

Sibling graphs which are k -distance magic

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Abstract: This paper investigates k -distance magic labeling, for a positive integer k , within the framework of sibling graphs, that is, graphs that are both self-centered and antipodal. A k -distance magic labeling is a vertex labeling in which the sum of the labels on all vertices at distance k from any given vertex is constant throughout the graph. We establish sufficient conditions for a sibling graph to admit a k -distance magic labeling, covering both regular and non-regular cases. Using these conditions, we show that several well-known families of regular sibling graphs of diameter k are k -distance magic, including cylindrical grid graphs, cyclic grid graphs, n -dimensional hypercubes, circulant graphs, weak Bruhat graphs, and the Möbius–Kantor graph. In addition, we construct examples of irregular sibling graphs that admit a 2-distance magic labeling.

Keywords: distance magic, labeling, sibling graph

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1. Introduction

Let $G = (V, E)$ be a graph with a (finite) set $V = V(G)$ of vertices and a set $E = E(G)$ of edges. The *distance* between vertices u and v in G , denoted by $d(u, v)$, is the length of a shortest path connecting u and v in G . The *eccentricity* $e(v)$ of a vertex v is the distance to a vertex that is farthest from v . Thus, $e(v) = \max d(u, v)$ for $u \in V(G)$. The *diameter* $\text{diam}(G)$ of G is the maximum eccentricity, and the *radius* $\text{rad}(G)$ of G is the minimum eccentricity in G . A vertex v is called a *center* of G if $e(v) = \text{radius}(G)$.

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A graph G is *self-centered* if every vertex is a center of G [2]. Thus, a self-centered graph is a graph whose diameter equals its radius. An antipode of any vertex v is the vertex farthest from v . An *antipodal graph* is a connected graph in which each vertex has exactly one antipode. A graph G is called a *sibling graph* if it is both self-centered and antipodal. For instance, even cycles, odd-length paths and trees with a unique odd-length longest path are antipodal graphs [5]. Among these, only even cycles are also self-centered, and hence qualify as sibling graphs. But odd-length paths and such trees are not self-centered. Therefore, they are not sibling graphs. The Petersen graph is not a sibling graph, since it is self-centered but not antipodal. We note that the term “sibling tree” was introduced in a different context by Arockiaraj et al. [1], where it refers to a graph obtained from the 1-rooted complete binary tree by adding edges (called sibling edges) between the left and right children of each parent node.

For any integer k , $1 \leq k \leq \text{diam}(G)$, and any vertex $x \in V(G)$ define $N_k(x)$ as the set of all vertices in G at distance k from x . If $k = 1$, we may write $N(v)$, instead. In graph theory, a *distance magic labeling* is a vertex labeling that assigns distinct integers to the vertices of a graph while enforcing a fixed “magic” sum around each vertex. More precisely, for a graph G with n vertices, a *distance magic labeling* is a bijection

$$f : V(G) \longrightarrow \{1, 2, \dots, n\}$$

such that there exists a constant M (called the *magic constant*) satisfying

$$\sum_{u \in N(v)} f(u) = M \quad \text{for every } v \in V(G).$$

See [10] for the original definition. Rinovia Simanjuntak [9] introduced the notion of distance magic labelings for a fixed distance other than one in a presentation at the 2010 International Workshop on Graph Labeling (IWOGL). Furthermore, O’Neil and Slater [8] generalized the notion of distance magic to be D -distance magic. Let G be a graph of order n with diameter d . Let $D \subset \{1, 2, \dots, d\}$. A labeling $f : V(G) \rightarrow \{1, 2, \dots, n\}$ is called a *D -distance magic labeling* if there exists a constant M such that $\sum_{w \in N_D(u)} f(w)$ is M for every $u \in V(G)$. Here, $N_D(x) := \{y \in V(G) : d(x, y) \in D\}$ denotes the set of vertices at distances in D from x in G . We say that G is *D -distance magic* (or *D -vertex magic*) if G admits a D -distance magic labeling. If $D = 1$ then we have the distance magic concept. If $D = \{k\}$ for some $k \in [1, d]$, the graph G is called *k -distance magic*. In [6], Vilfred studied the concept of k -distance magic labeling of a graph. He characterized all paths, cycles, a union of paths, a union of cycles and the long brush graphs to be k -distance magic. For a cycle, it is given in the following theorem.

Theorem 1. [6] Let $k \geq 2$, $n \geq 3$ and $k, n \in \mathbb{N}$. Then C_n is k -distance magic if and only if $n = 4k$.

In this paper, we investigate k -distance magic labeling for a positive integer k , typically chosen as half the diameter, in the framework of sibling graphs. We establish sufficient conditions under which a sibling graph admits a k -distance magic labeling, covering both regular and non-regular cases. Using these conditions, we demonstrate that numerous families of regular sibling graphs of diameter $2k$ are k -distance magic, including cylindrical grid graphs, cyclic grid graphs, hypercubes, circulant graphs, weak Bruhat graphs, and the Möbius–Kantor graph. In addition, we construct examples of irregular sibling graphs that admit a 2-distance magic labeling.

