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Research Article

Analyzing energy and defining new classes of borderenergetic graphs

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Abstract: In graph theory, the *energy* of a graph G, denoted as $\mathcal{E}(G)$, is quantified by $\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|$, where the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ derive from the adjacency matrix of G, and n represents the vertex total. This research investigates the conditions enabling line graphs of non-regular structures to transform into borderenergetic forms, emphasizing structural traits that drive this transition. The focus includes irregular graphs like the complete bipartite graphs $K_{a,b}$ across varying a and b. We also examine corona products, exemplified by $K_{a,b} \circ K_r$, to identify conditions for the emergence of borderenergetic graphs, thus enhancing comprehension of these graphs through structural and spectral perspectives.

Keywords: graphs, rank, eigenvalues, energy, topological index.

AMS Subject classification: 05C50, 05C12, 15A18

1. Introduction

This paper persistently assumes that graphs are simple, undirected, and finite. The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of a graph's adjacency matrix $A(G)$, where G is a graph of order$n$, define its spectrum $Sp(G)$. For more detail on spectral graph theory, see [5]. A graph is termed integral if its eigenvalues are all integers; further reading on integral graphs is available in [1, 4]. The spectrum serves as an energetic signature,

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highlighting how eigenvalue distribution indicates structural energy dynamics. In chemistry, graph spectra are crucial in Hückel Molecular Orbital (HMO) theory for modeling π-electrons energy in conjugated molecules. Spectral graph theory also interfaces with linear algebra and combinatorial optimization, using spectral properties to examine matrix theory concepts like determinants and Perron–Frobenius theory. Additionally, it links to classical combinatorial optimization in graph coloring, stable sets, routing, and embedding. Spectral methods, particularly via the Laplacian matrix and semi-definite programming, offer insights into the complexity of problems such as the traveling salesman problem (TSP).

1.1. Graph energy and borderenergetic graphs

The notion of *graph energy* has its roots in the 1940s, stemming from theoretical chemistry, particularly Erich Hückel's work on the *Hückel Molecular Orbital (HMO) theory*. This theory introduces an approximate approach for solving the Schrödinger equation for *unsaturated conjugated hydrocarbons*, such as benzene. Within the HMO framework, the π-electron energy levels in these molecules are derived from solving an eigenvalue problem involving a reduced *Hamiltonian matrix*. In this context, the Hamiltonian matrix is defined as:

$$H = \alpha I_n + \beta A(G),$$

where I_n is the identity matrix of size n, $A(G)$ is the adjacency matrix representing the molecule's graph G, and α, β are constants reflective of atomic interactions. The β-scaled eigenvalues of $A(G)$ correspond to the molecular orbital energy levels, thus linking molecular chemistry to graph spectra. In the 1970s, Ivan Gutman introduced the notion of *graph energy* within graph theory, defining it as the sum of the absolute values of the adjacency matrix's eigenvalues:

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of $A(G)$. Initially motivated by chemical applications, this energy concept has become pivotal in spectral graph theory, relevant to mathematics, computer science, physics, and combinatorial optimization. For comprehensive studies on this subject and its role in chemical graph theory, refer to [10, 15] and the surveys [8].

In graph theory, the energy of a complete graph K_n is $2(n-1)$. A graph G with order n is termed *borderenergetic* if $\mathcal{E}(G) = 2(n-1)$, following the definition by Gong et al. [7]. When G is not isomorphic to K_n, it is a *non-complete borderenergetic graph*. Significant works in this area include [11, 12, 14, 16, 17]. Given the scarcity of exact borderenergetic graphs, especially among families like line graphs, studying graphs with energy *nearly* $2(n-1)$ is insightful. A graph G is considered *almost borderenergetic* [6] if

$$|\mathcal{E}(G) - 2(n-1)| < 1.$$

These graphs, while not strictly borderenergetic, are of interest due to their close energy resemblance to complete graphs, making them significant in spectral graph theory. The study of borderenergetic graphs is motivated by the concept of *hyper-energetic graphs*, previously introduced by Ivan Gutman, which defines graphs as hyperenergetic if their energy surpasses that of the complete graph. Conversely, borderenergetic graphs define the precise energy threshold, equaling K_n without requiring completeness.
This concept serves as an effective means for assessing and contrasting graph structures that closely resemble complete graphs in terms of spectral properties, shedding light on both theoretical aspects and practical significance concerning graph energy. Numerous studies have investigated borderenergetic graphs. It was demonstrated in [13] that no non-complete borderenergetic graphs exist with a maximum degree $\Delta = 2$ or 3. According to [7], no borderenergetic graphs exist with an order $n \leq 6$, but a unique borderenergetic graph is present for $n = 7$, while graphs for $n \geq 7$ do exist. In [6], researchers analyzed the graph $G = pG_1 \cup qK_{k+1}$, a k-regular graph derived by merging p copies of G_1 and q copies of K_{k+1}, where G_1 is k-regular with order r and possesses t positive eigenvalues, and the rest being less than $2 - k$. They proved that if

$$pE(G_1) = kpr - 2(k-2)pt + (k^2 - 3k + 4)q - 2,$$

then $\mathcal{L}(G)$, the line graph of G, constitutes a non-complete borderenergetic graph. Recent contributions to borderenergetic graphs can be found in [17] and associated works.

2. Construction of some new borderenergetic graphs

This section details the creation of novel infinite families of non-complete borderenergetic graphs via the line graph of a complete bipartite graph. Consider two graphs, $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. The *union* graph $G = G_1 \cup G_2$ is characterized by: $V(G) = V_1 \cup V_2$ and $E(G) = E_1 \cup E_2$. If G_1 and G_2 are disjoint, G's edges exclusively link the original graphs' vertices. The *line graph* $\mathcal{L}(G)$ maps edges of G to vertices in $\mathcal{L}(G)$, with adjacency defined by shared vertices in G. These graphs are valuable in studying structural and spectral properties, as they reflect edge adjacencies. The following lemma, from [2, 9], addresses the eigenvalues of line graphs and signless Laplacian of G.

Lemma 1. *Given a graph G with n vertices and $m \geq 1$ edges, let q_i be the i^{th} largest signless Laplacian eigenvalue of G, and $\lambda_i(\mathcal{L}(G))$ the corresponding eigenvalue of its line graph. Then $q_i(G) = \lambda_i(\mathcal{L}(G)) + 2$, where $i = 1, 2, \ldots, k$ and $k = \min\{n, m\}$. For $m > n$, $\lambda_i(\mathcal{L}(G)) = -2$ for $i \geq n + 1$, and for $n > m$, $q_i = 0$ for $i \geq m + 1$.*

The subsequent result indicates that the line graph of a complete bipartite graph $K_{a,b}$ is borderenergetic only when $(a, b) = (3, 3)$ and otherwise remains non-borderenergetic.

