

## On unimodular tricyclic graphs with a unique perfect matching

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**Abstract:** A graph is called unimodular if the determinant of its adjacency matrix is  $\pm 1$ . Akbari and Kirkland [*Linear Algebra and its Applications*, 421(1):3–15, 2007.] proved that a unicyclic graph is unimodular precisely when it has a unique perfect matching. However, a bicyclic or tricyclic graph with a unique perfect matching need not be unimodular. Recently, Basumatary and Sarma [*Discrete Applied Mathematics*, 346:49-61, 2024.] characterized those bicyclic graphs with a unique perfect matching that are unimodular. In this article, we identify which tricyclic graphs with a unique perfect matching possess the unimodularity property. Our results provide a complete characterization of such graphs.

**Keywords:** Adjacency matrix, unique perfect matching, tricyclic graph, unimodular graph.

**AMS Subject classification:** 05C50, 05C05, 15A18

### 1. Introduction

This article focuses exclusively on simple and connected graphs. Consider  $G = (V(G), E(G))$  as a graph with vertex set  $V(G) = \{1, 2, \dots, n\}$  and edge set  $E(G)$ . The order of  $G$  is the number of vertices  $|V(G)|$ , and the size of  $G$  is the number of edges  $|E(G)|$ . The adjacency matrix  $A(G)$  of  $G$  is defined as  $A(G) = [a_{ij}]_{n \times n}$ , where  $a_{ij} = 1$  if  $i$  and  $j$  are adjacent in  $G$  and 0 otherwise.

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A matrix is classified as *totally unimodular* when all of its minors are limited to the values  $\pm 1$  or 0. This concept differs from a unimodular matrix, which is a square integer matrix with determinant equal to  $\pm 1$ . A *unimodular* graph is a graph whose adjacency matrix is unimodular.

A  $p$ -cyclic graph is defined as a graph  $G$  of size  $|V(G)| - 1 + p$ . The value of  $p$  determines the specific classification of the graph. When  $p = 0, 1, 2$  and 3, the graph is classified as a tree, a unicyclic graph, a bicyclic graph and a tricyclic graph, respectively. Note that a tricyclic graph may have more than three cycles, especially when some of its cycles share edges. In particular, tricyclic graphs are the graphs with cyclomatic number 3, where the cyclomatic number is also called the cycle rank or feedback edge set number in the literature. In a graph  $G$ , a *perfect matching* is a collection of edges that are vertex-disjoint and cover all vertices of  $G$ . For a graph  $G$  possessing a unique perfect matching  $\mathcal{M}$ , we say that a subgraph  $H$  is  $\mathcal{M}$ -*reducible* when the edges shared by  $\mathcal{M}$  and  $H$  form a perfect matching for  $H$  itself. A vertex of degree 1 is referred to as a *pendant vertex*. The *base* of a bicyclic graph  $G$  is a subgraph of  $G$  that contains non-pendant vertices from which  $G$  can be constructed by attaching some trees to it. There are three types of bases for a bicyclic graph:  $\infty(a, b)$ ,  $\infty(a, l, b)$  and  $\theta(p, q, r)$  (see Fig. 1).

In an oriented graph (where each edge has a specified direction), a *directed path* is a path in which all its edges follow the same direction. An *alternating cycle* is a cycle in which consecutive edges have opposite orientation. An *alternating graph* is an oriented graph in which every cycle is alternating.

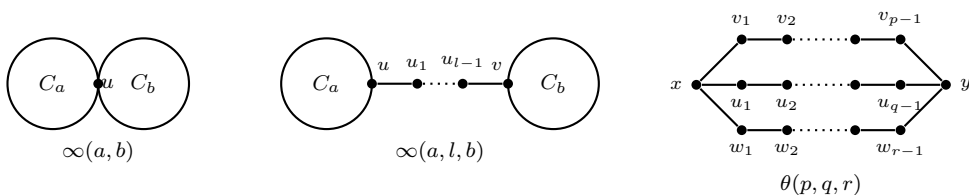


Figure 1: The bases of bicyclic graphs

Many significant problems in combinatorial optimization and related fields can be formulated as integer programs. The solvability of these programs is often closely dependent on the properties of the underlying constraint matrices. A crucial class of matrices in this context is the set of totally unimodular matrices. The unimodular matrices are closely related to totally unimodular matrices. Although the class of totally unimodular matrices has been extensively studied, the class of unimodular matrices has not been explored. Hoffman et al. [9] were the first to establish the necessary and sufficient conditions for a matrix to be totally unimodular. Subsequently, Heller et al. [8] provided additional necessary and sufficient conditions for total unimodularity. These foundational works inspired further research on the characterization of totally unimodular matrices. Numerous studies have contributed to this area, including those

in [3–7, 10] and the references therein. However, there remains limited research that specifically focuses on unimodular matrices.

In the context of unimodular matrices, Akbari and Kirkland [1] provided significant contributions by characterizing unicyclic unimodular graphs. They showed that a unicyclic graph  $G$  is unimodular if and only if it possesses a unique perfect matching. Later, Basumatary and Sarma [2] extended these ideas by characterizing bicyclic unimodular graphs with a unique perfect matching. They showed that a bicyclic graph  $G$  with a unique perfect matching  $\mathcal{M}$  is unimodular if and only if any one of the following conditions is satisfied:

1. the base of  $G$  is either  $\infty(a, b)$  or  $\theta(p, q, r)$ ,
2. the base of  $G$  is  $\infty(a, l, b)$  provided that  $\infty(a, l, b)$  is not  $\mathcal{M}$ -reducible.

These findings suggest possible extensions to general  $p$ -cyclic graphs; however, a complete analysis becomes significantly more complex as the number of cycles increases. Therefore, the study of tricyclic graphs is a natural and necessary intermediate step toward understanding unimodularity in graphs with more cycles. The presence of more than two cycles in a tricyclic graph creates additional structural complexity and new interactions that do not appear in unicyclic or bicyclic graphs. Moreover, tricyclic graphs have been studied in several contexts, including chemical graph theory (particularly in the study of polycyclic molecular structures), spectral graph theory, and matching theory, where the interaction among cycles plays an essential role.