2. Main Results

In this paper, we consider a sibling graph G , namely a self-centered antipodal graph of diameter $d = \text{diam}(G) \geq 2$. This graph has the property that for every vertex u in G there exists a unique vertex v at distance d from u , namely $N_d(u) = \{v\}$. This vertex v is called the *sibling* of u in G . Since v is a sibling of u implies that u is a sibling of v in G , then the order of the sibling graph G must be even. Define a function $\pi : V(G) \rightarrow V(G)$ such that $\pi(x) = y$ if y is a sibling of x in G . Then, π is a permutation on G with the property that $\pi^2 = I$. Furthermore, π is an automorphism in G , as shown in the following theorem.

Theorem 2. *Let G be a sibling graph. Then, the sibling function on G is an automorphism.*

Proof. Let G be a sibling graph of diameter $d = \text{diam}(G)$ and π be its sibling function. We show that x is adjacent to y if and only if $\pi(x)$ is adjacent to $\pi(y)$. Before we do so, we shall prove that $\delta(G) \geq 2$. For a contradiction, we assume that there is a vertex x of degree one in G . Then, there is a unique path of length d from x to its sibling x' , namely $x' = \pi(x)$. Let $P := (x, y, \dots, x')$ be such a path. Then, a geodesic path connecting y and its sibling $y' (= \pi(y))$ will not pass through x . Let $Q := (y, \dots, z, y')$. But now $z = \pi(x)$, a contradiction. Therefore, $\delta(G) \geq 2$.

Let x be adjacent to y in G . We have $d(x, \pi(x)) = d$ implies $d(y, \pi(x)) = d - 1$. Let $a_1, a_2, \dots, a_r \in N(\pi(x))$, for $r \geq 2$, then $d(x, a_i) = d - 1$ and $d(y, a_i) \neq d - 1$ for each i . If there is a geodesic path from y to a_i that does not include x or $\pi(x)$, then $d(y, a_i) = d - 2$. Since $d(y, \pi(y)) = d$ the geodesic path from y to $\pi(y)$ passes through x or $\pi(x)$. Then $d(\pi(y), x) = d - 1$ and $\pi(y) \in N(\pi(x))$. If there exists another geodesic path from y to $a_i \neq \pi(y)$ including x or $\pi(x)$ then $d(y, a_i) = d$, a contradiction. Therefore, $\pi(y) \in N(\pi(x))$ is the only vertex at a distance d from y . Therefore, π is an automorphism in G . \square

Since a sibling function is an automorphism on a sibling graph G , then the existence of a path of length k (for some k) from x to y in G will imply a corresponding path of length k from $\pi(x)$ to $\pi(y)$. Thus, we immediately have the following corollary.

Corollary 1. For any positive integer t and any vertex x in a sibling graph G , $|N_t(x)| = |N_t(\pi(x))|$.

Note that a sibling graph is not necessarily regular. In Figure 1, the graph G_1 is an example of a non-regular sibling graph with a sibling function $\pi := (a_1 b_1)(a_2 b_2) \cdots (a_6 b_6)$, and G_2 is a regular sibling graph with a sibling function $\pi := (a_1 b_1)(a_2 b_2) \cdots (a_{12} b_{12})$.

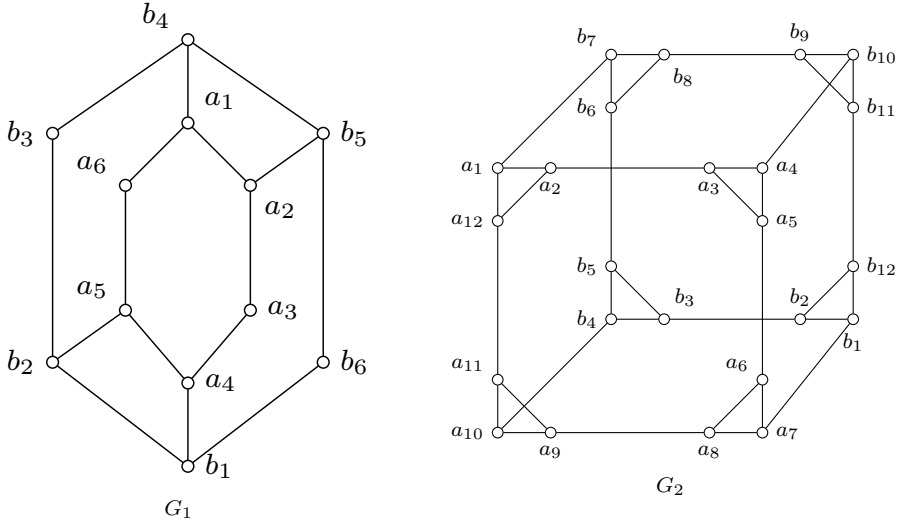


Figure 1. Two sibling graphs.

Corollary 2. If G is a sibling graph of diameter d , then for every $x \in V(G)$ there exists a cycle C_{2d} without a chord edge containing x , its sibling $\pi(x)$, and $(d - 1)$ other vertices with their siblings.