Theorem 1. *Consider the line graph denoted by $\mathcal{L}(K_{a,b})$, where $K_{a,b}$ represents the complete bipartite graph with the condition $1 < a \leq b$. For the specific case when $a = 3$ and $b = 3$, the graph $\mathcal{L}(K_{a,b})$ is identified as a non-complete borderenergetic graph. In contrast, for all other parameter values where $a \neq 3$ and $b \neq 3$, the graph does not possess the borderenergetic property.*

Proof. Consider the complete bipartite graph denoted as $K_{a,b}$ where the integers satisfy $1 < a \leq b$, and observe that it has an overall order of $n = a + b$. Denote the line graph of $K_{a,b}$ by $\mathcal{L}(K_{a,b})$, comprising N vertices and M edges, with calculations following $N = ab$ and $2M = ab(a+b-2)$. It is well-documented within graph theory studies that the signless Laplacian spectrum for $K_{a,b}$ comprises several eigenvalues: specifically, the eigenvalue $n = a+b$ occurs with a multiplicity of one; an eigenvalue of a with a multiplicity of $b-1$; an eigenvalue of b repeating $a-1$ times; and an eigenvalue of 0, which appears once. Invoking Lemma 1, we discern that the eigenvalues of the line graph $\mathcal{L}(K_{a,b})$ are composed such that, there is an eigenvalue $a + b - 2$ with a singular occurrence, an eigenvalue $a - 2$ repeating $b - 1$ times, an eigenvalue $b - 2$ appearing $a - 1$ times, and an eigenvalue of -2 occurring with a multiplicity of $(a - 1)(b - 1)$. Therefore, we compute:

$$\begin{aligned} \mathcal{E}(\mathcal{L}(K_{a,b})) &= a + b - 2 + (b-1)(a-2) + (a-1)(b-2) + 2(a-1)(b-2) \\ &= 4ab - 4b - 4a + 4. \end{aligned} \tag{1}$$

Additionally, referring to a well-established result in the context of complete graphs:

$$\mathcal{E}(K_N) = 2N - 2 = 2ab - 2. \tag{2}$$

If we posit that $\mathcal{L}(K_{a,b})$ qualifies as a borderenergetic graph, the equality $\mathcal{E}(\mathcal{L}(K_{a,b})) = \mathcal{E}(K_N)$ must hold. Consequently, by equating results from equations (1) and (2):

$$4ab - 4b - 4a + 4 = 2ab - 2.$$

This calculation implies the following equation:

$$2ab - 4a - 4b + 6 = 0,$$

From which, it follows that:

$$b = \frac{2a-3}{a-2} = 2 + \frac{1}{a-2}.$$

Given that b represents a positive integer, only one set of values satisfies this equation precisely, namely $(a, b) = (3, 3)$. Thus, $\mathcal{L}(K_{a,b})$ is conclusively borderenergetic solely if $(a, b) = (3, 3)$. Furthermore, note that the degree of any vertex, denoted as v, in $\mathcal{L}(K_{a,b})$ is expressed by $d(v) = a + b - 2 < ab - 1$, which is less than the degree of any vertex in a complete graph with ab vertices. This conclusion, valid for all $a > 1$, indicates that $\mathcal{L}(K_{a,b})$ cannot be a complete graph. Therefore, the proof is completed. □

The compilation of results provided herein elucidates the characteristics of borderenergetic graphs within the context of the graph family denoted by $\mathcal{L}(K_{a,b}) \cup K_r$.

Theorem 2. *Consider the graph $\mathcal{L}(K_{a,b})$, which represents the line graph derived from the complete bipartite graph $K_{a,b}$, where $1 < a \leq b$. Additionally, let K_r denote a complete graph with an order of r, where $r \geq 2$. The graph $G = \mathcal{L}(K_{a,b}) \cup K_r$ manifests as a non-complete borderenergetic graph specifically for the cases where either $a = 3$, $b = 4$, with r being any positive integer, or alternately $a = 4$, $b = 3$, with r remaining arbitrary. For all other combinations of a and b values, the resultant graph G does not exhibit the properties of a borderenergetic graph.*

Proof. According to the foundational definition, the energy associated with the graph G is represented by the equation:

$$\mathcal{E}(G) = \mathcal{E}(\mathcal{L}(K_{a,b}) \cup K_r) = \mathcal{E}(\mathcal{L}(K_{a,b})) + \mathcal{E}(K_r). \tag{3}$$

Utilizing Theorem 1, we establish that:

$$\mathcal{E}(\mathcal{L}(K_{a,b})) = 4ab - 4b - 4a + 4.$$

Additionally, the energy of the complete graph K_r is given by:

$$\mathcal{E}(K_r) = 2r - 2.$$

By combining these results according to equation (3), we deduce:

$$\mathcal{E}(G) = 4ab - 4b - 4a + 2r + 2.$$

The cardinality of the vertex set in the line graph $\mathcal{L}(K_{a,b})$ can be quantified as $|V(\mathcal{L}(K_{a,b}))| = ab$, and for the complete graph K_r, it is r. Consequently, the total number of vertices in the union of these graphs is expressed by:

$$|V(\mathcal{L}(K_{a,b}) \cup K_r)| = ab + r.$$

To ascertain that the graph $G = \mathcal{L}(K_{a,b}) \cup K_r$ is borderenergetic, we need the condition

$$\mathcal{E}(\mathcal{L}(K_{a,b}) \cup K_r) = 2(ab + r) - 2 = 2ab + 2r - 2.$$

By substituting, the expression for $\mathcal{E}(\mathcal{L}(K_{a,b}) \cup K_r)$ becomes

$$4ab - 4b - 4a + 2r + 2 = 2ab + 2r - 2.$$

Through simplification, we derive

$$2ab - 4a - 4b + 4 = 0.$$

This equation can be rephrased as $b = \frac{2(a-1)}{a-2} = 2 + \frac{2}{a-2}$. Given that a and b are positive integers, the solutions that satisfy the equation are exclusively $a = 3, b = 4$ or $a = 4, b = 3$. Thus, the graph $G = \mathcal{L}(K_{a,b}) \cup K_r$ is borderenergetic when $a = 3$, $b = 4$ with r arbitrary, or when $a = 4$, $b = 3$ with r being arbitrary. Additionally, the graph $G = \mathcal{L}(K_{a,b}) \cup K_r$ is neither a complete graph nor a union of complete graphs, which completes the proof. □

The observations presented below are immediate.