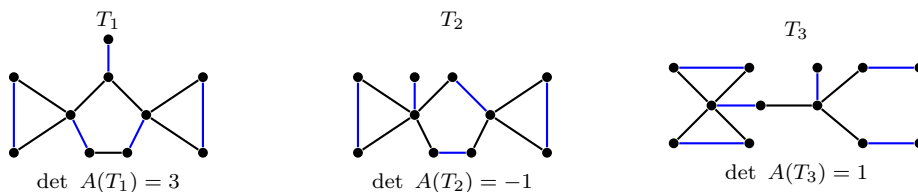


Figure 2: The edges colored blue are the matching edges

Motivated by these considerations, we focus on the class  $\mathcal{T}$  consisting of all tricyclic graphs possessing a unique perfect matching. The exploration of the graphs within  $\mathcal{T}$  indicates the presence of both unimodular and non-unimodular varieties within this collection. For example, as shown in Fig. 2, the tricyclic graphs  $T_2$  and  $T_3$  are unimodular, whereas the graph  $T_1$  is not. This observation shows that determining the unimodularity of the graphs in  $\mathcal{T}$  is a non-trivial task.

In this article, we provide a complete characterization of all unimodular graphs within  $\mathcal{T}$  by establishing a necessary and sufficient condition for a tricyclic graph  $T \in \mathcal{T}$  to be unimodular. Although our arguments do not directly extend to general  $p$ -cyclic graphs, our results provide valuable insight into how cycle structure influences unimodularity. By understanding how cycles interact and affect the determinant of the

adjacency matrix, we obtain structural patterns that can guide further investigation. These insights form a necessary foundation for studying more complex  $p$ -cyclic graphs, where the number of cycles and the complexity of their interactions increase.

The article is structured in three sections. Section 2 establishes the foundational results, notation, and definitions needed. Section 3 presents a necessary and sufficient condition for a tricyclic graph  $T \in \mathcal{T}$  to be unimodular and uses this condition to completely characterize all such unimodular graphs.

## 2. Preliminaries

Consider a graph  $G$  that possesses a unique perfect matching, which we denote by  $\mathcal{M}$  throughout this article. The edges in  $\mathcal{M}$  are called matching edges, and all other edges are non-matching edges. A path  $P$  is said to be an  $mm$ -alternating path if its edges alternate between matching and non-matching edges, beginning and ending with matching edges. Removing an edge  $e$  from a graph  $G$  means that only the edge  $e$  is removed, whereas the incident vertices remain part of the graph. A subgraph of  $G$  that can be formed by choosing some vertices from  $G$  and keeping all the edges between them that are in  $G$  is called an *induced subgraph* of  $G$ .

Suppose  $C_a$  and  $C_b$  are two cycles in a graph  $G$  that share some common edges. Then the new cycle created by combining  $C_a$  and  $C_b$  excluding some of their overlapping vertices is denoted by  $C_a^b$  and we call it the overlap cycle of  $C_a$  and  $C_b$ . We sometimes use the notation  $n(C_a)$  instead of  $|V(C_a)|$  to represent the number of vertices of the cycle  $C_a$ .

A *spanning elementary subgraph* is a subgraph that covers all the vertices of the original graph and consists of connected components that are either single edges or cycles. In the case of a graph possessing a unique perfect matching, there exists a unique spanning elementary subgraph that consists solely of individual edges. We denote the set of all spanning elementary subgraphs of a graph  $G$  by  $\mathcal{S}(G)$  or sometimes simply by  $\mathcal{S}$ . For any  $H \in \mathcal{S}(G)$ , let  $C_H$  denote the subgraph formed by its cycles and  $M_H$  the subgraph formed by its independent edges (edges with no common vertices). Therefore,  $H = C_H \cup M_H$ . For convenience, we consider  $C_H$  and  $M_H$  as sets:  $C_H$  is the set of cycles and  $M_H$  is the set of independent edges within  $H$ . The cardinalities of these sets are denoted by  $|C_H|$  and  $|M_H|$ , respectively.

The following result states that, in a simple connected graph possessing exactly one perfect matching, any spanning elementary subgraph cannot contain precisely one cycle component.

**Lemma 1.** [2, Lemma 4] *Let  $G$  be a simple connected graph possessing a unique perfect matching. Suppose  $H \in \mathcal{S}(G)$  with  $C_H \neq \emptyset$ . Then  $|C_H| \geq 2$ .*

The determinant of  $A(G)$  of a graph  $G$  can be computed using its spanning elementary subgraphs (see [11]).

**Theorem 1.** [11, Theorem 2.3] Consider a simple graph  $G$  with adjacency matrix  $A(G)$ . Then

$$\det A(G) = \sum_{H \in \mathcal{S}(G)} 2^{|C_H|} (-1)^{|C_H| + |E(H)|}.$$

The following result, provided by Basumatary and Sarma [2], gives a necessary and sufficient condition for determining whether a simple connected graph possessing a unique perfect matching is unimodular.

**Theorem 2.** [2, Theorem 5] Let  $G$  be a simple connected graph possessing a unique perfect matching  $\mathcal{M}$ . Then  $G$  is unimodular if and only if

$$\sum_{H_0 \in \mathcal{S} \setminus \{\mathcal{M}\}} 2^{|C_{H_0}|} (-1)^{|C_{H_0}| + |E(H_0)|} = 0.$$

The base of a graph  $G$  is a subgraph  $H$  that fulfills the following conditions:

1.  $H$  does not contain any pendant vertices,
2. any subgraph of  $G$  containing non-pendant vertices is also a subgraph of  $H$ .

Observe that every graph  $G$  possesses a unique base, which we denote by  $G^B$ . Note that, any graph  $G$  can be constructed by connecting trees to the vertices of its base  $G^B$ .

Consider  $\mathcal{Q}(n)$  as the collection of bicyclic graphs containing  $n$  vertices. For any arbitrary graph  $G \in \mathcal{Q}(n)$ , there are three possible types of bases, described as follows:

1.  $\infty(a, b)$ : formed by identifying a vertex  $u$  from each of two cycles  $C_a$  ( $a \geq 3$ ) and  $C_b$  ( $b \geq 3$ ) (see Fig. 3a).
2.  $\infty(a, l, b)$ : formed from two cycles  $C_a$  ( $a \geq 3$ ) and  $C_b$  ( $b \geq 3$ ) with a path of length  $l \geq 1$  connecting a vertex from  $C_a$  to a vertex from  $C_b$  (see Fig. 3b).
3.  $\theta(p, q, r)$ : formed from three mutually exclusive paths connecting vertices  $u$  and  $v$ , with path lengths  $p \geq 1$ ,  $q \geq 2$  and  $r \geq 2$  (see Fig. 3c).