Proof. Let G be a sibling graph of diameter d . Let π be a sibling function on G . Let x be a vertex in $V(G)$ and let a path $P := (x = x_0, x_1, \dots, x_{d-1}, x_d = \pi(x))$ of length d from x to $\pi(x)$. Then, by Theorem 2 there is a path $Q := (\pi(x) = x_d, \pi(x_1), \dots, \pi(x_{d-1}), \pi(x_d) = x)$ of length d in G . Since each $\pi(x_i) \notin P$ for $i = 1, 2, \dots, x_{d-1}$, then P and Q are paths having no common internal vertices. Therefore, P together with Q will form a cycle C of length $2d$ containing d vertices as well as their siblings, including x and $\pi(x)$. Now, if there is a chord edge (u, v) in C with $u \in P$ and $v \in Q$, then $\pi(u)$ is not in Q , a contradiction. \square

Corollary 2 immediately implies the following corollary.

Corollary 3. If G is a sibling graph of diameter $d = 2k$ with a sibling function π , then $N_k(u) \cap N_k(\pi(u)) \neq \emptyset$.

Note that in a sibling graph of diameter $2k$ (for some k), it is not necessarily that $N_k(x) = N_k(\pi(x))$, for any vertex x . For example, in the graph G_2 (Figure 1), $a_{10} \in N_3(a_1)$, but $a_{10} \notin N_3(b_1)$. Thus, $N_3(a_1) \neq N_3(\pi(a_1))$. However, in graph G_1 (Figure 1), it can be verified that $N_2(x) = N_2(\pi(x))$, for any vertex x .

Theorem 3. *Let G be a sibling graph of even diameter $2k$, for some integer $k \geq 1$. Let π be a sibling function on G . If G satisfies the following conditions:*

1. $N_k(x) = N_k(\pi(x))$, for each $x \in V(G)$, and
2. $|N_k(x)| = |N_k(y)|$, for any $x, y \in V(G)$,

then G is a k -distance magic graph.

Proof. Let G be a sibling graph of even diameter, say $d = 2k$ for some integer $k \geq 1$. Let π be a sibling function on G , namely $\pi(x) = y$ if y is a sibling of x . If $N_k(x) = N_k(\pi(x))$ for each $x \in V(G)$ and $|N_k(x)| = |N_k(y)|$ for any $x, y \in V(G)$, then we will show that G is a k -distance magic graph.

Let x be any vertex of G and let $a \in N_k(x)$. By the first condition, we have $a \in N_k(\pi(x))$. Since G is a sibling graph, by Theorem 2, $a \in N_k(x)$ implies $\pi(a) \in N_k(\pi(x))$, and so $\pi(a) \in N_k(x)$, according to the first condition. Therefore, for each $x \in V(G)$ both $N_k(x)$ and $N_k(\pi(x))$ contain the same pairs of vertices and their siblings. Now, define a 1-1 mapping $\alpha : V(G) \rightarrow \{1, 2, \dots, n\}$ such that if $\alpha(x) = a$ then $\alpha(\pi(x)) = n + 1 - a$, where n is the order of the graph G , $1 \leq a \leq n$. By the second condition, such a mapping α gives us a k -distance magic labeling of G with the magic constant $t(n + 1)$, if $|N_k(x)| = 2t$ for each $x \in V(G)$ and for some integer t . \square

By Theorem 3, being a sibling graph does not guarantee to be a k -distance magic graph. A sibling graph G must satisfy the above two conditions to be a k -distance magic graph, with k being half the diameter.

There are many examples of sibling graphs that satisfy only one of these two conditions of Theorem 3, and so they are not k -distance magic. In Figure 2, graph G_3 is an example of a sibling graph (b_i is the sibling of a_i for each i) satisfying only the first condition, and graph G_4 is an example of a sibling graph (b_i is the sibling of a_i for each i) that satisfies the second condition but not the first one. But, G_3 is not 2-distance magic. For a contradiction, assume that G_3 is a 2-distance magic graph. Then, there exists a 2-distance magic labeling $f : V(G_3) \rightarrow \{1, 2, \dots, |V(G)|\}$. Then,

$$\sum_{j \in N_2(a_1)} f(j) = \sum_{j \in N_2(a_5)} f(j) = \sum_{j \in N_2(a_7)} f(j).$$

This implies that

$$f(a_5) + f(b_5) = f(a_6) + f(b_6) = f(a_7) + f(b_7). \quad (2.1)$$

Now, from the following equalities:

$$\sum_{j \in N_2(a_1)} f(j) = \sum_{j \in N_2(a_2)} f(j) = \sum_{j \in N_2(a_4)} f(j),$$

we also have

$$f(a_2) + f(b_2) = f(a_3) + f(b_3) = f(a_4) + f(b_4). \quad (2.2)$$

Next, we have

$$\sum_{j \in N_2(a_3)} f(j) = \sum_{j \in N_2(a_6)} f(j). \quad (2.3)$$

From (2.1), (2.2), (2.3), we have $f(a_5) + f(b_5) = f(a_2) + f(b_2) = f(a_3) + f(b_3)$. But, then

$$\sum_{j \in N_2(a_3)} f(j) \neq \sum_{j \in N_2(a_5)} f(j),$$

a contradiction. Therefore, we conclude that G_3 is a sibling graph, but not a 2-distance magic graph.

