - i, && i = 1, 2, \dots, j,\\
f(c_{2i}) &= n + \tfrac{k+1}{2} - i, && i = j+1, j+2, \dots, \tfrac{n}{2} - 1,\\
f(c_{2i+1}) &= \tfrac{k-1}{2} + i, && i = j+1, j+2 \dots, \tfrac{n}{2} - 1,
\end{aligned}$$

where j is such that:

$$\left(n + \tfrac{k+1}{2} - j\right) - \left(\tfrac{k-1}{2} + j+1\right) = \tfrac{n}{2} \Leftrightarrow j = \tfrac{n}{4}.$$

As outlined in Remark 6, this labelling for P' is injective as well the corresponding edge-labelling with edge labels $1, 2, \dots, n/2-1, n/2+1, \dots, n+k$. The tadpole G

is obtained from P' by adding the edge c_nc_1 and since $|f(c_n) - f(c_1)| = n/2$, f is a graceful labelling for G. By construction, all edges in M have the weight $n + k$, and consequently, f is nicely graceful of weight $n + k$ for G. This concludes the proof of Case 2.

Case 3. $n = 4$, $k \equiv 1 \pmod 2$ for the weights $n + k$ and $n + k \pm 2$.
The previous method for the second part of Case 2 works as well if $n = 4$, and for the weight $k + 4$. Assume now that the weight is $k + 2$ and we then derive the case $k + 6$ by using the complementary labelling. Note that the tadpole $T_{[4,k]}$ can be obtained from a path $P = x_1x_2 \ldots x_{k+3}$ by adding a vertex x_0 linked to x_1 and x_3. Consider the canonical labelling of P from x_1 to x_{k+3} which is strongly graceful. The related maximum matching is $M = \{x_{2i+1}x_{2i+2} : i = 0, 1, \ldots, (k + 1)/2\}$ and the weight is $k + 2$. Complete this labelling by assigning to x_0 the label $k + 4$ to get the required labelling. This concludes the proof of the first three cases.
The following case directly illustrates Remark 6.

Case 4. $n \equiv 1 \pmod 4$, $k \equiv 0 \pmod 2), k \geq (n - 1)/2$, for the weight $n + k$.
Note that, according to Theorem 1, the case $k = 0$ is not nicely graceful, so a condition on k is required. This time, we suppose that $t = c_n$ and tx_1 is an edge of G and consequently, x_k is of degree 1.
We add one vertex x_{k+1} linked to x_k to create a path P with a perfect matching. We then apply Remark 6 using an out-labelling and the maximum matching

$$M = \left\{c_{2i+1}c_{2i+2} : i = 0, 1, \ldots, \tfrac{n-3}{2}\right\} \cup \{tx_1\} \cup \left\{x_{2i}x_{2i+1} : i = 1, 2, \ldots, \tfrac{k}{2} - 1\right\},$$

such that all vertices but x_k are matched.
Let $P = c_1c_2 \ldots c_nx_1 \ldots x_{k+1}$. We start from the canonical labelling of P with $n + k$ edges. Note that, since $n \equiv 1 \pmod 4$, in this labelling $t = c_n$ has the label $(n - 1)/2$ and the edge tx_1 has the label $k + 1$. So, in the edge labelling of G induced by the canonical labelling of P, the edge c_1t has the label $(n - 1)/2$, which is even. Therefore the same label appears on an edge not in the matching. Since $k \geq (n - 1)/2$, this is on the path $x_1x_2 \ldots x_k$. Consequently, we can use Remark 6 with a swap along the path, leading to a nicely graceful labelling of weight $n + k$, which concludes the case.