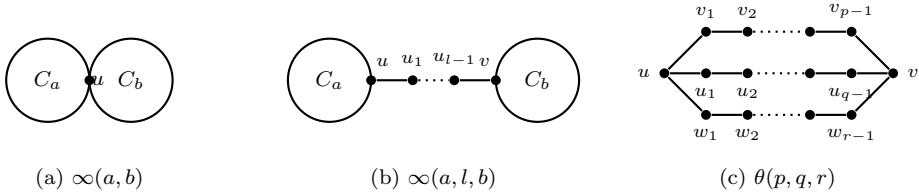


Figure 3: The graphs  $\infty(a, b)$ ,  $\infty(a, l, b)$  and  $\theta(p, q, r)$

We classify the collection  $\mathcal{Q}(n)$  of all bicyclic graphs of order  $n$  into three distinct classes based on their base structures:

- (a)  $\mathcal{Q}_1(n)$  = class of all graphs in  $\mathcal{Q}(n)$  whose base is  $\infty(a, b)$ , where  $a \geq 3$  and  $b \geq 3$ ,
- (b)  $\mathcal{Q}_2(n)$  = class of all graphs in  $\mathcal{Q}(n)$  whose base is  $\infty(a, l, b)$ , where  $a \geq 3, b \geq 3$  and  $l \geq 1$ ,
- (c)  $\mathcal{Q}_3(n)$  = class of all graphs in  $\mathcal{Q}(n)$  whose base is  $\theta(p, q, r)$ , where  $p \geq 1$  and  $\min\{q, r\} \geq 2$ .

Therefore,  $\mathcal{Q}(n) = \mathcal{Q}_1(n) \cup \mathcal{Q}_2(n) \cup \mathcal{Q}_3(n)$ .

The following theorem due to Basumatary and Sarma [2], provides a complete characterization of the class of all bicyclic unimodular graphs that have exactly one perfect matching.

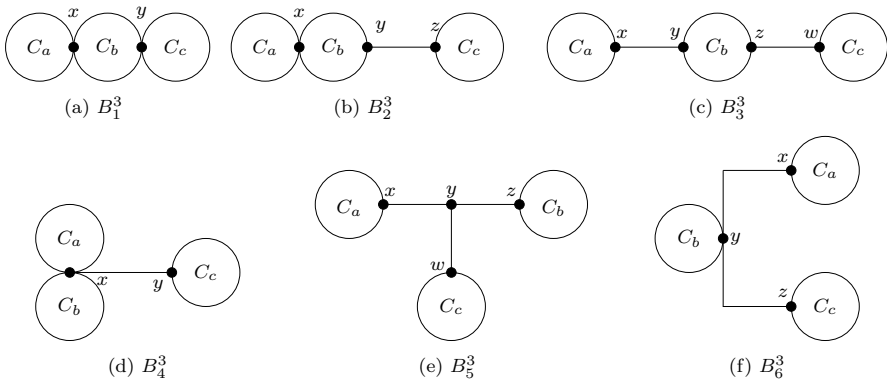
**Theorem 3.** [2, Theorem 10] *Let  $G \in \mathcal{Q}(n)$  be a bicyclic graph possessing a unique perfect matching  $\mathcal{M}$ . Then  $G$  is unimodular if and only if any one of the following conditions is satisfied:*

- (a)  $G \in \mathcal{Q}_1(n)$ ,
- (b)  $G \in \mathcal{Q}_2(n)$  and the base  $G^B = \infty(a, l, b)$  is not  $\mathcal{M}$ -reducible,
- (c)  $G \in \mathcal{Q}_3(n)$ .

**Remark 1.** By Theorem 3, we conclude that a graph  $G \in \mathcal{Q}(n)$  is not unimodular if and only if  $G \in \mathcal{Q}_2(n)$  provided its base graph  $G^B = \infty(p, l, q)$  is  $\mathcal{M}$ -reducible. Equivalently, this means that the structure of the base graph  $G^B = \infty(p, l, q)$  is of the form  $\mathcal{Q}^*$  as shown in Fig. 6. In other words, there exists a spanning elementary subgraph  $H \in \mathcal{S}(G)$  such that  $C_H$  includes both cycles of  $G$ .

### 3. Unimodular tricyclic graphs

Let  $\mathcal{T}$  denote the set of all tricyclic graphs possessing a unique perfect matching  $\mathcal{M}$ . As shown in Figure 5 and detailed in [12], there are 15 distinct types of bases for tricyclic graphs. These bases are denoted by  $B_i^3$  for  $i = 1, \dots, 7$ ,  $B_j^4$  for  $j = 1, \dots, 4$ , and  $B_k^6$  for  $k = 1, \dots, 4$ .



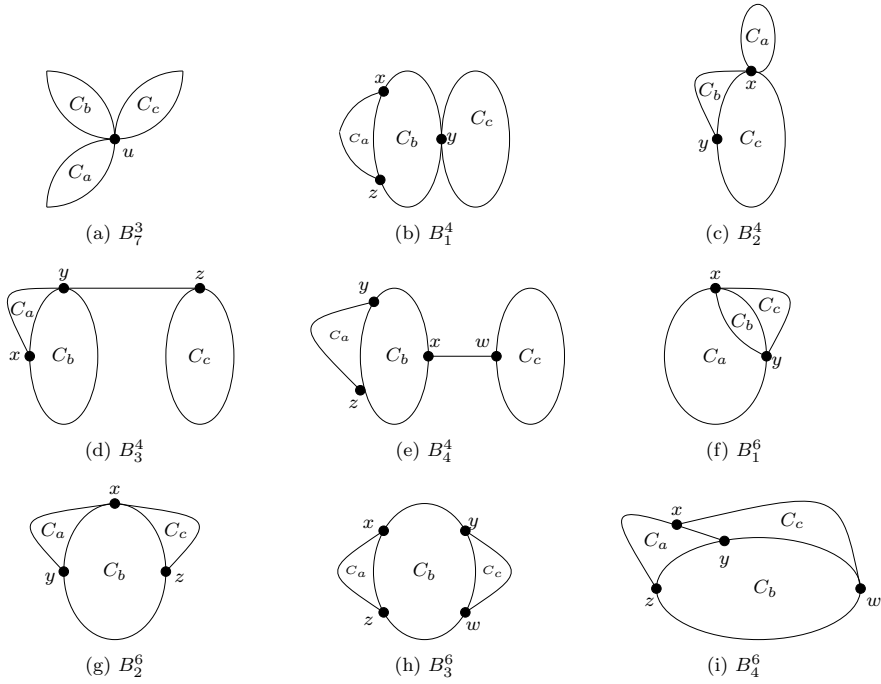


Figure 5: The bases of tricyclic graphs.