Now, for a contradiction, assume that G_4 is 3-distance magic. There exists a 3-distance magic labeling $f : V(G_3) \rightarrow \{1, 2, \dots, |V(G)|\}$. Then

$$\sum_{j \in N_3(a_1)} f(j) = \sum_{j \in N_3(a_7)} f(j) \text{ implies } f(a_8) + f(b_8) = f(a_{10}) + f(b_{10}). \quad (2.4)$$

$$\sum_{j \in N_3(a_3)} f(j) = \sum_{j \in N_3(b_3)} f(j) \text{ implies } f(a_8) + f(b_{10}) = f(b_8) + f(a_{10}). \quad (2.5)$$

From (2.4) and (2.5), we obtain $f(a_8) = f(b_{10}) = f(b_8) = f(a_{10})$, a contradiction. Therefore, G_4 is a sibling graph, but not a 3-distance magic graph.

Some vertex-transitive graphs are sibling graphs and k -distance magic. However, in Figure 1, graph G_2 is an example of a vertex-transitive sibling graph, but it is not k -distance magic, showing that vertex transitivity is not mandatory for k -distance magic, with k being half the diameter.

It is known that cylindrical grid graphs $P_m \square C_n$ do not admit a 1-distance-magic labeling [7]. The following theorem characterizes the k -distance-magic property of $P_2 \square C_n$ for even n , where k equals half the diameter.

Theorem 4. *The cylindrical grid graph $P_2 \square C_{2t}$ admits a k -distance-magic labeling, where k is half the diameter, if and only if t is odd and $t \geq 3$.*

Proof. Let $G \cong P_2 \square C_{2t}$, $t \geq 2$. Let $V(G) = \{u_1, u_2, \dots, u_{2t}, v_1, v_2, \dots, v_{2t}\}$ and $E(G) = \{u_i u_j, v_i v_j : i - j \equiv \pm 1 \pmod{2t}, 1 \leq i, j \leq 2t\} \cup \{u_i v_i : 1 \leq i \leq 2t\}$. It is obvious that the diameter of the graph G is $t + 1$. Since k is half the diameter and k is an integer, then t must be odd.

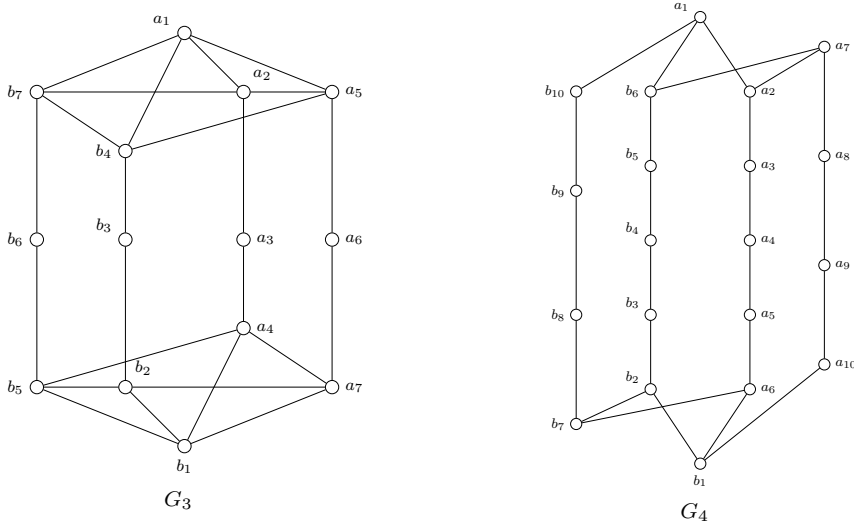


Figure 2. Graphs G_3 and G_4 are sibling, but G_3 is not 2-distance, and G_4 is not 3-distance magic.

Further, for odd t and $t \geq 3$, it is easy to verify that G is a sibling graph with the sibling function π mapping u_i to v_{i+t} , where all indices are calculated in modulo $2t$. Let $k = \frac{t+1}{2}$. Now, for any vertex $u_i \in V(G)$, consider all vertices at distance k from u_i and from its sibling, namely v_{i+t} . Then,

$$\begin{aligned}
 N_k(u_i) &= \{u_{i+k}, u_{i-k}, v_{i+(k-1)}, v_{i-(k-1)}\}, \text{ and} \\
 N_k(v_{i+t}) &= \{v_{i+t+k}, v_{i+t-k}, u_{i+t+(k-1)}, u_{i+t-(k-1)}\} \\
 &= \{v_{i+2t+k-t+1-1}, v_{i+t+1-k-1}, u_{i+2t-t+k-1}, u_{i+t-k+1}\} \\
 &= \{v_{i+2t-k+1}, v_{i+k-1}, u_{i+2t-k}, u_{i+k}\} \\
 &= \{v_{i-(k-1)}, v_{i+(k-1)}, u_{i-k}, u_{i+k}\} \\
 N_k(v_{i+t}) &= N_k(u_i).
 \end{aligned}$$

Furthermore, G satisfies the two conditions of Theorem 3, namely $N_k(x) = N_k(\pi(x))$ for each x and $|N_k(x)| = |N_k(y)| = 4$ for any $x, y \in V(G)$. Therefore, by Theorem 3, G is a k -distance magic graph. \square

Theorem 5. For integers $m \geq n \geq 1$, the cyclic grid graph $C_{2m} \square C_{2n}$ admits a k -distance-magic labeling, where k is half the diameter, if and only if $m+n$ is even.