Case 5. $n \equiv 2 \pmod 4$, $k \equiv 1 \pmod 2$, $k \leq n/2 - 2$, for the weight $n + k$ or $n + k \pm 1$.
We first consider the weight $n + k$ and let us first assume $k \leq n/2 - 4$. As in the previous case, we suppose that $t = c_n$ and tx_1 is an edge of G. We add one vertex x_{k+1} linked to x_k to create a path P with a perfect matching. We then apply Remark 6 using an out-labelling and the maximum matching

$$M = \left\{c_{2i+1}c_{2i+2} : i = 0, 1, \ldots, \tfrac{n}{2} - 1\right\} \cup \left\{x_{2i+1}x_{2i+2} : i = 0, 1, \ldots, \tfrac{k-3}{2}\right\},$$

such that all vertices but x_k are matched. Let $P = c_1c_2 \ldots c_nx_1 \ldots x_{k+1}$. We start from the canonical labelling of P with $n + k$ edges and the weight $n + k$, using the out-labelling method from Remark 6. In this labelling, the edge in the matching

$c_{n-1}c_n$ has the label $k+2$ and the vertex c_{n-1} has the label $n/2-1$. If there is a swap on the cycle, then $t=c_n$ will get the label $n/2-1$ as well as the edge c_1t. Since $n/2-1$ is even, an edge not in the matching has this label in the canonical labelling. Since $k \leq n/2-4$, we have $k+2 < n/2-1$, which confirms that a swap in the cycle will allow to replace the second edge-label $n/2-1$ with 1 using Remark 6. It produces a nicely graceful labelling of weight $n+k$.

To conclude Case 5 for the weight $n+k$, we finally assume that $k=n/2-2$ (recall that k is odd). For this case, we choose $t=c_1$ and

$$M = \left\{c_{2i+1}c_{2i+2} : i=0,1,\dots,\tfrac{n}{2}-1\right\} \cup \ \left\{x_{2i+1}x_{2i+2} : i=0,1,\dots,\tfrac{k-3}{2}\right\}.$$

We use a similar but slightly different method than Remark 6 with an out-labelling for the weight $n+k$. The difference is that we label the vertices from c_1 to c_n and then from x_1 to x_k, which corresponds to a walk and not to a path. A swap on the cycle leads to the labelling described below. We then choose j to avoid two identical edge labels:

$$\begin{aligned}
f(c_{2i+1}) &= i, && i=0,1,\dots,j,\\
f(c_{2i+2}) &= n+k-i, && i=0,1,\dots,j,\\
f(c_{2i+1}) &= n+k-i, && i=j+1,j+2,\dots,\tfrac{n}{2}-1,\\
f(c_{2i+2}) &= i, && i=j+1,j+2,\dots,\tfrac{n}{2}-1,\\
f(x_{2i+1}) &= \tfrac{n}{2}+k-i, && i=0,1,\dots,\tfrac{k-3}{2},\\
f(x_{2i+2}) &= \tfrac{n}{2}+i, && i=0,1,\dots,\tfrac{k-3}{2},\\
f(x_k) &= \tfrac{n+k+1}{2}.
\end{aligned}$$

Note that $f(c_n)=n/2-1=k+1$, $f(x_1)-f(c_1)=n/2+k$ and the edge $c_{n-1}c_n$ has the label $n/2=k+2$. As a consequence, we choose j such that:

$$f(c_{2j+2})-f(c_{2j+1}) = \tfrac{n}{2}+k+1 \Leftrightarrow j=\tfrac{n-2}{4}.$$

Then, using the same arguments as in Remark 6, we verify that the labelling f is injective, the labels of edges in the cycle are $\{n+k, n+k-1,\dots,k+2\} \cup \{k+1,1\} \setminus \{n/2+k\}$ and the labels of the edges in the path are $\{n/2+k\} \cup \{k,k-1,\dots,2\}$. This shows that the labelling f is graceful and edges in M have all the weight $n+k$, which concludes the proof for the weight $n+k$.