Corollary 1. *Consider the two graphs $G_1 = \mathcal{L}(K_{3,4}) \cup K_r$ and $G_2 = \mathcal{L}(K_{4,3}) \cup K_r$. These graphs are identified as non-complete borderenergetic graphs with an order denoted by $n = r + 7$, where the condition $r \geq 2$ is satisfied.*

The statement in Corollary 1 provides confirmation of the existence of borderenergetic graphs for which the order n meets or exceeds 9.

Corollary 2. *For any given values of a and b, except for the specific cases where $a = 3$ and $b = 4$ or where $a = 4$ and $b = 3$, the graph denoted as $G = \mathcal{L}(K_{m,n}) \cup K_r$ is determined to be a non-borderenergetic graph.*

The following findings effectively characterize the borderenergetic graphs within the graph family represented by $\mathcal{L}(K_{a,b}) \cup pK_r$.

Theorem 3. *Consider $a, b, r \in \mathbb{N}$ with the condition $b \geq a > 2$. We introduce the variable*

$$p = ab - 2a - 2b + 3 = (a - 2)(b - 2) - 1.$$

Define the graph G as $G = \mathcal{L}(K_{a,b}) \cup pK_r$, in which $\mathcal{L}(K_{a,b})$ represents the line graph of the complete bipartite graph $K_{a,b}$, and pK_r indicates the disjoint union of p instances of the complete graph K_r. It is established that for any given $r \geq 2$, the graph G is considered a borderenergetic graph according to its properties.

Proof. In light of the fact that the energy of a graph is additive over disjoint unions, we can assert that the energy of the graph $G = \mathcal{L}(K_{a,b}) \cup pK_r$ is determined by the expression

$$\mathcal{E}(G) = \mathcal{E}(\mathcal{L}(K_{a,b})) + 2p(r - 1).$$

Applying Theorem 1, we deduce that the energy associated with the graph $\mathcal{L}(K_{a,b})$ is expressed as

$$\mathcal{E}(\mathcal{L}(K_{m,n})) = 4ab - 4b - 4a + 4.$$

From this result, one can derive that the energy of the combined graph G is effectively represented by the equation

$$\mathcal{E}(G) = 4ab - 4b - 4a + 4 + 2p(r-1).$$

Furthermore, analyzing the vertex count, we find that the number of vertices within $\mathcal{L}(K_{a,b})$ is given by $|V(\mathcal{L}(K_{a,b}))| = ab$, while in the case of pK_r, it comprises pr vertices. Therefore, the total number of vertices in the graph formed by the union is

$$|V(\mathcal{L}(K_{a,b}) \cup pK_r)| = ab + pr.$$

To establish that the graph $G = \mathcal{L}(K_{a,b}) \cup pK_r$ is borderenergetic, we need to satisfy the condition

$$\mathcal{E}(\mathcal{L}(K_{a,b}) \cup pK_r) = 2(ab + pr) - 2 = 2ab + 2pr - 2.$$

By substituting the known formula for $\mathcal{E}(\mathcal{L}(K_{a,b}) \cup pK_r)$, we obtain

$$4ab - 4b - 4a + 4 + 2p(r-1) = 2ab + 2pr - 2.$$

Upon further simplification of the terms on both sides of the equation, we deduce that

$$ab - 2a - 2b + 3 = p.$$

Thus, it is evident that when $p = ab - 2a - 2b + 3$ holds true, with r being any integer, the graph $G = \mathcal{L}(K_{a,b}) \cup pK_r$ will consistently be classified as a borderenergetic graph. □

Example 1. Let us consider the graph $G = \mathcal{L}(K_{3,5}) \cup 2K_4$. Here, $\mathcal{L}(K_{3,5})$ is defined as the line graph derived from the complete bipartite graph $K_{3,5}$, and $2K_4$ represents the disjoint union of two distinct instances of the complete graph K_4.
First, calculate the number of vertices in the line graph $\mathcal{L}(K_{3,5})$, which is the product of its bipartite sets: $3 \times 5 = 15$. Therefore, the overall vertex count in G is given by:

$$|V(G)| = 15 + 2 \times 4 = 23.$$

Next, we assess the energy of the complete graph K_{23}, which is computed using its vertex count:

$$\mathcal{E}(K_{23}) = 2(23-1) = 44.$$

Let us proceed to calculate the energy associated with graph G. The spectral representation of $\mathcal{L}(K_{3,5})$ is characterized by the following eigenvalues and their corresponding multiplicities: the eigenvalue 6 with a multiplicity of 1, the eigenvalue 1 occurring with a multiplicity of 4, the eigenvalue 3 present with a multiplicity of 2, and finally, the eigenvalue -2 with a multiplicity of 8.

Consequently, the energy of $\mathcal{L}(K_{3,5})$ is determined by evaluating the absolute values of its eigenvalues summed according to their multiplicities, resulting in:

$$\mathcal{E}(\mathcal{L}(K_{3,5})) = |6| + 4|1| + 2|3| + 8|-2| = 6 + 4 + 6 + 16 = 32.$$

The calculated energy of the complete graph K_4 is given by the expression $2(4-1) = 6$. Consequently, the energy value for the disjoint union of two identical complete graphs, denoted as $2K_4$, is calculated as $2 \times 6 = 12$. Therefore, the resultant energy of the composite graph G can be expressed as follows:

$$\mathcal{E}(G) = \mathcal{E}(\mathcal{L}(K_{3,5})) + \mathcal{E}(2K_4) = 32 + 12 = 44.$$

Given the condition $\mathcal{E}(G) = \mathcal{E}(K_{23})$, this implies that the graph G qualifies as a borderenergetic graph.

It is evident that for the graph construction $G = \mathcal{L}(K_{3,5}) \cup 2K_r$, it persistently remains a borderenergetic graph regardless of the value of parameter $r \geq 2$. Similarly, by selecting the parameters $a = 4$ and $b = 4$, we deduce that $p = 3$, thereby confirming that the graph $\mathcal{L}(K_{4,4}) \cup 3K_r$ is also classified as a borderenergetic graph. The ensuing theorem provides a formal characterization of the borderenergetic graphs within the broader class $q\mathcal{L}(K_{a,b}) \cup pK_r$.