Proof. For integers $m \geq n \geq 1$, let $G \cong C_{2m} \square C_{2n}$. Let $V(G) = \{u_{i,j} : 1 \leq i \leq 2m, 1 \leq j \leq 2n\}$ and $E(G) = \{u_{i,j}u_{i,k} : j-k \equiv \pm 1 \pmod{2n}, 1 \leq i \leq 2m, 1 \leq j, k \leq 2n\} \cup \{u_{i,j}u_{s,j} : i-s \equiv \pm 1 \pmod{2m}, 1 \leq i, s \leq 2m, 1 \leq j \leq 2n\}$. The order of G is $4mn$ and the diameter G is $m+n$. Since k is half the diameter, then $m+n$ must be even.

Furthermore, G is a sibling graph, since for any $u_{i,j} \in V(G)$ there exists one vertex $y = u_{i+m,j+n}$ at a distance exactly $m+n$ from $u_{i,j}$. Therefore, the function $\pi(u_{i,j}) = u_{i+m,j+n}$ is a sibling function of G where the first indices are calculated in modulo $2m$, and the second ones are calculated in modulo $2n$.

Now, let $m+n$ be an even number, and $m+n = 2k$ for some integer k . We show that G satisfies the two conditions of Theorem 3. It can be verified that the distance between two vertices $u_{i,j}$ and $u_{s,t}$ in G is

$$d(u_{i,j}, u_{s,t}) = \min\{i-s, s-i \bmod 2m\} + \min\{j-t, t-j \bmod 2n\}.$$

Therefore, if $m+n = 2k$ and $m \geq n \geq 1$ then we obtain that $N_k(u_{i,j}) = N_k(u_{i+m,j+n})$ and $|N_k(u_{i,j})| = 2n+2$ for any vertex $u_{i,j}$ in G . Note that the first indices are calculated in modulo $2m$, and the second ones are calculated in modulo $2n$. Therefore, by Theorem 3, G is a k -distance magic graph. \square

Theorem 6. *The hypercube Q_n admits a k -distance-magic labeling, where k is half the diameter, precisely when n is even.*

Proof. Let $G \cong Q_n$ be the n -dimensional hypercube. Its vertex set is

$$V(G) = \{x_1x_2 \dots x_n : x_i \in \mathbb{Z}_2, i = 1, 2, \dots, n\}.$$

Two vertices $x = x_1x_2 \dots x_n$ and $y = y_1y_2 \dots y_n$ are adjacent whenever

$$\sum_{i=1}^n |x_i - y_i| = 1,$$

that is, when their binary strings differ in exactly one coordinate. Consequently, the distance between any two vertices is the number of coordinates in which they differ, and hence the diameter of G is n , since two binary strings of length n may differ in at most n positions. Therefore, since k is half the diameter, then n must be even.

First, we will show that G is a sibling graph. Let $x = x_1x_2 \dots x_n$ be any vertex of G . Then the vertex $\bar{x} = \bar{x}_1\bar{x}_2 \dots \bar{x}_n$ is a unique vertex at distance n from x , where $\bar{x}_i = x_i + 1$. Therefore, \bar{x} is the sibling of x , and so G is a sibling graph.

Second, we will show that G satisfies the first condition of Theorem 3. If $y \in N_k(x)$ then the representation of y differs in k digits from the one of x , and so they both have k common digits (since $k = n/2$). This means that the representation of y will also differ in k digits from that of \bar{x} . Therefore, $y \in N_k(\bar{x})$. This concludes that $N_k(x) = N_k(\bar{x})$ for any $x \in V(G)$.

Third, we will show that G satisfies the second condition. By the adjacency definition of G , the representation of each vertex at distance k from x will differ in k digits from the one of x . Therefore, $|N_k(x)| = \binom{n}{k}$ for any $x \in V(G)$.

By Theorem 3, this leads to the conclusion that G is a k -distance magic graph. \square

In [3], a study was conducted on the existence of distance magic labeling in circulant graphs $C_n(1, a)$. We investigate k -distance magic labeling in circulant graphs, specifically among sibling graphs, through the following theorems.

Theorem 7. *For any positive integer t , a circulant graph $C_n(1, 2, \dots, t)$ is a k -distance graph if $n = 2t(2k - 1) + 2$, with $t \geq 2$ and $k \geq 1$.*

Proof. Let G be a circulant graph $C_n(1, 2, \dots, t)$ with $n = 2t(2k - 1) + 2$ for any $t \geq 2$ and $k \geq 1$. Let $V(G) = \{v_1, v_2, \dots, v_n\}$ and $E(G) = \{v_i v_j : i - j \equiv \pm a \pmod{n}, 1 \leq a \leq t, 1 \leq i, j \leq n\}$. It is easy to verify that the diameter of the graph G is $2k$. For any vertex v_i in G , there is a unique vertex $v_{i+t(2k-1)+1}$ at distance $2k$ from v_i . Therefore, G is a sibling graph with the sibling function $f(v_i) = v_{i+t(2k-1)+1}$ for each i with all indices calculated in modulo n .