We will show that G has a nicely graceful labelling for the weight $n+k+1$ and the case with weight $n+k-1$ will be deduced using the complementary labelling. We choose again $t=c_n$ and we assume that tx_1 is an edge of G. Consider the matching

$$M = \left\{c_{2i}c_{2i+1} : i=1,2,\dots,\tfrac{n}{2}-1\right\} \cup \{tx_1\} \cup \left\{x_{2i}x_{2i+1} : i=1,2,\dots,\tfrac{k-1}{2}\right\},$$

which is a maximum matching such that all vertices of G but c_1 are matched.

We apply the method in Remark 6 for the weight $n + k + 1$ using an out-labelling method for the weight $n + k + 1$. We consider the path $P = c_1c_2 \dots c_nx_1 \dots x_kx_{k+1}$ for a virtual additional vertex x_{k+1}.
The reasoning is similar to the argument for the weight $n+k$ with the difference that the edge $c_{n-1}c_n$ is now an edge not in the matching, tx_1 is in the matching and all edges in the matching have even labels. In the canonical labelling for P, c_1 has the label 0, c_{n-1} has the label $n/2 - 1$, $t = c_n$ has the label $n/2 + k + 1$, x_1 has the label $n/2$ and the edge tx_1 has the label $k + 1$. Using Remark 6, if we do a swap on the cycle, then t and the edge c_1t will both get the label $n/2$, which is odd. So, another edge not in the matching has the label $n/2$ and since $n/2 > k + 1$, this is an edge on the cycle. So, using Remark 6, a swap from this edge will produce the expected labelling, which concludes the proof. □

Combining the sufficient conditions and the necessary conditions stated before leads to necessary and sufficient conditions for strong gracefulness of tadpoles but it leaves open some cases with $n \equiv 1, 2 \pmod 4$ for nice gracefulness.

Corollary 2. *A tadpole $T_{[n,k]}$ is strongly graceful if and only if $n \equiv 0 \pmod 4$, $n > 4$ or $n \equiv 3 \pmod 4$.*

We conclude this section with the characterisation of nicely graceful tadpoles $T_{[n,k]}$ for n odd and the weight $n + k \pm 2$.

Theorem 3. *A tadpole $T_{[n,k]}$ for odd $n \geq 3$ is nicely graceful for the weight $n + k \pm 2$ if and only if either $n \equiv 1 \pmod 4$, $n \geq 5$ and $k = (n-1)/2$ or $n \equiv 3 \pmod 4$ and $k \in \{(n-3)/2, (n+1)/2\}$.*

Proof. The proof is different than the proof techniques we used in the previous theorems. The idea is as follows: let $G(V, E)$ be a tadpole $T_{[n,k]}$ for an odd n and assume it has a nicely graceful labelling f, for the weight $n + k - 2$ and the maximum matching M. The case of the weight $n+k+2$ will be derived using the complementary labelling of f. Then, k is necessarily even to ensure that a maximum matching lets one vertex unmatched. If $k = 0$, then G is a cycle and Theorem 2 ensures that only C_3 is nicely graceful for the weight $n + k - 2$. So, we can assume that $k \geq 2$. Edges in M necessarily have odd labels from 1 to $n+k-2$, one edge not in M has the label $n+k$ and all other edges not in M have even labels from 2 to $n+k-1$. We know, using Remark 5, that G is not a cycle and we denote t the unique vertex of degree 3; Remark 5 induces that t is the only unmatched vertex. We know as well that $f(t) = n + k$ and that $n + k - 1$ is not the label of a vertex. So, the set of vertex labels is $\{n + k\} \cup L^- \cup L^+$, where $L^- = \{0, 1, \dots, (n+k-3)/2\}$ and $L^+ = \{(n+k-1)/2, (n+k+1)/2, \dots, n+k-2\}$; moreover $M = \{u_iv_i : f(u_i) = i, f(v_i) = (n + k - 2 - i), i \in L^-\}$ and for all $i \in L^-$, $n + k - 2 - i \in L^+$. We then use necessary conditions to exhibit a subgraph of G with the necessary related vertex-labels.