Theorem 4. *Consider the natural numbers a, b, r such that $b \geq a > 3$ and a positive integer $p > 1$. Define the quantity*

$$q = \frac{p-1}{ab - 2b - 2a + 2} = \frac{p-1}{(a-2)(b-2) - 2}.$$

Let the graph G be represented as $G = q\mathcal{L}(K_{a,b}) \cup pK_r$. Here, $q\mathcal{L}(K_{a,b})$ corresponds to the combination of q instances of the line graph of the complete bipartite graph $K_{a,b}$, and pK_r symbolizes the union of p separate instances of the complete graph K_r. The graph G qualifies as a borderenergetic graph by satisfying the condition

$$\mathcal{E}(G) = \mathcal{E}(K_N),$$

where the integer N is defined as the count of vertices in G, expressed by the equation $N = |V(G)| = qmn + pr$.

Proof. Considering that the energy of a graph is cumulative when taking the disjoint union of graphs, the total energy $\mathcal{E}(G)$ for the graph $G = q\mathcal{L}(K_{a,b}) \cup pK_r$ is derived by the following expression:

$$\mathcal{E}(G) = q\mathcal{E}(\mathcal{L}(K_{a,b})) + 2p(r-1).$$

Referring to Theorem 1, we find that the energy associated with the graph $\mathcal{L}(K_{a,b})$ is represented by:

$$\mathcal{E}(\mathcal{L}(K_{m,n})) = 4ab - 4b - 4a + 4.$$

By substituting the previously mentioned expression, it deduces that the energy of the entire graph G is calculated as:

$$\mathcal{E}(G) = q(4ab - 4b - 4a + 4) + 2p(r - 1).$$

Computing the number of vertices in the graph $q\mathcal{L}(K_{a,b})$, we have $|V(q\mathcal{L}(K_{a,b}))| = qab$ and for the graph pK_r, we find $|V(pK_r)| = pr$. Consequently, the collective number of vertices in the union of these disjoint graphs is:

$$|V(q\mathcal{L}(K_{a,b}) \cup pK_r)| = qab + pr.$$

For the graph $G = q\mathcal{L}(K_{a,b}) \cup pK_r$ to exhibit the characteristic of being borderenergetic, it is essential that:

$$\mathcal{E}(q\mathcal{L}(K_{a,b}) \cup pK_r) = 2(qab + pr) - 2 = 2qab + 2pr - 2.$$

Upon substitution of the known expression for $\mathcal{E}(q\mathcal{L}(K_{a,b}) \cup pK_r)$, we realize:

$$4abq - 4bq - 4aq + 4q + 2p(r - 1) = 2abq + 2pr - 2.$$

Upon further reduction, we find

$$q = \frac{p-1}{ab - 2b - 2a + 2} = \frac{p-1}{(a-2)(b-2) - 2}.$$

This expression elucidates that when $q = \frac{p-1}{(a-2)(b-2)-2}$ and r is any given parameter, the composition of graph $G = q\mathcal{L}(K_{a,b}) \cup pK_r$ exhibits the property of being a bordenergetic graph at all times. □

Example 2. Consider the graph $G = q\mathcal{L}(K_{a,b}) \cup pK_r$ as specified in the preceding theorem, where we set

$$q = \frac{p-1}{(a-2)(b-2) - 2}.$$

Consequently, for these specific pairs (a, b, p, q), the graph G maintains bordenergetic characteristics:

$$(a, b, p, q) = (3, 14, 81, 8),$$
$$(a, b, p, q) = (3, 14, 91, 9),$$
$$(a, b, p, q) = (5, 8, 49, 3),$$
$$(a, b, p, q) = (8, 3, 17, 4),$$
$$(a, b, p, q) = (9, 7, 100, 3),$$
$$(a, b, p, q) = (19, 3, 46, 3).$$

Considering each specific set of values for a, b, p, and q, the associated graph G adheres to the condition

$$\mathcal{E}(G) = \mathcal{E}(K_N),$$

where N denotes the cardinality of the vertex set $V(G)$, calculated as $qmn + pr$. Consequently, the graph G is classified as borderenergetic. □

Given that q is a positive integer, the expression $q = \frac{p-1}{(a-2)(b-2)-2}$ leads to the equation $p - 1 = k\Big((a-2)(b-2) - 2\Big)$ for some positive integer k, which directly results in $q = k$. As an illustrative case, when $a = 4$ and $b = 4$, substituting these values into the expression yields $p - 1 = k\Big((4-2)(4-2) - 2\Big)$, simplifying to $p = 2k + 1$ where k is a positive integer. Thus, this establishes $q = k$, indicating that the composite graph $k\mathcal{L}(K_{4,4}) \cup (2k+1)K_r$ remains borderenergetic for every choice of $k, r \in \mathbb{N}$.

The notion of the *corona product* of two graphs, a significant operation in graph theory, was initially introduced by Frucht and Harary in the year 1970. This concept serves as a structured and recursive method for constructing new graphs by combining existing ones in a systematic manner. Consider two graphs, denoted as G and H. The resulting graph, known as the corona product $G \circ H$, is constructed by taking a single copy of graph G along with $|V(G)|$ copies of graph H. The construction involves connecting each i-th vertex of the graph G to every vertex within the respective i-th copy of graph H. This operation produces a new graph that seamlessly integrates the comprehensive structure of graph G while incorporating multiple replicas of graph H, yielding a composite formation that embodies both the central characteristics of G and the peripheral properties of H. Specifically, the spectrum of the corona product, denoted as $G_1 \circ G_2$, has been derived under the condition where G_2 is a regular graph, as detailed in the results found in [2].

Lemma 2. *Consider G_1 to be an arbitrary graph with a vertex count of n_1 and G_2 to be an r_2-regular graph with n_2 vertices. Define $G = G_1 \circ G_2$ as the corona product of these two graphs. Let $\{\mu_1, \mu_2, \ldots, \mu_{n_1}\}$ denote the eigenvalues comprising the spectrum of G_1, and $\{\eta_1, \eta_2, \ldots, \eta_{n_2}\}$ denote the signless Laplacian spectrum of G_2. In this context, the signless Laplacian spectrum of the graph G is composed of eigenvalues expressed as*

$$\frac{\mu_i + n_2 + 1 + 2r_2 \pm \sqrt{(\mu_i + n_2 + 1 + 2r_2)^2 - 4(2r_2n_2 + (2r_2+1)\mu_i)}}{2}$$

each of which appears with a multiplicity of 1, for $i = 1, 2, \ldots, n_1$. Additionally, the eigenvalues $\eta_j + 1$ have a multiplicity of n_1 for each $j = 1, 2, \ldots, n_2 - 1$.