Let  $T \in \mathcal{T}$  be a tricyclic graph. For any cycle  $C_i$  in  $T$  and an edge  $e$  in  $C_i$ , let  $T(C_i, e)$  represent the subgraph of  $T$  which is obtained from  $T$  by removing the edge  $e$  from the cycle  $C_i$ . Clearly,  $T(C_i, e)$  is a bicyclic subgraph of  $T$ . For a cycle  $C_i$  in  $T$ , an edge  $e \in E(C_i)$  is termed a *modular edge* of  $C_i$  if it is a non-matching edge of  $C_i$  such that  $T(C_i, e)$  is a unimodular bicyclic subgraph of  $T$ . Conversely,  $e \in E(C_i)$  is a *non-modular edge* of  $C_i$  if it is a non-matching edge such that  $T(C_i, e)$  is not unimodular. We call the cycle  $C_i$  of  $T$  a *modular cycle* if every non-matching edge of  $C_i$  is a modular edge; otherwise, we call that cycle a *non-modular cycle*. Observe that, the unique perfect matching  $\mathcal{M}$  of  $T$  is preserved in  $T(C_i, e)$ , where  $e$  is a non-matching edge of the cycle  $C_i$  of  $T$ .

To proceed further, we need the following definition.

**Definition 1.** [2, Definition 11] Consider a graph  $G$  possessing a unique perfect matching. Suppose  $C$  is a cycle of  $G$ . An edge  $e \in E(G)$  is said to be a *peg* of  $C$  if  $e$  is a matching edge of  $G$  such that it is incident with exactly one vertex of  $C$ .

The following lemma gives the information on the pegs of the two cycle components of a spanning elementary subgraph.

**Lemma 2.** *Consider a tricyclic graph  $T \in \mathcal{T}$ . Suppose there exists a spanning elementary subgraph  $H \in \mathcal{S}(T)$  such that  $C_H = \{C_\alpha, C_\beta\}$ . Then each of  $C_\alpha$  and  $C_\beta$  contains exactly one peg. Moreover, these two pegs are pendent edges of an  $mm$ -alternating path joining the cycles  $C_\alpha$  and  $C_\beta$ .*

*Proof.* Since  $T$  has a unique perfect matching and  $C_\alpha, C_\beta \in C_H$ , both  $C_\alpha$  and  $C_\beta$  must be odd cycles. Hence, each of them contains at least one peg and in fact, the total number of pegs in each cycle is odd. Also all such pegs must be the pendent edges of an  $mm$ -alternating path joining the cycles  $C_\alpha$  and  $C_\beta$ . Note that there are at most two distinct paths between any two cycles in  $T$ . It follows that each of  $C_\alpha$  and  $C_\beta$  contains exactly one peg and these two pegs are pendent edges of an  $mm$ -alternating path joining the cycles  $C_\alpha$  and  $C_\beta$ .  $\square$

The following lemma, which directly follows from Remark 1 and Lemma 2, provides a criterion for identifying a non-modular cycle in a tricyclic graph  $T \in \mathcal{T}$ .

**Lemma 3.** *Consider a tricyclic graph  $T \in \mathcal{T}$ . Let  $e$  be a non-matching edge of the cycle  $C_\alpha$  in  $T$ . Then, the following statements are equivalent:*

1.  $C_\alpha$  is a non-modular cycle with a non-modular edge  $e$ .
2. The base of  $T(C_\alpha, e)$  takes the form  $\mathcal{Q}^*$  (Fig. 6).
3. There exists a spanning elementary subgraph  $H \in \mathcal{S}(T)$  such that  $C_H = \{C_\beta, C_\gamma\}$ , where  $C_\beta$  and  $C_\gamma$  are cycles in  $T$  distinct from  $C_\alpha$ , each having exactly one peg and each of these pegs is a pendent edge of an  $mm$ -alternating path  $\mathcal{P}_{\beta\gamma}$  joining the cycles  $C_\beta$  and  $C_\gamma$  and  $e \notin C_\beta \cup \mathcal{P}_{\beta\gamma} \cup C_\gamma$ .

**Remark 2.** Lemma 3 implies that a tricyclic graph  $T \in \mathcal{T}$  contains at least one non-modular cycle if and only if  $T$  has a spanning elementary subgraph containing exactly two cycle components.

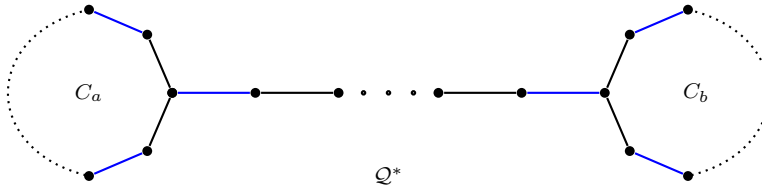


Figure 6: The edges colored blue are the matching edges.

The following lemma establishes an upper bound on the number of spanning elementary subgraphs with exactly two cycle components in case of a tricyclic graph  $T \in \mathcal{T}$ .

**Lemma 4.** [2, Lemma 14] Consider a tricyclic graph  $T \in \mathcal{T}$ . Then  $T$  can have at most two spanning elementary subgraphs containing exactly two cycle components each.

The following lemma provides the structure of tricyclic graphs  $T \in \mathcal{T}$  that have a spanning elementary subgraph  $H$  such that  $|C_H| = 3$ .

**Lemma 5.** [2, Lemma 15] Let  $T \in \mathcal{T}$  be a tricyclic graph that contains a spanning elementary subgraph  $H$  such that  $|C_H| = 3$ . Then the base of  $T$  is of the form  $\mathcal{B}_3^3$  (Fig. 7).