To this aim, we define, for any positive even integers a, b, c, the tree $T_{a,b,c}$ formed by a vertex t and three paths of length $a-1, b-1, c-1$ attached to t using an edge incident to one of their endpoints. Note that adding one edge between two leaves of $T_{a,b,c}$ creates a tadpole $T_{[n,k]}$ with n odd. For convenience, we see $T_{a,b,c}$ as a rooted tree with the root on t, of degree 3, the left branch denoted $t1_1 1_2 \dots 1_a$, the middle branch denoted $t2_1 2_2 \dots 2_b$ and the right branch denoted $t3_1 3_2 \dots 3_c$.
Using Remark 5, we know that G has a subgraph $T_{2,2,2}$ with root t. We necessarily have $f(t) = n+k$ and without loss of generality, we can assume $f(1_1) = 1$, $f(2_1) = 0$ and $f(3_1) = 3$. In addition, $\{1_1 1_2, 2_1 2_2, 3_1 3_2\} \subset M$ and using the weight $n+k-2$ of edges in M, we have $f(1_2) = n+k-3, f(2_2) = n+k-2$ and $f(3_2) = n+k-5$. From here, we use similar arguments as in Remark 5 to add, at each step, a new edge in M and the related vertex labels. The process continues until $(n+k-1)/2$ edges are added in M. The successive subgraphs of G will be $T_{2,2,2}, T_{4,2,2}, T_{4,4,2}, \dots$
For a positive integer p, we call *step* p the step in our process where the pth matching edge is included in the current subgraph $T_{a,b,c}$ of G. At any step, the following edges of $T_{a,b,c}$ for even numbers a, b, c are in M:

$$\begin{aligned}\{1_{2i+1}1_{2i+2} : i = 0, 1, \dots, \tfrac{a}{2}-1\} \cup \{2_{2i+1}2_{2i+2} : i = 0, 1, \dots, \tfrac{b}{2}-1\}\\ \cup \{3_{2i+1}3_{2i+2} : i = 0, 1, \dots, \tfrac{c}{2}-1\}.\end{aligned}$$

We have as well necessary values for the labels of the vertices in $T_{a,b,c}$. There are $|M| = (n+k-1)/2$ steps and if $p < (n+k-1)/2$, we use necessary conditions to add to $T_{a,b,c}$ two new vertices, endpoints of an edge in M. The new graph is either $T_{a+2,b,c}, T_{a,b+2,c}$ or $T_{a,b,c+2}$. We start at the end of step 3 and, as we will see, the subgraph obtained after step p is $T_{2p/3,2p/3,2p/3}$ if $p \equiv 0 \pmod 3$, $T_{2(p-1)/3+2,2(p-1)/3,2(p-1)/3}$ if $p \equiv 1 \pmod 3$ and $T_{2(p-2)/3+2,2(p-2)/3+2,2(p-2)/3}$ if $p \equiv 2 \pmod 3$. At each step, we have necessary values for labels of vertices of the current $T_{a,b,c}$. Any label already allocated to a vertex is called *used* and it is called *closed* if the corresponding vertex in the current $T_{a,b,c}$ is not a leaf. It means that all edges incident to this vertex in G are edges of the current $T_{a,b,c}$. For convenience, we also say that $n+k-1$ is closed.
We claim that, after the step $p = 6i+3$, $i \geq 0$, if it exists, the following properties hold:

1. The current $T_{a,b,c}$ is $T_{2p/3,2p/3,2p/3}$.
2. The set of closed vertex-labels is:
$\{n+k\} \cup \{0, 1, \dots, p-2, p\} \cup \{n+k+2-p, n+k+3-p, \dots, n+k-1\}$.
3. The labels of the three leaves are:
$f\left(1_{2p/3}\right) = n+k-p$, $f\left(2_{2p/3}\right) = n+k+1-p$, $f\left(3_{2p/3}\right) = n+k-2-p$.
4. The set of even edge-labels already allocated to edges not in M is:
$\{n+k-1, n+k-3, \dots, n+k+3-2p\}$.