Since $d(v_i, v_j) = k$ if $j \equiv i \pm s \pmod{n}$ for all s with $(k - 1)t < s \leq kt$, then $N_k(v_i) = N_k(v_{i+t(2k-1)+1})$ and $|N_k(v_i)| = |N_k(v_l)|$ for any i and l . Therefore by Theorem 3 G is a k -distance graph. \square

Corollary 4. *The cocktail party graph $K_{\underbrace{2, 2, \dots, 2}_n}$ of the order $2n$ is a 1-distance magic graph, for any $n \geq 2$.*

Proof. The cocktail party graph $K_{\underbrace{2, 2, \dots, 2}_n}$ of order $2n$ is isomorphic to the circulant graph $C_{2n}(1, 2, \dots, n - 1)$. By Theorem 7, it is a 1-distance magic graph. \square

Theorem 8. *For any odd integer $a \geq 2$ and any integer $k \geq 1$, a circulant graph $C_n(1, a)$ is a k -distance magic graph if $n = 4ak - (a^2 - 1)$ and $k \geq \lceil a/2 \rceil$.*

Theorem 9. *Let a be an odd integer with $a \geq 3$ and let k be an integer with $k \geq \lceil a/2 \rceil$. Then the circulant graph $C_n(1, a)$ is a k -distance magic graph whenever*

$$n = 4ak - (a^2 - 1).$$

Proof. For any odd integer $a \geq 2$ and $k \geq 1$, let G be a circulant graph $C_n(1, a)$ with $n = 4ak - (a^2 - 1)$. Let $V(G) = \{v_0, v_1, v_2, \dots, v_{n-1}\}$, and $E(G) = \{v_i v_{i \pm t} : 1 \leq i \leq n, t = 1, a\}$, the indices being taken modulo n . For $0 \leq i, j \leq n - 1$ define

$$z = \min\{(j - i) \bmod n, (i - j) \bmod n\}, \quad 0 \leq z \leq \lfloor n/2 \rfloor.$$

Write

$$z = pa + r,$$

with integers $p \geq 0$ and $0 \leq r \leq a - 1$. Then the distance between v_i and v_j in G is

$$d(v_i, v_j) = \begin{cases} p + r, & 0 \leq r \leq \lceil a/2 \rceil, \\ p + 1 + a - r, & \lceil a/2 \rceil < r \leq a - 1. \end{cases}$$

It can be verified that the diameter of the graph G is $d = 2k$. Since $n = 4ak - (a^2 - 1)$, for any vertex v_i , there exists a unique vertex v_{i+2k} at distance $2k$ from v_i , where the indices are calculated in modulo n . Therefore, G is a sibling graph with the sibling function $f(v_i) = v_{i+2k}$ for each i with all indices calculated in modulo n .

Now, we will determine the number of vertices at distance k from vertex v_0 . Let v_i be a vertex in $N_k(v_0)$. By the distance formula, we have $(k - \lceil a/2 \rceil + s)a \leq i \leq (k - \lceil a/2 \rceil + s + 1)a$ for $s = 0, 1, 2, \dots, \lceil a/2 \rceil$. Furthermore, there are exactly two i s in the first interval (that is, for $s = 0$) and four i s in each of the other intervals. Therefore, there are $4\lceil a/2 \rceil + 2 = 2(2\lceil a/2 \rceil + 1) = 2a$ vertices. Then $|N_k(v_0)| = 2a$. Since G is vertex-transitive, then $|N_k(v)| = 2a$ for any $v \in V(G)$. Thus, the second condition of Theorem 3 is satisfied.

Now, we show that the first condition is also satisfied. Let v_i be a vertex at a distance k from v_0 . We will show that v_i is also at distance k from $v_{\frac{n}{2}}$.

Let $z = \min\{i, n - i\}$ be the circular distance between v_0 and v_i along the cycle C_n . Write $z = pa + r$ with integers $p \geq 0$ and $0 \leq r \leq a - 1$. By the distance formula for $C_n(1, a)$,

$$d(v_0, v_i) = \begin{cases} p + r, & 0 \leq r \leq \lceil \frac{a}{2} \rceil, \\ p + 1 + a - r, & \lceil \frac{a}{2} \rceil < r \leq a - 1. \end{cases} \quad (6)$$

Now consider the distance from $v_{n/2}$ to v_i along the cycle. Since v_0 and $v_{n/2}$ are antipodal, the corresponding circular distance is

$$z' = \frac{n}{2} - z.$$

Write $z' = p'a + r'$ with $0 \leq r' \leq a - 1$. Applying the same above formula, we obtain

$$d(v_{n/2}, v_i) = \begin{cases} p' + r', & 0 \leq r' \leq \lceil \frac{a}{2} \rceil, \\ p' + 1 + a - r', & \lceil \frac{a}{2} \rceil < r' \leq a - 1. \end{cases} \quad (7)$$

Using the identity

$$n = 4ak - (a^2 - 1),$$

one checks directly (by computing $z' = \frac{n}{2} - z$ and expanding both expressions in (6) and (7) that for every vertex v_i ,

$$d(v_0, v_i) + d(v_{n/2}, v_i) = 2k. \quad (8)$$

Finally, if $d(v_0, v_i) = k$, substituting into (8) yields

$$k + d(v_{n/2}, v_i) = 2k,$$

and hence

$$d(v_{n/2}, v_i) = k.$$

This shows that $N_k(v_0) = N_k(\pi(v_{n/2}))$. Since G is vertex-transitive, then the first condition of Theorem 3 is satisfied. Therefore, G is a k -distance magic graph. \square

Let S_n be the symmetric group on n elements and let T be the set of transpositions $\{(i, j) : 1 \leq i < j \leq n\}$. The *Bruhat graph* is the graph whose vertex set is the symmetric group S_n and two permutations π and σ are connected by an edge if $\pi\sigma^{-1}$ is a transposition. The Bruhat graph is intimately related to the strong Bruhat order on the symmetric group [4].