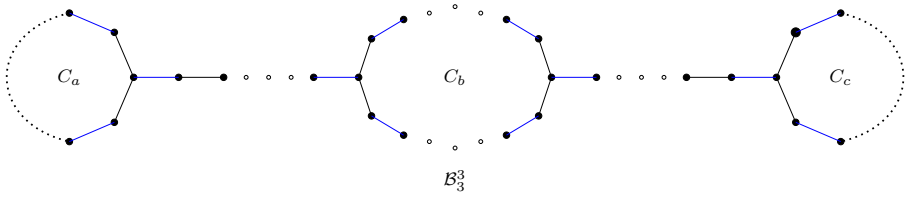


Figure 7: The edges colored blue are the matching edges.

The following result describes about the spanning elementary subgraphs of a tricyclic graph  $T \in \mathcal{T}$  whose base is of the form  $\mathcal{B}_3^3$  (Fig. 7).

**Remark 3.** [2, Remark 16] Consider a tricyclic graph  $T \in \mathcal{T}$  whose base is of the form  $\mathcal{B}_3^3$  (Fig. 7). Then  $T$  can have exactly four spanning elementary subgraphs, namely,  $\mathcal{M}, H_1, H_2$  and  $H_3$ , where (i)  $H_1$  and  $H_2$  each contain two odd cycles such that  $C_{H_1} = C_{H_2}$  and  $|E(H_1)| = |E(H_2)|$  but  $M_{H_1} \neq M_{H_2}$  and (ii)  $H_3$  contains two odd cycles and one even cycle. Note that, in this case,  $T$  contains exactly one non-modular cycle, namely,  $C_b$ .

The following lemma provides the possible forms of the set  $\mathcal{S}(T)$  for a tricyclic graph  $T \in \mathcal{T}$ .

**Lemma 6.** Consider a tricyclic graph  $T \in \mathcal{T}$ . Then  $\mathcal{S}(T)$  is one of the following forms:

1.  $\mathcal{S}(T) = \{\mathcal{M}\}$ ,
2.  $\mathcal{S}(T) = \{\mathcal{M}, H_1\}$ , where  $|C_{H_1}| = 2$ ,
3.  $\mathcal{S}(T) = \{\mathcal{M}, H_1, H_2\}$ , where  $|C_{H_1}| = |C_{H_2}| = 2$ ,
4.  $\mathcal{S}(T) = \{\mathcal{M}, H_1, H_2, H_3\}$ , where  $|C_{H_1}| = |C_{H_2}| = 2$  and  $|C_{H_3}| = 3$ .

*Proof.* The set  $\mathcal{S}(T)$  is nonempty since it contains the perfect matching  $\mathcal{M}$ . For any  $H \in \mathcal{S}(T)$ , we have either  $C_H = \emptyset$  or  $C_H \neq \emptyset$ . If  $C_H = \emptyset$  for all  $H \in \mathcal{S}(T)$ , then  $T$  possesses only one spanning elementary subgraph, namely  $\mathcal{M}$ , and  $\mathcal{S}(T) = \{\mathcal{M}\}$ . Otherwise,  $C_H \neq \emptyset$  for every  $H \in \mathcal{S}(T) \setminus \{\mathcal{M}\}$ . By Lemma 1, we obtain the following two possibilities:

1.  $|C_H| = 2$  for every  $H \in \mathcal{S}(T) \setminus \{\mathcal{M}\}$ . Lemma 4 implies that no more than two such spanning elementary subgraphs can exist. Thus,  $\mathcal{S}(T) = \{\mathcal{M}, H_1\}$  with  $|C_{H_1}| = 2$ , or  $\mathcal{S}(T) = \{\mathcal{M}, H_1, H_2\}$  with  $|C_{H_1}| = |C_{H_2}| = 2$ .
2.  $|C_H| = 3$  for at least one  $H \in \mathcal{S}(T)$ . Lemma 5 shows that the base of  $T$  takes the form  $\mathcal{B}_3^3$ . Remark 3 then gives  $\mathcal{S}(T) = \{\mathcal{M}, H_1, H_2, H_3\}$ , where  $|C_{H_1}| = |C_{H_2}| = 2$  and  $|C_{H_3}| = 3$ .

This completes the proof.  $\square$

Using Lemma 6 and Remark 2, the following result can be deduced.

**Lemma 7.** *Let  $T \in \mathcal{T}$  be a tricyclic graph. Then, every cycle of  $T$  is a modular cycle if and only if  $\mathcal{M}$  is the only one spanning elementary subgraph of  $T$ .*

*Proof.* By Lemma 6, we see that if  $T$  contains a spanning elementary subgraph  $H$  such that  $C_H \neq \emptyset$ , then there exists at least one  $H_1 \in \mathcal{S}(T)$  such that  $|C_{H_1}| = 2$ . Thus, by Remark 2, it follows that  $T$  contains at least one non-modular cycle if and only if there exists at least one  $H \in \mathcal{S}(T)$  such that  $C_H \neq \emptyset$ . Consequently, the final equivalence follows immediately.  $\square$

The following result identifies the possible bases of a tricyclic graph  $T \in \mathcal{T}$  that has exactly three spanning elementary subgraphs, two of which contain exactly two cycle components.

**Lemma 8.** *Consider a tricyclic graph  $T \in \mathcal{T}$ . Let  $H_1$  and  $H_2$  be two spanning elementary subgraphs of  $T$ . Then the following statements are equivalent.*

1.  $\mathcal{S} = \{\mathcal{M}, H_1, H_2\}$  with  $|C_{H_1}| = 2$  and  $|C_{H_2}| = 2$ .
2.  $C_{H_1} = \{C_\alpha, C_\gamma\}$  and  $C_{H_2} = \{C_\beta, C_\gamma\}$ , where  $C_\alpha$  and  $C_\beta$  are non-modular cycles of  $T$  having exactly one peg each.
3. The base of  $T$  is one of the forms  $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4, \mathcal{H}_5, \mathcal{H}_6, \mathcal{H}_7$ , or  $\mathcal{H}_8$  (see Figs. 8–15).
4.  $T$  contains exactly two non-modular cycles having precisely one peg each.