We prove these properties by induction on i considering, one after the other, the following 6 steps, if they exist, i.e., while $2p \leq n+k-1$. Remark 5 shows that these properties are satisfied for $i = 0$. Assume they are satisfied for any $i \geq 0$ and $2p < n+k-1$.

Step $\boldsymbol{p+1 = 6i+4}$: Let us consider the possibilities for the edge of label $n+k+1-2p$. This label is not yet allocated due to Property 4 and since it is even, it is the label of an edge not in M. The $2p$ possible cases for labels for the end vertices of this edge are $(n+k, 2p-1), (n+k-1, 2p-2), \ldots, (n+k-2p+1, 0)$, but cannot use closed labels. The first $(p-1)$ ordered pairs $(n+k, 2p-1), (n+k-1, 2p-2), \ldots, (n+k-p+2, p+1)$, the pair $(n+k-p+1, p)$ and the $(p-1)$ last pairs, $(n+k-p+3, p-2), \ldots, (n+k-2p+1, 0)$ are not possible since they all involve a closed label due to Property 2 and consequently, the only possible pair is $(p-1, n+k-p)$ corresponding to a new edge between the vertex of label $n+k-p$, i.e., the vertex $1_{2p/3}$, and the vertex $1_{2p/3+1}$ that takes the label $p-1$. We need to add the vertex $1_{2p/3+2}$, the new leaf in the branch 1, with $1_{2p/3+1}1_{2p/3+2} \in M$ and $f\left(1_{2p/3+2}\right) = n+k-1-p$. Labels $n+k-p$ and $p-1$ are added to the list of closed labels and thus, the set of closed labels is now

$$\{n+k\} \cup \{0, 1, \ldots, p\} \cup \{n+k-p, n+k+2-p, n+k+3-p, \ldots, n+k-1\},$$

with the second set contained in L^- and the third contained in L^+. The three leaves have labels $f\left(1_{2p/3+2}\right) = n+k-1-p$, $f\left(2_{2p/3}\right) = n+k+1-p$ and $f\left(3_{2p/3}\right) = n+k-2-p$.

Step $\boldsymbol{p+2 = 6i+5}$: Similarly, we consider the possibilities for the edge of label $n+k-1-2p$ and note that the only possibility is the pair $(n+k+1-p, p+2)$ corresponding to the edge $2_{2p/3}2_{2p/3+1}$ with $f\left(2_{2p/3+1}\right) = p+2$ and consequently $f\left(2_{2p/3+2}\right) = n+k-p-4$ since $2_{2p/3+1}2_{2p/3+2} \in M$. The set of closed labels is now

$$\{n+k\} \cup \{0, 1, \ldots, p, p+2\} \cup \{n+k-p, n+k+1-p, \ldots, n+k-1\},$$

and the three leaves have labels $f\left(1_{2p/3+2}\right) = n+k-1-p$, $f\left(2_{2p/3+2}\right) = n+k-p-4$ and $f\left(3_{2p/3}\right) = n+k-2-p$.
Note that the sets of closed labels of the three leaves can be deduced from the ones after the step $p = 6i+3$ by replacing p with $p+2$. The only difference is the branches of the corresponding leaves. The new pair of vertex-labels added can be deduced in the next step from the analysis of step $p+1 = 6i+4$.