In the theorem presented below, we aim to determine the energy associated with the line graph of the corona product of the complete bipartite graph $K_{a,b}$ with the complete graph K_r.

Theorem 5. *Consider $G = K_{a,b}$ and $H = K_r$, where the conditions $b \geq a \geq r \geq 3$ are met. We examine the corona product $G \circ H$, and denote the line graph of this product* *as $\mathcal{L}(G \circ H)$. We define the parameters $2\beta_1 = -(a-4) + \sqrt{a^2 - 2a + 4}$, $2\beta_2 = -(b-4) + \sqrt{b^2 - 2b + 4}$, and $2\beta_3 = -(a+b-4) + \sqrt{(a+b)^2 - 2(a+b) + 4}$. Under the condition that $r < \beta_3$, the energy of the line graph is given by*

$$\mathcal{E}(\mathcal{L}(G \circ H)) = (2a+2b)r^2 - (5a+5b-3)r + 2ab + a + b - 5 + \gamma_1 + (b-1)\gamma_2 + (a-1)\gamma_3.$$

Let $\beta_3 \le r < \beta_2$ be the case, then the energy expression becomes

$$\mathcal{E}(\mathcal{L}(G \circ H)) = (2a+2b)r^2 - (5a+5b-6)r + 2ab + 2a + 2b - 10 + (b-1)\gamma_2 + (a-1)\gamma_3.$$

For the scenario where $\beta_2 \le r < \beta_1$, the resulting energy is formulated as

$$\mathcal{E}(\mathcal{L}(G \circ H)) = (2a+2b)r^2 - (2a+5b+3)r + 3ab + b - 3a - 5 + (b-1)\gamma_2.$$

Finally, when the condition $r \ge \beta_1$ is satisfied, the energy takes the form

$$\mathcal{E}(\mathcal{L}(G \circ H)) = (2a+2b)r^2 - (2a+2b)r + 4ab - 4a - 4b.$$

Where

$$\gamma_1 = \sqrt{(a+b+3r-1)^2 - 4(2r^2 + (2a+2b-2)r - (a+b))},$$
$$\gamma_2 = \sqrt{(a+3r-1)^2 - 4[2r^2 - 2r + 2(r-1)a + a]},$$
$$\gamma_3 = \sqrt{(b+3r-1)^2 - 4[2r^2 - 2r + 2(r-1)b + b]}.$$

Proof. Consider the corona product of the complete bipartite graph $K_{a,b}$ and the complete graph K_r, denoted as $G \circ H$. Let us denote its line graph as $\mathcal{L}(G \circ H)$. By definition, constructing the corona product $G \circ H$ involves taking one instance of the graph G, and for each vertex v in $V(G)$, appending a new instance of the graph H. Vertex v is then connected to all vertices in this corresponding instance of H. To begin, we calculate the total number of vertices present in the graph $G \circ H$. The expression for this, reflecting both the vertices from G and the additional vertices from r copies of H, is:

$$|V(G \circ H)| = a + b + (a+b)r = (a+b)(r+1).$$

Subsequently, we need to compute the total number of edges within $G \circ H$. These encompass the following components:

- The initial ab edges which originate from the bipartite graph $K_{a,b}$,
- An additional component of $(a+b) \times \binom{r}{2}$ edges found within each newly attached copy of K_r,

- Further, $(a+b)\times r$ edges are formed by connecting each vertex from G to all r vertices present in its respective copy of H.

This comprehensive enumeration captures all the components contributing to the edge count in $G\circ H$. As a result, we have the following expression for the number of edges in $G\circ H$:

$$\begin{aligned}|E(G\circ H)| &= ab+(a+b)\binom{r}{2}+(a+b)r\\ &= ab+(a+b)\left(\binom{r}{2}+r\right)\\ &= ab+(a+b)r+\frac{r(r-1)(a+b)}{2}.\end{aligned}$$

In connection to this, the signless Laplacian spectrum of the complete bipartite graph $K_{a,b}$ is observed to contain certain eigenvalues. Specifically, it includes the eigenvalue $a+b$ with multiplicity 1, alongside the eigenvalue a which appears with a multiplicity of $b-1$, as well as the eigenvalue b repeating $a-1$ times, in addition to the eigenvalue 0. Furthermore, considering the complete graph K_r, its signless Laplacian spectrum consists of the eigenvalue $2r-2$ with multiplicity 1, along with the eigenvalue $r-2$ which has a multiplicity of $r-2$. By applying Lemma 2, we can articulate the signless Laplacian spectrum of the composed graph $G\circ H$. This spectrum involves the eigenvalue expressed as:

$$\frac{\alpha\pm\sqrt{\alpha^2-4[2r^2-2r+2(r-1)(a+b)+a+b]}}{2},$$

where $\alpha=a+b+3r-1$, each occurring with multiplicity 1. Additionally, it comprises the eigenvalue:

$$\frac{a+3r-1\pm\sqrt{(a+3r-1)^2-4[2r^2-2r+2(r-1)a+a]}}{2},$$

shown with multiplicity $b-1$; and

$$\frac{b+3r-1\pm\sqrt{(b+3r-1)^2-4[2r^2-2r+2(r-1)b+b]}}{2},$$

having a multiplicity of $a-1$. Also included is the eigenvalue $2r$, appearing once, and the eigenvalue $r-1$ with a multiplicity calculated as $(a+b)(r-1)$. Applying Lemma 1, we can determine the spectrum of the graph $\mathcal{L}(G\circ H)$ by considering the following eigenvalues: The first eigenvalue is specified as $\frac{\alpha-4\pm\sqrt{\alpha^2-4[2r^2-2r+2(r-1)(a+b)+a+b]}}{2}$, where $\alpha=a+b+3r-1$, which appears with a multiplicity of 1. Secondly, the eigenvalue $\frac{a+3r-5\pm\sqrt{(a+3r-1)^2-4[2r^2-2r+2(r-1)a+a]}}{2}$ is repeated $b-1$ times. Thirdly, the eigenvalue $\frac{b+3r-5\pm\sqrt{(b+3r-1)^2-4[2r^2-2r+2(r-1)b+b]}}{2}$ is replicated $a-1$ times. Additionally, there is the eigenvalue $2r-2$ which appears once, and there is the eigenvalue $r-3$, repeated $(a+b)(r-1)+1$ times. Lastly, the eigenvalue -2 comes with a multiplicity of $ab+(a+b)\left(\frac{r(r-1)-2}{2}\right)$. Here, when we consider the scenario where the expression $\frac{\alpha-4-\sqrt{\alpha^2-4[2r^2-2r+2(r-1)(a+b)+a+b]}}{2}$ becomes negative, it leads us to the inequality $2r^2+(2a+2b-8)r-3(a+b-2)<0$. This inequality has solutions within the range $r\leq\frac{-(a-b-4)+\sqrt{(a+b)^2-2(a+b)+4}}{2}=\beta_3$. Notably, one can observe that β_3 is less than a. Hence, this illustrates that the eigenvalue $\frac{\alpha-4-\sqrt{\alpha^2-4[2r^2-2r+2(r-1)(a+b)+a+b]}}{2}$ is indeed negative when $r<\beta_3$, while it becomes non-negative when $r\geq\beta_3$. Moreover, considering the case where $\frac{a+3r-5-\sqrt{(a+3r-1)^2-4[2r^2-2r+2(r-1)a+a]}}{2}<0$, we arrive at the inequality:

$$2r^2+(2a-8)r-(3a-6)<0.$$

Consider the condition $r<\beta_1$, where β_1 is given by $\frac{-(a-4)+\sqrt{a^2-2a+4}}{2}$. It can be readily verified that this expression satisfies $\frac{-(a-4)+\sqrt{a^2-2a+4}}{2}\leq a$. This ensures that the eigenvalue $\frac{a+3r-5-\sqrt{(a+3r-1)^2-4[2r^2-2r+2(r-1)a+a]}}{2}$ is negative when $r<\frac{-(a-4)+\sqrt{a^2-2a+4}}{2}$ and non-negative when r reaches or exceeds this threshold. In a similar manner, the eigenvalue $\frac{1}{2}\Big(b+3r-5-\sqrt{(b+3r-1)^2-4[2r^2-2r+(2r-1)b+b]}\Big)$ turns out to be negative for $r<\beta_2$, defined as $\frac{-(b-4)+\sqrt{b^2-2b+4}}{2}$, and becomes non-negative for $r\geq\frac{-(b-4)+\sqrt{b^2-2b+4}}{2}$. Furthermore, given the constraints $a\leq b\leq a+b$, it can be deduced that $\beta_3=\frac{-(a-b-4)+\sqrt{(a+b)^2-2(a+b)+4}}{2}$ is such that $\beta_3\leq\beta_2=\frac{-(b-4)+\sqrt{b^2-2b+4}}{2}\leq\frac{-(a-4)+\sqrt{a^2-2a+4}}{2}=\beta_1$. Consequently, this establishes that if $r<\beta_3$, then the energy of the graph $\mathcal{L}(G\circ H)$ is represented by

$$\begin{aligned}\mathcal{E}(\mathcal{L}(G\circ H)) &= 2\Big[\frac{a+b+3r-5+\gamma_1}{2}+\frac{(b-1)}{2}(a+3r-5+\gamma_2)+2r-2\\ &\quad+\frac{(a-1)}{2}(b+3r-5+\gamma_3)+((a+b)(r-1)+1)(r-3)\Big]\\ &= (2a+2b)r^2-(5a+5b-3)r+2ab+a+b-5+\gamma_1\\ &\quad+(b-1)\gamma_2+(a-1)\gamma_3.\end{aligned}$$

If $\beta_3 \leq r < \beta_2$, then energy of graph $\mathcal{L}(G \circ H)$ is given by

$$\begin{aligned}\mathcal{E}(\mathcal{L}(G \circ H)) &= 2\Big[\frac{a+b+3r-5 \pm \gamma_1}{2} + \frac{(b-1)}{2}(a+3r-5+\gamma_2) + 2r - 2 \\ &+ \frac{(a-1)}{2}(b+3r-5+\gamma_3) + ((a+b)(r-1)+1)(r-3)\Big] \\ &= (2a+2b)r^2 - (5a+5b-6)r + 2ab + 2a + 2b - 10 \\ &+ (b-1)\gamma_2 + (a-1)\gamma_3.\end{aligned}$$

If $\beta_2 \leq r < \beta_1$, then energy of graph $\mathcal{L}(G \circ H)$ is given by

$$\begin{aligned}\mathcal{E}(\mathcal{L}(G \circ H)) &= 2\Big[\frac{a+b+3r-5 \pm \gamma_1}{2} + \frac{(b-1)}{2}(a+3r-5+\gamma_2) + 2r - 2 \\ &+ \frac{(a-1)}{2}(b+3r-5 \pm \gamma_3) + ((a+b)(r-1)+1)(r-3)\Big] \\ &= (2a+2b)r^2 - (2a+5b+3)r + 3ab + b - 3a - 5 + (b-1)\gamma_2.\end{aligned}$$

If $r \geq \beta_1$, then energy of graph $\mathcal{L}(G \circ H)$ is given by

$$\begin{aligned}\mathcal{E}(\mathcal{L}(G \circ H)) &= 2\Big[\frac{a+b+3r-5 \pm \gamma_1}{2} + \frac{(b-1)}{2}(a+3r-5 \pm \gamma_2) + 2r - 2 \\ &+ \frac{(a-1)}{2}(b+3r-5 \pm \gamma_3) + ((a+b)(r-1)+1)(r-3)\Big] \\ &= (2a+2b)r^2 - (2a+2b)r + 4ab - 4a - 4b.\end{aligned}$$

This completes the proof. □

The following result gives the condition under which the graph $\mathcal{L}(G \circ H)$ is borderenergetic.

Theorem 6. *Consider the graphs $G = K_{a,b}$ and $H = K_r$, where we have $b \geq a \geq r \geq 3$. Investigate the corona product $G \circ H$, with $\mathcal{L}(G \circ H)$ symbolizing the line graph derived from this corona product. If the inequality $2r \geq -(a-4) + \sqrt{a^2 - 2a + 4}$ holds, then it follows that the graph $\mathcal{L}(G \circ H)$ is borderenergetic exclusively when $a = 3$ and $b = 5$. In contrast, for any other values, the graph will not exhibit borderenergetic properties.*

Proof. As stated in Theorem 5, the condition $2r \geq -(a-4) + \sqrt{a^2 - 2a + 4}$ determines the energy of the graph $\mathcal{L}(G \circ H)$ to be

$$\mathcal{E}(\mathcal{L}(G \circ H)) = (2a+2b)r^2 - (2a+2b)r + 4ab - 4a - 4b. \tag{4}$$