*Proof.* (1)  $\Rightarrow$  (2). Let  $\mathcal{S} = \{\mathcal{M}, H_1, H_2\}$  with  $|C_{H_1}| = 2$  and  $|C_{H_2}| = 2$ . Then, at least one cycle of  $T$  must be vertex disjoint from the other cycles. Consequently, the possible bases of  $T$  are  $B_2^3, B_3^3, B_4^3, B_5^3, B_6^3, B_3^4$  and  $B_4^4$  (see Figs. 4b– 4f, 5d and 5e).

1. Base of  $T$  is any one of  $B_2^3, B_3^3, B_4^3, B_5^3$  or  $B_6^3$ . Then  $T$  contains exactly three distinct cycles  $C_\alpha, C_\beta$  and  $C_\gamma$ , no two of which share a common edge. Since  $|C_{H_1}| = 2$  and  $|C_{H_2}| = 2$ , it implies that  $C_{H_1}$  and  $C_{H_2}$  must have a common cycle. Let  $C_\gamma \in C_{H_1} \cap C_{H_2}$ . Then, we have  $C_{H_1} = \{C_\alpha, C_\gamma\}$  and  $C_{H_2} = \{C_\beta, C_\gamma\}$ . By Lemma 3, it follows that  $C_{H_1} = \{C_\alpha, C_\gamma\}$  implies  $C_\beta$  is a non-modular cycle and  $C_{H_2} = \{C_\beta, C_\gamma\}$  implies  $C_\alpha$  is a non-modular cycle. Moreover, by Lemma 2, it follows that each of the cycles  $C_\alpha$  and  $C_\beta$  contains exactly one peg.

2. Base of  $T$  is either  $B_3^4$  or  $B_4^4$ . Then  $T$  contains exactly four distinct cycles, say,  $C_i, C_j, C_k$  and  $C_l$ , where one of  $C_i, C_j$  and  $C_k$  is the overlap cycle of other two and  $C_l$  is vertex disjoint from  $C_i, C_j$  and  $C_k$ . Since  $|C_{H_1}| = 2$  and  $|C_{H_2}| = 2$ , it implies that  $C_l \in C_{H_1} \cap C_{H_2}$ . Let  $C_{H_1} = \{C_\alpha, C_\gamma\}$  and  $C_{H_2} = \{C_\beta, C_\gamma\}$ , where  $\gamma = l$  and  $\alpha, \beta \in \{i, j, k\}$  with  $\alpha \neq \beta$ . Let  $\delta \in \{i, j, k\} \setminus \{\alpha, \beta\}$ . By Lemma 3,  $C_{H_1} = \{C_\alpha, C_\gamma\}$  implies  $C_\beta$  or  $C_\delta$  is non-modular and  $C_{H_2} = \{C_\beta, C_\gamma\}$  implies  $C_\alpha$  or  $C_\delta$  is non-modular. Since one of  $C_\alpha, C_\beta$  and  $C_\delta$  is the overlap cycle of other two, it follows that all the three cycles  $C_\alpha, C_\beta$  and  $C_\delta$  are non-modular. Moreover, by Lemma 2, it follows that each of the cycles  $C_\alpha$  and  $C_\beta$  contains exactly one peg. (2)  $\Rightarrow$  (3). Since  $C_{H_1} = \{C_\alpha, C_\gamma\}$  and  $C_{H_2} = \{C_\beta, C_\gamma\}$ , the cycle  $C_\gamma$  is vertex disjoint from both cycles  $C_\alpha$  and  $C_\beta$ . Hence, the possible bases of  $T$  are  $B_2^3, B_3^3, B_4^3, B_5^3, B_6^3, B_3^4$  and  $B_4^4$  (see Figs. 4b–4f, 5d and 5e). By Lemma 2, each of the cycles  $C_\alpha, C_\beta$  and  $C_\gamma$  contains exactly one peg. These pegs are the pendent edges of an  $mm$ -alternating path joining the cycles  $C_\alpha$  to  $C_\gamma$  and  $C_\beta$  to  $C_\gamma$ , respectively. Moreover, since  $C_\alpha$  and  $C_\beta$  are non-modular cycles, it follows from Lemma 3 that there exist non-modular edges  $e_1 \in E(C_\alpha)$  and  $e_2 \in E(C_\beta)$  such that the bases of  $T(C_\alpha, e_1)$  and  $T(C_\beta, e_2)$  are of the form  $\mathcal{Q}^*$  (Fig. 6). This implies that the base of  $T$  must be one of the forms  $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4, \mathcal{H}_5, \mathcal{H}_6, \mathcal{H}_7$  and  $\mathcal{H}_8$  as shown in Figs. 8–15. In the bases  $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4, \mathcal{H}_5, \mathcal{H}_6$  and  $\mathcal{H}_7$ , the cycles  $C_a, C_b$  and  $C_c$  as shown in Figs. 8–14 correspond to  $C_\alpha, C_\beta$  and  $C_\gamma$ , respectively. In case of the base  $\mathcal{H}_8$ , the cycles  $C_a^b, C_b$  and  $C_c$  as shown in Fig. 15 correspond to  $C_\alpha, C_\beta$  and  $C_\gamma$ , respectively.

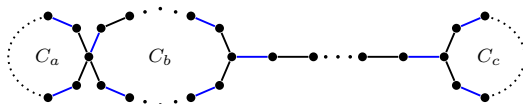
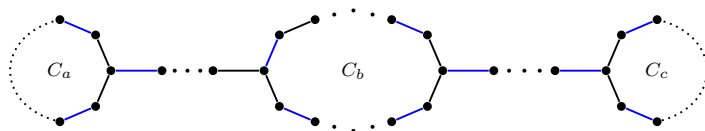
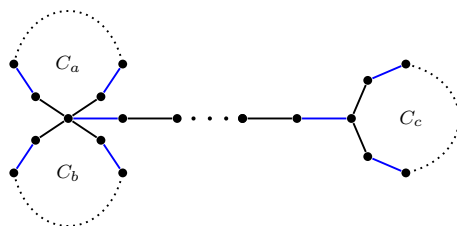
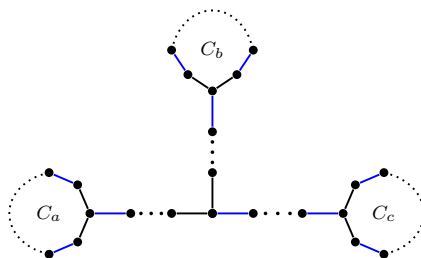
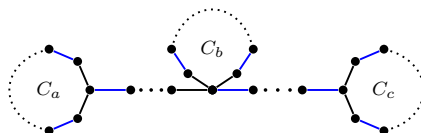
(3)  $\Rightarrow$  (4). In all possible bases of  $T$ , we see that the cycles  $C_\alpha$  and  $C_\beta$  are only the cycles which are non-modular cycles of  $T$  having exactly one peg each.