In particular, the *weak Bruhat graph* B_n is defined as the graph whose vertex set is the symmetric group S_n and two permutations are connected by an edge if they differ by an adjacent transposition. The associated graph is known as the 1-skeleton of the permutahedron [4]. The weak Bruhat graph has $n!$ vertices.

The distance between two vertices $\sigma_1 := [a_1, a_2, \dots, a_n]$ and $\sigma_2 := [b_1, b_2, \dots, b_n]$ in the weak Bruhat graph B_n is calculated using the Kendall tau distance. The *Kendall tau distance* $K(\sigma_1, \sigma_2)$ is the minimum number of adjacent transpositions necessary to transform σ_1 into σ_2 . It can also be considered as the minimum number of discordant pairs between σ_1 and σ_2 . For example, if $\sigma_1 = [2, 4, 1, 3]$ and $\sigma_2 = [4, 1, 3, 2]$, then the discordant pairs are $D = \{(1, 2), (1, 3), (1, 4)\}$. Thus, $K(\sigma_1, \sigma_2) = 3$. Therefore, the farthest vertex from the vertex $[1, 2, 3, 4]$ in B_4 is $[4, 3, 2, 1]$, namely its distance is $3 + 2 + 1 = 6$. It can be shown that the diameter of the weak Bruhat graph B_n is $(n - 1) + (n - 2) + \dots + 2 + 1 = n(n - 1)/2$. The following theorem deals with the weak Bruhat graph.

Theorem 10. *Let B_n be the weak Bruhat graph on the symmetric group S_n . If $n \equiv 0$ or $1 \pmod{4}$, then B_n is a k -distance magic graph with*

$$k = \frac{n(n-1)}{4}.$$

Proof. Let G be the weak Bruhat graph B_n . The diameter of B_n is $d = (n-1) + (n-2) + \dots + 2 + 1 = n(n-1)/2$. This diameter is even if $n \equiv 0, 1 \pmod{4}$. For any vertex $\sigma := [a_1, a_2, \dots, a_{n-1}, a_n]$ in G , there is a unique vertex $\phi = [a_n, a_{n-1}, \dots, a_2, a_1]$ at distance $n(n-1)/2$, the diameter of G , from σ . Therefore, G is a sibling graph with a sibling function $\pi([a_1, a_2, \dots, a_{n-1}, a_n]) = [a_n, a_{n-1}, \dots, a_2, a_1]$.

Let $\sigma := [a_1, a_2, \dots, a_n]$ be any vertex in G . Let k be half the diameter of G , that is, $k = n(n-1)/4$. Every vertex ϕ at distance k from σ will have exactly k discordant pairs with respect to σ . This means that ϕ will have exactly concordant pairs with

respect to the sibling $\pi(\sigma)$ of σ . Therefore, $\pi(\sigma)$ has exactly $d - k = k$ discordant pairs with respect to ϕ , so the distance between ϕ and $\pi(\sigma)$ is k . Thus, $N_k(\sigma) = N_k(\pi(\sigma))$. We conclude that the first condition of Theorem 3 is satisfied.

From the definition of Kendall tau distance, it is clear that the number of vertices at distance k from any vertex σ is invariant with respect to σ . Therefore, the second condition of Theorem 3 is satisfied. This concludes that G is a k -distance magic graph. \square

In Figure 3, we present the weak Bruhat graph B_4 with a 3-distance magic labeling. This graph is also called the truncated octahedral graph.

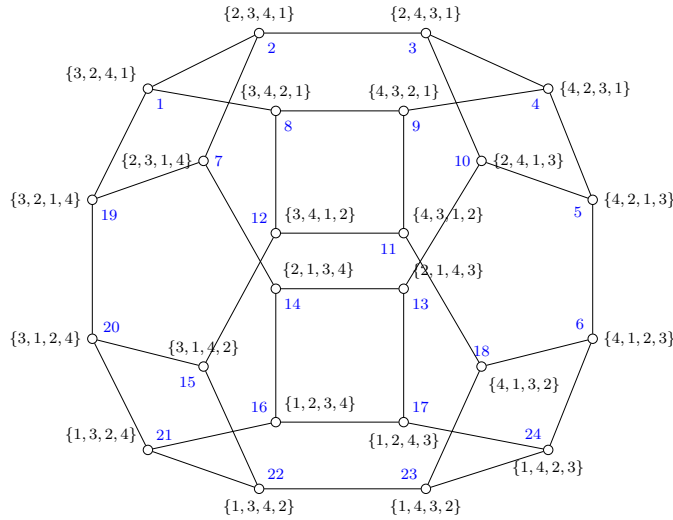


Figure 3. The weak Bruhat graph B_3

The *Möbius-Kantor graph* is a symmetric bipartite cubic graph with 16 vertices and 24 edges. It can be defined as the generalized Petersen graph $P(8, 3)$.