The quantity of vertices, denoted as N, in the graph $\mathcal{L}(G \circ H)$ is equivalent to the edge count in the graph $G \circ H$. By definition, the graph $G \circ H$ results in

$$N = ab + (a+b)r + \frac{r(r-1)(a+b)}{2}.$$

Subsequently, the energy of a complete graph with N vertices is expressed as

$$\mathcal{E}(K_N) = 2N - 2 = (a+b)r^2 + (a+b)r + 2ab - 2. \tag{5}$$

In the scenario where $\mathcal{L}(G \circ H)$ qualifies as a borderenergetic graph, it holds that $\mathcal{E}(\mathcal{L}(G \circ H)) = \mathcal{E}(K_N)$. Comparing equations (4) and (5) yields

$$(a+b)r^2 - 3(a+b)r + 2ab - 4(a+b) + 2 = 0. \tag{6}$$

Rewriting Equation (6) gives $(a+b)(r^2 - 3r - 4) + 2ab + 2 = 0$, with its right-hand side remaining positive for all $r \geq 4$. Hence, the condition specified by Equation (6) is not satisfied for $r \geq 4$. As a consequence, $r = 3$ emerges as the solitary condition under which Equation (6) holds true. Substituting $r = 3$ into Equation (6) results in

$$9(a+b) - 9(a+b) + 2ab - 4a - 4b + 2 = 0,$$

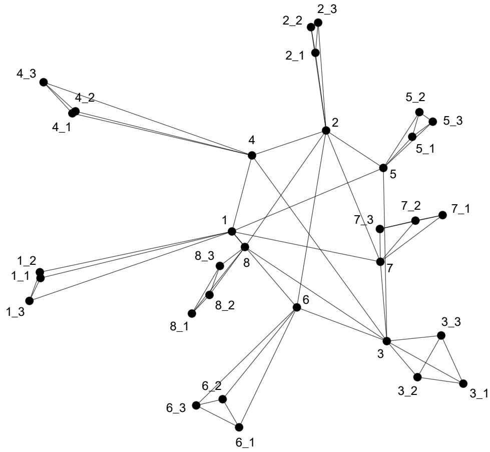
leading to $b = \frac{2a-1}{a-2} = 2 + \frac{3}{a-2}$. The integer solutions that are positive for this equation are $a = 3$ and $b = 5$, or alternatively $a = 5$ and $b = 3$. This deduces that the graph $\mathcal{L}(K_{3,5} \circ K_3)$ indeed forms a borderenergetic graph. □

In fact, one can verify that $\mathcal{E}(\mathcal{L}(K_{3,5} \circ K_3)) = 124 = 2(63) - 2 = \mathcal{E}(K_{63})$, given that the graph $\mathcal{L}(K_{3,5} \circ K_3)$ consists of 63 vertices. Consequently, the following observation is derived.

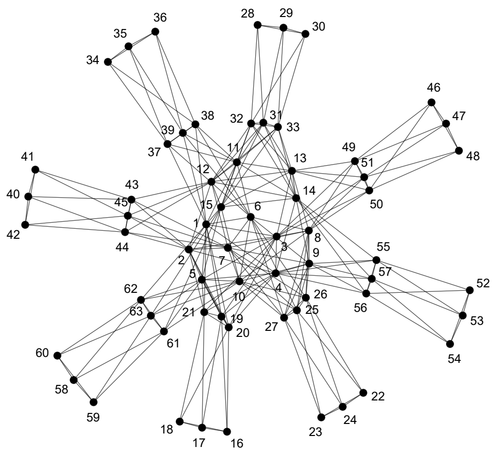
Theorem 7. *The graph represented as $\mathcal{L}(K_{3,5} \circ K_3)$, which arises as the line graph of the corona product involving the complete bipartite graph $K_{3,5}$ and a complete graph K_3, is classified as a borderenergetic graph that is not complete. This specific graph has an order of* 63*, denoting the total number of vertices it contains, as illustrated in 1.*

Let us delve into the scenario where $r = 2$. Focus on the complete bipartite graph denoted by $G = K_{3,4}$ and the complete graph $H = K_2$. We will construct the corona product $G \circ H$, followed by evaluating the eigenvalue spectrum of the line graph $\mathcal{L}(G \circ H)$.

- The complete bipartite graph $G = K_{3,4}$ comprises 7 vertices, attributed to the sum of partitions, and possesses $3 \times 4 = 12$ edges due to the bipartite structure.
- Each of these 7 vertices in G is augmented by a connected copy of $H = K_2$, which itself contains 2 vertices and a single edge.
- As a result, the corona graph $G \circ H$ manifests with a total of $7 + 7 \times 2 = 21$ vertices, accounting for the vertices of both graphs and added connections, and a sum of $12 + 7 \times 1 + 7 \times 2 = 33$ edges due to combined connections.
- The line graph $\mathcal{L}(G \circ H)$ inherits 33 vertices, as each edge in the corona product is transformed into a vertex in the line graph.



$K_{3,5} \circ K_3$



$\mathcal{L}(K_{3,5} \circ K_3)$

Figure 1. **Visualizations of the Corona Product and Its Line Graph**

According to Theorem 5, the eigenvalues characterizing $\mathcal{L}(K_{3,4} \circ K_2)$ are enumerated as follows:

$$
\begin{aligned}
&4+\sqrt{11}, \quad \frac{5+\sqrt{17}}{2}, \quad \frac{5+\sqrt{17}}{2}, \quad 2+\sqrt{3}, \quad 2+\sqrt{3}, \quad 2+\sqrt{3}, \\
&4-\sqrt{11}, \quad \frac{5-\sqrt{17}}{2}, \quad \frac{5-\sqrt{17}}{2}, \quad 2-\sqrt{3}, \quad 2-\sqrt{3}, \quad 2-\sqrt{3}, \\
&2, \quad -1, \quad -1, \quad -1, \quad -1, \quad -1, \quad -1, \quad -1, \quad -1, \quad -2, \quad -2, \\
&-2, \quad -2, \quad -2, \quad -2, \quad -2, \quad -2, \quad -2, \quad -2, \quad -2, \quad -2
\end{aligned}
$$

Hence, the energy of the line graph $\mathcal{L}(G \circ H)$ is computed to be $\mathcal{E}(\mathcal{L}(G \circ H)) = 64$. With $|V(\mathcal{L}(G \circ H))| = 33$ vertices in the line graph, the energy of a complete graph with an equivalent number of vertices is

$$\mathcal{E}(K_{33}) = 2(33 - 1) = 64.$$

Consequently, $\mathcal{L}(K_{3,4} \circ K_2)$ achieves the status of being borderenergetic.
It is imperative to emphasize that while some graphs are termed borderenergetic, not all of them possess integer eigenvalues. An illustrative case is the line graph of the corona product $\mathcal{L}(K_{3,4} \circ K_2)$, which is identified as borderenergetic but includes eigenvalues that are not integers, thus disqualifying it from being classified as an integral graph. Numerous borderenergetic graphs, particularly those generated from irregular graph structures, line graphs, or sophisticated operations like the corona product, often display non-integer eigenvalues.