Step $\boldsymbol{p+3 = 6i+6}$: Using the reasoning in step $p+1 = 6i+4$, we get that the only possibility for the pair of vertex-labels of the edge of label $n+k-3-2p$ is $(p+1, n+k-2-p)$ corresponding to the edge $3_{2p/3}3_{2p/3+1}$ with $f\left(3_{2p/3+1}\right) = p+1$ and we add the vertex $3_{2p/3+2}$ with $3_{2p/3+1}3_{2p/3+2} \in M$. The set of closed labels is now:

$$\{n+k\} \cup \{0, 1, \ldots, p+2\} \cup \{n+k-2-p, n+k-p, n+k+1-p, \ldots, n+k-1\},$$

and the three leaves have labels $f\left(1_{2p/3+2}\right) = n+k-1-p$, $f\left(2_{2p/3+2}\right) = n+k-p-4$ and $f\left(3_{2p/3+2}\right) = n+k-p-3$.

Step $\boldsymbol{p+4=6i+7}$: We are now in a similar state as at the end of step $p+1=6i+4$ and we can deduce that the next ordered pair of vertex-labels corresponding to the edge of label $n+k-5-2p$ is the pair $(n+k-1-p, p+4)$ corresponding to the edge $1_{2p/3+2}1_{2p/3+3}$ with $f\left(1_{2p/3+3}\right)=p+4$. We then add the vertex $1_{2p/3+4}$ with $1_{2p/3+3}1_{2p/3+4}\in M$. The set of closed labels is now:

$$\{n+k\}\cup\{0,1,\ldots,p+2,p+4\}\cup\{n+k-2-p,n+k-1-p,\ldots,n+k-1\},$$

and the three leaves have labels $f\left(1_{2p/3+4}\right)=n+k-p-6$, $f\left(2_{2p/3+2}\right)=n+k-p-4$ and $f\left(3_{2p/3+2}\right)=n+k-p-3$.

Step $\boldsymbol{p+5=6i+8}$: Using step $p+3=3i+6$, the new ordered pair of vertex-labels corresponding to the edge of label $n+k-7-2p$ is the pair $(p+3, n+k-4-p)$ corresponding to the edge $2_{2p/3+2}2_{2p/3+3}$ with $f\left(2_{2p/3+3}\right)=p+3$ and we add $2_{2p/3+4}$ with $2_{2p/3+3}2_{2p/3+4}\in M$. The set of closed labels is now:

$$\{n+k\}\cup\{0,1,\ldots,p+4\}\cup\{n+k-4-p,n+k-2-p,n+k-1-p,\ldots,n+k-1\},$$

and the three leaves have labels $f\left(1_{2p/3+4}\right)=n+k-p-6$, $f\left(2_{2p/3+4}\right)=n+k-p-5$ and $f\left(3_{2p/3+2}\right)=n+k-p-3$.

Step $\boldsymbol{p+6=6i+9=6(i+1)+3}$: Finally, we use step $p+4=3i+7$ to determine that the new ordered pair of vertex-labels corresponding to the edge of label $n+k-9-2p$ is the pair $(p+6, n+k-3-p)$ corresponding to the edge $3_{2p/3+2}3_{2p/3+3}$ with $f\left(3_{2p/3+3}\right)=p+6$ and we add $3_{2p/3+4}$ with $3_{2p/3+3}3_{2p/3+4}\in M$. The set of closed labels is now:

$$\{n+k\}\cup\{0,1,\ldots,p+4,p+6\}\cup\{n+k-4-p,n+k-3-p,\ldots,n+k-1\},$$

and the three leaves have labels $f\left(1_{2p/3+4}\right)=n+k-p-6$, $f\left(2_{2p/3+4}\right)=n+k-p-5$ and $f\left(3_{2p/3+4}\right)=n+k-p-8$. It corresponds exactly to the claimed properties for the step $p+6=6(i+1)+3$. By induction, it proves that these 6 patterns repetitively occur every 6 steps.