(4)  $\Rightarrow$  (1). Let  $C_\alpha$  and  $C_\beta$  be two non-modular cycles of  $T$  having exactly one peg each. By Lemma 3, there exists two spanning elementary subgraphs  $H_1$  and  $H_2$  such that  $|C_{H_1}| = |C_{H_2}| = 2$ . If possible suppose that  $C_{H_1} = C_{H_2} = \{C_\gamma, C_\delta\}$ . Then Lemma 3 implies that  $C_\alpha, C_\beta, C_\gamma$  and  $C_\delta$  are four distinct cycles, no two of which overlap. However, from the possible bases of  $T$  as shown in Fig. 5, no such configuration exists, which yields a contradiction. Hence,  $H_1$  and  $H_2$  must be distinct spanning elementary subgraphs of  $T$  such that  $|C_{H_1}| = |C_{H_2}| = 2$ . Further, since a tricyclic graph with base  $\mathcal{B}_3^3$  (Fig. 7) does not contain two non-modular cycles, therefore, Lemma 5 implies that  $T$  contains no spanning elementary subgraph containing exactly three cycle components. Hence, by Lemma 6, we have  $\mathcal{S} = \{\mathcal{M}, H_1, H_2\}$  and  $|C_{H_1}| = 2$  and  $|C_{H_2}| = 2$ .

□

In the graphs  $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4, \mathcal{H}_5, \mathcal{H}_6, \mathcal{H}_7$  and  $\mathcal{H}_8$  shown in Figs. 8–15, the blue colored edges represent the matching edges.

For a tricyclic graph  $T \in \mathcal{T}$  with  $\mathcal{S}(T) = \{\mathcal{M}, H_1, H_2\}$ , where  $|C_{H_1}| = |C_{H_2}| = 2$ , the following result provides a condition under which the expression  $\sum_{H_0 \in \mathcal{S} \setminus \{\mathcal{M}\}} 2^{|C_{H_0}|} (-1)^{|C_{H_0}| + |E(H_0)|}$  is equal to zero.

Figure 8:  $\mathcal{H}_1$ Figure 9:  $\mathcal{H}_2$ Figure 10:  $\mathcal{H}_3$ Figure 11:  $\mathcal{H}_4$ Figure 12:  $\mathcal{H}_5$

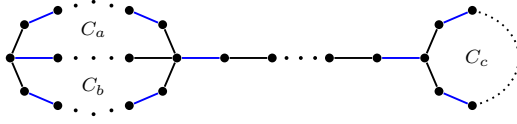


Figure 13:  $\mathcal{H}_6$

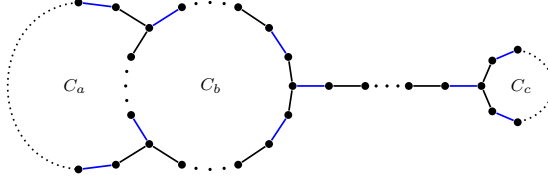


Figure 14:  $\mathcal{H}_7$

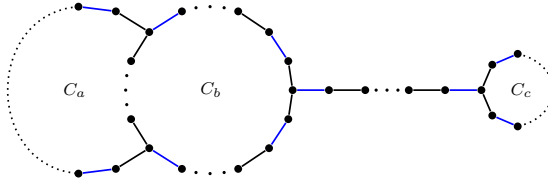


Figure 15:  $\mathcal{H}_8$

**Lemma 9.** Consider a tricyclic graph  $T \in \mathcal{T}$  with  $\mathcal{S} = \{\mathcal{M}, H_1, H_2\}$ , where  $C_{H_1} = \{C_\alpha, C_\gamma\}$  and  $C_{H_2} = \{C_\beta, C_\gamma\}$ , where  $C_\alpha$  and  $C_\beta$  are two non-modular cycles having precisely one peg each. Then  $\sum_{H_0 \in \mathcal{S} \setminus \{\mathcal{M}\}} 2^{|C_{H_0}|} (-1)^{|C_{H_0}| + |E(H_0)|} = 0$  if and only if  $n(C_\alpha) + n(C_\beta)$  is a multiple of 4.

*Proof.* We have

$$\begin{aligned}
 \sum_{H_0 \in \mathcal{S} \setminus \{\mathcal{M}\}} 2^{|C_{H_0}|} (-1)^{|C_{H_0}| + |E(H_0)|} &= 2^{|C_{H_1}|} (-1)^{|C_{H_1}| + |E(H_1)|} \\
 &\quad + 2^{|C_{H_2}|} (-1)^{|C_{H_2}| + |E(H_2)|} \\
 &= 4 \left( (-1)^{|E(H_1)|} + (-1)^{|E(H_2)|} \right) \\
 &= 4(-1)^{|M_{H_1}| + |E(C_\alpha)| + |E(C_\gamma)|} \\
 &\quad + 4(-1)^{|M_{H_2}| + |E(C_\beta)| + |E(C_\gamma)|}. \tag{3.1}
 \end{aligned}$$

Since all the cycles  $C_\alpha, C_\beta$  and  $C_\gamma$  are odd cycles, (3.1) reduces to

$$\sum_{H_0 \in \mathcal{S} \setminus \{\mathcal{M}\}} 2^{|C_{H_0}|} (-1)^{|C_{H_0}| + |E(H_0)|} = 4 \left( (-1)^{|M_{H_1}|} + (-1)^{|M_{H_2}|} \right). \quad (3.2)$$

We can decompose  $M_{H_1}$  as follows:

$$M_{H_1} = (M_{H_1} \setminus E(T^B)) \cup (M_{H_1} \cap E(T^B)), \text{ where } T^B \text{ denotes the base of } T. \quad (3.3)$$