Theorem 8. *The graph $\mathcal{L}(K_{3,4} \circ K_2)$ represents a non-complete borderenergetic graph with an order of* 33.

For the case where $r = 2$, an analysis based on Theorem 5 reveals that the eigenvalues of the graph $\mathcal{L}(K_{a,b} \circ K_2)$ can be described as follows: the eigenvalue $\frac{a+b+1 \pm \sqrt{(a+b+5)^2 - 4(3a+3b+4)}}{2}$ manifests with a multiplicity of 1; $\frac{a+1 \pm \sqrt{(a+5)^2 - 4(3a+4)}}{2}$ appears with a multiplicity of $b-1$; $\frac{b+1 \pm \sqrt{(b+5)^2 - 4(3b+4)}}{2}$ demonstrates a multiplicity of $a-1$; the eigenvalue 2 is present with a multiplicity of 1; -1 has a multiplicity of $a+b+1$; and -2 exhibits a multiplicity of ab. Further scrutiny and calculations reveal that when $b \geq a \geq 2$, almost all eigenvalues are non-negative with the exceptions being -1 and -2. Consequently, direct computation yields:

$$\mathcal{E}(\mathcal{L}(K_{a,b} \circ K_2)) = 4ab + 2a + 2b + 2.$$

Additionally, the calculation of the total number of vertices, N, in the graph $\mathcal{L}(K_{a,b} \circ K_2)$ leads to:

$$N = ab + 3(a + b).$$

Thus, the energy of a complete graph on N vertices is expressed as:

$$\mathcal{E}(K_N) = 2N - 2 = 6(a + b) + 2ab - 2.$$

If $K_{a,b} \circ K_2$ qualifies as a borderenergetic graph, then it must satisfy the condition $\mathcal{E}(K_{a,b} \circ K_2) = \mathcal{E}(K_N)$, thereby deducing that $b = \frac{2a-2}{a-2} = 2 + \frac{2}{a-2}$. Solving this equation for positive integers yields the solutions $a = 3$ and $b = 4$ or $a = 4$ and $b = 3$. This computation leads us to make the following inference.

Theorem 9. *Consider the graph defined as $G = K_{a,b}$ and another graph denoted $H = K_2$, where it holds true that $b \geq a \geq 2$. Under these conditions, the graph given by $\mathcal{L}(G \circ H)$ can only be described as borderenergetic when $a = 3$ and $b = 4$. In contrast, for all other values of a and b, the graph exhibits non-borderenergetic properties.*

The concept of a *double graph* $D[G]$ emerges through the creation of two identical copies of a given graph G. In this construction, each vertex in one of these copies directly connects with the adjacent vertices that correspond to its equivalent vertex situated in the secondary copy. This idea can be extended in a straightforward manner to include the *k-fold graph* $D^k[G]$. Such a graph is formed by taking k identical replicas of the graph G. Within this framework, a vertex located in any particular replica is linked to the neighboring vertices of its corresponding vertex that are distributed across each of the other remaining $k-1$ replicas. The description of the adjacency spectrum pertaining to $D^k[G]$ is delineated by the subsequent result referenced from [3].

Theorem 10. *Consider a graph G with n vertices, and denote the eigenvalues of its adjacency matrix by $\lambda_1, \lambda_2, \ldots, \lambda_n$. For the k-fold graph, denoted by $D^k[G]$, the set of eigenvalues comprises $k\lambda_1, k\lambda_2, \ldots, k\lambda_n$, supplemented by the eigenvalue 0, which appears with a multiplicity of $(k-1)n$.*

By referring to Theorem 10, it can be concluded that the total energy of the k-fold graph $D^k[G]$ can be expressed mathematically as follows:

$$\mathcal{E}(D^k[G]) = k\left(|\lambda_1| + \cdots + |\lambda_n|\right) + (k-1)n \cdot 0 = k\mathcal{E}(G). \tag{7}$$

Given that the number of vertices in $D^k[G]$ is represented as $N = kn$, if the graph G exhibits an energy level such that $\mathcal{E}(G) = 2n - \frac{2}{k}$, then it follows that $\mathcal{E}(D^k[G]) = k\left(2n - \frac{2}{k}\right) = 2kn - 2 = \mathcal{E}(K_{kn})$. This demonstrates that $D^k[G]$ is classified as a borderenergetic graph. Specifically, in the scenario where $\mathcal{E}(G) = 2n - 1$, it can be deduced that $\mathcal{E}(D[G]) = 2(2n-1) = 4n-2 = \mathcal{E}(K_{2n})$, which implies that $D[G]$ retains the borderenergetic graph status. Moreover, if G itself qualifies as a borderenergetic graph, then $\mathcal{E}(G) = 2n - 2$, leading to $\mathcal{E}(D[G]) = 2(2n-2) = 4n - 4 < 4n - 2 = \mathcal{E}(K_{2n})$. This infers that $D[G]$ does not reach the threshold of a hyperenergetic graph. Consequently, we arrive at the following deduction.

Theorem 11. *Let Γ denote the collection of borderenergetic graphs with order n. If $G \in \Gamma$, then the double graph $D[G]$ associated with G is not hyperenergetic.*

3. Conclusion

The exploration of borderenergetic graphs—those which possess equivalent energy levels to that of complete graphs—yields profound insights within the realm of spectral graph theory. However, the documented instances of such graphs are conspicuously sparse. This paucity served as the impetus for our investigation, which focused on the constructions derived from line graphs of irregular graphs, with a particular emphasis on complete bipartite graphs $K_{m,n}$. Our study further extended to encompass corona products and graph unions. Through comprehensive spectral analysis, we identified novel classes of borderenergetic graphs and delineated the necessary conditions for their emergence. Importantly, we uncovered that borderenergetic graphs do not universally possess integer spectra, with instances like the line graph of $K_{3,4} \circ K_2$ providing evidence of non-integral spectrums. These results significantly augment the understanding of graph energy theory and highlight promising avenues for future research in the area of graph operations and the broader field of structural graph theory.

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