Starting now from one value for $|M|=(n+k-1)/2\geq 3$, we can pursue the process described above during $|M|-3$ steps from 3 to $|M|$. For each case, it leads us to one of the 6 possible configurations we have described, depending on the value of $|M|$ modulo 6. In each case, this process leads to the tree $T_{a,b,c}$ together with the necessary vertex-labels for all vertices of the tree. By construction, this vertex-labelling is injective and after $|M|-3$ steps, the set of used vertex-labels is $\{0,1,\ldots,n+k-2,n+k\}$. The set of edge-labels of edges in M is $\{2i+1 : i=0,1,\ldots,|M|-1\}$ and the edge $t2_1$ has the label $n+k=2|M|+1$, which covers all possible odd edge-labels. By construction, the set of edge-labels of edges of $T_{a,b,c}$ not in M corresponds to the $2+|M|-3=|M|-1$ largest even integers less than $n+k$, so it is $\{2i : i=2,3,\ldots,|M|\}$. It means that the last edge of G links two leaves of $T_{a,b,c}$ which labels differ by 2. For each of the 6 possible configuration, there is exactly one such edge. Adding this edge to the corresponding tree $T_{a,b,c}$ leads to the unique tadpole $T_{[n,k]}$, with n odd, k even such that $(n+k-1)/2 = |M|$ and that is nicely graceful for the weight $n+k-2$. In each case, we can find such a nicely graceful labelling using the above construction; we just need to know which of the three branches is the path. We then have two possible labellings obtained from one to the other by reversing the labels on the cycle. Figure 6 shows the construction for $n = 11, k = 6$ and $n = 13, k = 6$. Table 3 describes, for each possible case, the tree $T_{a,b,c}$, the edge of label 2, the corresponding tadpole and the first vertex x_1 of the path $x_1x_2\dots x_k$ attached to t asw well as its label $f(x_1)$. For instance, the two cases represented in Figure 6 correspond to $|M| = 8$ ($|M| \equiv 2 \pmod 6$) and $|M| = 9$ ($|M| \equiv 3 \pmod 6$).

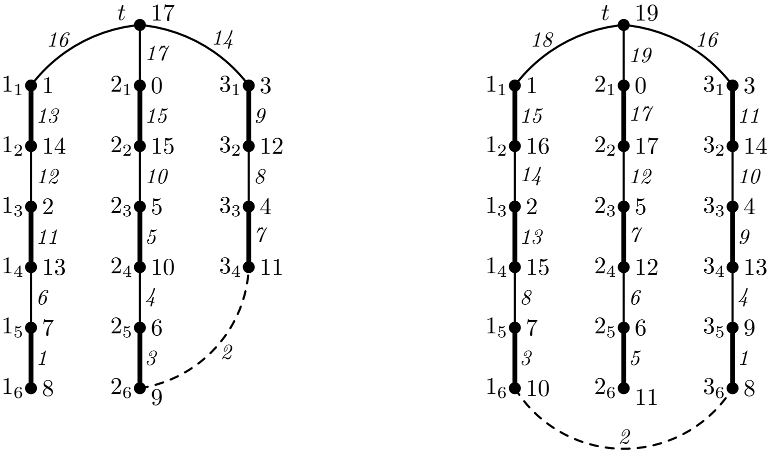

Figure 6. Nicely graceful labellings for $T_{[11,6]}$ (left) and $T_{[13,6]}$ (right). Solid lines correspond to edges of the trees $T_{6,6,4}$ (left) and $T_{6,6,6}$ (right). For each case, the dashed line is the additional edge to close the cycle. Bold lines correspond to edges in M.