Theorem 11. *The Möbius-Kantor graph is a 2-distance magic graph.*

Proof. Let G be the Möbius-Kantor graph. The order of G is 16 and its diameter is 4. Let $V(G) = \{u_1, u_2, \dots, u_8\} \cup \{v_1, v_2, \dots, v_8\}$ and $E(G) = \{u_i v_i : 1 \leq i \leq 8\} \cup \{u_i u_j : i - j \equiv \pm 3 \pmod{8}, 1 \leq i, j \leq 8\} \cup \{v_i v_j : i - j \equiv \pm 1 \pmod{8}, 1 \leq i, j \leq 8\}$. Then, G is a sibling graph with the sibling function π mapping u_i to u_{i+4} and v_i to v_{i+4} , where all indices are calculated modulo 8. Now, define a 2-distance magic labeling $\alpha : V(G) \rightarrow \{1, 2, \dots, 16\}$ such that if $\alpha(x) = a$ then $\alpha(\pi(x)) = 17 - a$, $1 \leq a \leq 16$. Then, we obtain 51 as the 2-distance magic constant of G . Therefore, the Möbius-Kantor graph is a 2-distance magic graph. \square

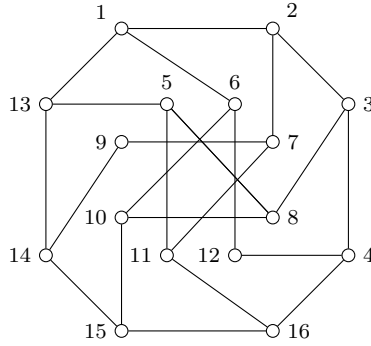


Figure 4. the Möbius-Kantor graph.

In Figure 4, we present a 2-distance magic labeling in the Möbius-Kantor graph. Even though all k -distance magic graphs obtained from the sibling graph family above are regular, we may have ones which are not regular. Consider the following graph F_n . Let n be any positive integer. Define F_n as a graph constructed by connecting a cycle C_8 with $(2n)K_2$ so that we have the set of vertices and the set of edges as follows. Let

$$V(F_n) = \{u_i, v_i : i = 1, 2, 3, 4\} \cup \{a_i, b_i : i = 1, 2, \dots, 2n\},$$

$$E(F_n) = \{u_i u_{i+1}, v_i v_{i+1} : i = 1, 2, 3\} \cup \{u_4 v_1, v_4 u_1\} \cup$$

$$\{u_1 b_j, v_1 a_j : \text{for all odd } j\} \cup$$

$$\{u_4 b_j, v_2 b_j, u_2 a_j, v_4 a_j : \text{for all even } j\}.$$

Note that for any $n \geq 1$ the graph F_n is not regular of order $4n + 8$ and diameter 4. For example, the graph F_3 is depicted in Figure 5.

Theorem 12. *The graph F_n is a 2-distance magic graph for all $n \geq 1$.*

Proof. For any $n \geq 1$, the graph F_n has order $4n + 8$ and diameter 4. Define a permutation $\pi : V(F_n) \rightarrow V(F_n)$ as follows:

$$\pi := (u_1 v_1)(u_2 v_2)(u_3 v_3)(u_4 v_4)(a_1 b_1)(a_2 b_2) \cdots (a_{2n} b_{2n}).$$

This permutation π is a sibling function of F_n . Thus, F_n is a sibling graph. Now, we shall prove that F_n satisfies the first and second conditions of Theorem 3. For any vertex $x \in F_n$, consider all vertices at distance 2 from x . We will have:

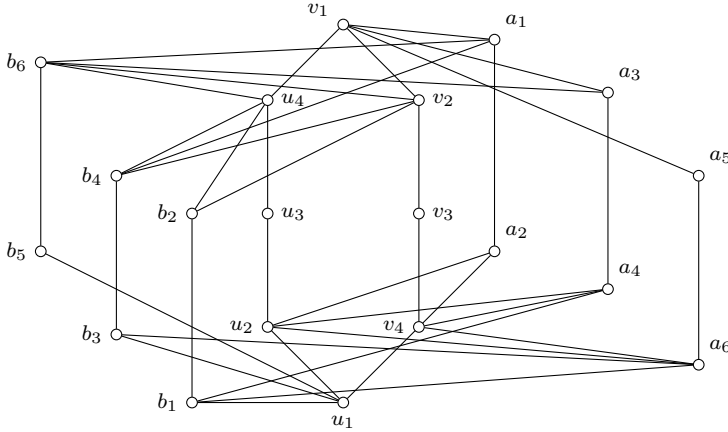


Figure 5. The irregular graph F_3 .

$$N_2(x) = N_2(\pi(x)) = \begin{cases} \{u_3, v_3, a_i, b_i : \text{all even } i\} & , x = u_1 \\ \{u_4, v_4, a_i, b_i : \text{all odd } i\} & , x = u_2 \\ \{u_1, v_1, a_i, b_i : \text{all even } i\} & , x = u_3 \\ \{u_2, u_4, v_2, v_4, a_i, b_i : \text{all odd } i, i \neq t\} & , x = a_t \text{ for odd } t \\ \{u_2, u_4, v_2, v_4, a_i, b_i : \text{all even } i, i \neq t\} & , x = a_t \text{ for even } t. \end{cases}$$

Thus, $N_2(x) = N_2(\pi(x))$ for every $x \in V(F_n)$ and $|N_2(x)| = 2(n+1)$ for every $x \in F_n$. By Theorem 3, F_n is a 2-distance magic graph. \square

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