$\lvert M\rvert$	$T_{a,b,c}$ ($m = \lfloor \lvert M\rvert/3 \rfloor$)	Edge of label 2	$T_{[n,k]}$	x_1	$f(x_1)$
$\lvert M\rvert \equiv 0 \pmod 6$	$T_{2m,2m,2m}$	$1_{2m}3_{2m}$	$T_{[4m+1,2m]}$	2_1	0
$\lvert M\rvert \equiv 1 \pmod 6$	$T_{2m+2,2m,2m}$	$1_{2m+2}2_{2m}$	$T_{[4m+3,2m]}$	3_1	3
$\lvert M\rvert \equiv 2 \pmod 6$	$T_{2m+2,2m+2,2m}$	$2_{2m+2}3_{2m}$	$T_{[4m+3,2m+2]}$	1_1	1
$\lvert M\rvert \equiv 3 \pmod 6$	$T_{2m,2m,2m}$	$1_{2m}3_{2m}$	$T_{[4m+1,2m]}$	2_1	0
$\lvert M\rvert \equiv 4 \pmod 6$	$T_{2m+2,2m,2m}$	$1_{2m+2}2_{2m}$	$T_{[4m+3,2m]}$	3_1	3
$\lvert M\rvert \equiv 5 \pmod 6$	$T_{2m+2,2m+2,2m}$	$2_{2m+2}3_{2m}$	$T_{[4m+3,2m+2]}$	1_1	1

Table 3. For each of the 6 possible cases, the table gives the tree, the edge of label 2, the corresponding tadpole, the first vertex x_1 on the path attached to the cycle and its label.

As a conclusion, for $n \equiv 1 \pmod 4$, $T_{[n,(n-1)/2]}$ is nicely graceful for the weight $n+k-2$ and for $n \equiv 3 \pmod 4$, both $T_{[n,(n-3)/2]}$ and $T_{[n,(n+1)/2]}$ are. In addition, for each case, there are exactly two different such nicely graceful labellings. Since we worked using necessary conditions, these are the only tadpoles $T_{[n,k]}$, with n odd that are nicely graceful for the weight $n + k \pm 2$. It concludes the proof. $\square$

5. Conclusion

In this work, we focus on gracefulness of tadpoles (including cycles). We introduce the notion of nicely graceful which generalises the notion of strongly graceful. We primary introduce it so as to study the strong gracefulness of tadpoles. However, the notion reveals to be interesting by itself.

We completely characterise nicely graceful cycles and strongly graceful tadpoles. We also obtain some sufficient conditions for a tadpole to be nicely graceful, which leaves some cases open, as shown in Table 1. The three open cases are for $n \equiv 1 \pmod 4$, $k \equiv 0 \pmod 2$, $k < (n-1)/2$ and the weight $n + k$ as well as for $n \equiv 2 \pmod 4$, $k \equiv 1 \pmod 2$, $k > n/2 - 2$ and the weights $n + k$ and $n + k \pm 1$. We conjecture that nicely graceful labellings exist for $n \equiv 2 \pmod 4$, $k \equiv 1 \pmod 2$, $k > n/2 - 2$ and the weights $n + k$ and $n + k \pm 1$. So far, we have nicely graceful labellings for several cases to support this conjecture.

In addition to these open cases for tadpoles, a further work will be to extend the study to other classes of hairy cycles.

In term of methods, the paper proposes some criteria for non-existence of a nicely graceful labelling for a given weight that can be used for other classes of graphs. We also firmly believe that the techniques we used to obtain strongly/nicely graceful labellings in tadpoles can be transferred to other classes of graphs.

On a longer perspective, this work opens a few questions for future projects. One of the motivations of introducing strong gracefulness in [4] is the result that all trees are graceful if and only if all trees with a perfect matching are strongly graceful, leading to a new formulation of the long-standing graceful tree conjecture in terms of strong gracefulness. The case of unicyclic graphs is completely different: our results outline classes of unicyclic graceful graphs with a perfect matching that are not strongly graceful. For instance, this is the case for tadpoles $T_{[n,k]}$, with $n \equiv 1 \pmod 4$ and k odd or $n \equiv 2 \pmod 4$ and k even. However, for even cycles of size at least 6, strong gracefulness and gracefulness are equivalent and an interesting question is to find some other classes of unicyclic graphs for which this happens. A second question is to study the case of graphs obtained by attaching some kinds of trees instead of paths to a cycle and in particular, graphs obtained from a caterpillar of degree 3 by adding one edge.

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