Since  $C_\beta \notin C_{H_1}$ , the set  $M_{H_1} \cap E(T^B)$  can be further decomposed as:

$$M_{H_1} \cap E(T^B) = (M_{H_1} \cap E(C_\beta)) \cup (M_{H_1} \cap (E(T^B) \setminus E(C_\beta))). \quad (3.4)$$

From (3.3) and (3.4), we have

$$M_{H_1} = (M_{H_1} \setminus E(T^B)) \cup (M_{H_1} \cap E(C_\beta)) \cup (M_{H_1} \cap (E(T^B) \setminus E(C_\beta))). \quad (3.5)$$

Similarly, we can decompose  $M_{H_2}$  as follows:

$$M_{H_2} = (M_{H_2} \setminus E(T^B)) \cup (M_{H_2} \cap E(C_\alpha)) \cup (M_{H_2} \cap (E(T^B) \setminus E(C_\alpha))). \quad (3.6)$$

According to Lemma 8, the possible bases of  $T$  are  $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4, \mathcal{H}_5, \mathcal{H}_6, \mathcal{H}_7$  and  $\mathcal{H}_8$  (Figs. 8- 15). Based on these structures, the following observations can be easily verified:  $|M_{H_1} \setminus E(T^B)| = |M_{H_2} \setminus E(T^B)|$  and  $|M_{H_1} \cap (E(T^B) \setminus E(C_\beta))| = |M_{H_2} \cap (E(T^B) \setminus E(C_\alpha))|$ . Furthermore, we have

$$|M_{H_1} \cap E(C_\beta)| = \begin{cases} \frac{n(C_\beta)-1}{2} & \text{if } E(C_\alpha) \cap E(C_\beta) = \emptyset, \\ \frac{n(C_\beta)-|V(C_\alpha) \cap V(C_\beta)|}{2} & \text{if } E(C_\alpha) \cap E(C_\beta) \neq \emptyset. \end{cases}$$

Similarly,

$$|M_{H_2} \cap E(C_\alpha)| = \begin{cases} \frac{n(C_\alpha)-1}{2} & \text{if } E(C_\alpha) \cap E(C_\beta) = \emptyset, \\ \frac{n(C_\alpha)-|V(C_\alpha) \cap V(C_\beta)|}{2} & \text{if } E(C_\alpha) \cap E(C_\beta) \neq \emptyset. \end{cases}$$

Finally, we can verify that  $|V(C_\alpha) \cap V(C_\beta)|$  is always odd.

Now if  $E(C_\alpha) \cap E(C_\beta) = \emptyset$ , then from (3.5) and (3.6), we have

$$|M_{H_1}| + |M_{H_2}| = 2|M_{H_1} \setminus E(T^B)| + 2|M_{H_1} \cap (E(T^B) \setminus E(C_\beta))| + \frac{n(C_\alpha) + n(C_\beta) - 1}{2}. \quad (3.7)$$

From (3.7), we can conclude that  $|M_{H_1}| + |M_{H_2}|$  is odd if and only if  $\frac{n(C_\alpha) + n(C_\beta)}{2}$  is even. Thus,  $|M_{H_1}|$  and  $|M_{H_2}|$  have opposite parity if and only if  $n(C_\alpha) + n(C_\beta)$  is a multiple of 4. Consequently, from (3.2), we conclude that

$$\sum_{H_0 \in \mathcal{S} \setminus \{\mathcal{M}\}} 2^{|C_{H_0}|} (-1)^{|C_{H_0}| + |E(H_0)|} = 0 \text{ if and only if } n(C_\alpha) + n(C_\beta) \text{ is a multiple of 4.}$$

Similarly, if  $E(C_\alpha) \cap E(C_\beta) \neq \emptyset$ , then from (3.5) and (3.6), we have

$$\begin{aligned} |M_{H_1}| + |M_{H_2}| &= 2|M_{H_1} \setminus E(T^B)| + 2|M_{H_1} \cap (E(T^B) \setminus E(C_\beta))| \\ &\quad + \frac{n(C_\alpha) + n(C_\beta)}{2} - |V(C_\alpha) \cap V(C_\beta)|. \end{aligned}$$

Since  $|V(C_\alpha) \cap V(C_\beta)|$  is always odd, by similar argument, we have

$$\sum_{H_0 \in \mathcal{S} \setminus \{\mathcal{M}\}} 2^{|C_{H_0}|} (-1)^{|C_{H_0}| + |E(H_0)|} = 0 \text{ if and only if } n(C_\alpha) + n(C_\beta) \text{ is a multiple of 4.} \quad \square$$

From Lemma 8 and 9, we obtain the following result.

**Lemma 10.** *Consider a tricyclic graph  $T \in \mathcal{T}$ . Let  $H_1$  and  $H_2$  denote the spanning elementary subgraphs of  $T$  such that each contain exactly two cycle components. Then  $\mathcal{S} = \{\mathcal{M}, H_1, H_2\}$ , where  $|C_{H_1}| = |C_{H_2}| = 2$  and  $\sum_{H_0 \in \mathcal{S} \setminus \{\mathcal{M}\}} 2^{|C_{H_0}|} (-1)^{|C_{H_0}| + |E(H_0)|} = 0$  if and only if  $T$  contains exactly two non-modular cycles having precisely one peg each, say  $C_\alpha$  and  $C_\beta$ , such that  $n(C_\alpha) + n(C_\beta)$  is a multiple of 4.*

For a tricyclic graph  $T \in \mathcal{T}$  with base  $\mathcal{B}_3^3$ , the subsequent result establishes a condition under which the expression  $\sum_{H_0 \in \mathcal{S} \setminus \{\mathcal{M}\}} 2^{|C_{H_0}|} (-1)^{|C_{H_0}| + |E(H_0)|}$  equals zero.

**Lemma 11.** *Consider a tricyclic graph  $T \in \mathcal{T}$ . Suppose that base of  $T$  is  $\mathcal{B}_3^3$  (Fig. 7) with three distinct cycles  $C_a, C_b$  and  $C_c$ , where  $C_b$  is a non-modular cycle containing exactly two pegs. Then*

$$\sum_{H_0 \in \mathcal{S} \setminus \{\mathcal{M}\}} 2^{|C_{H_0}|} (-1)^{|C_{H_0}| + |E(H_0)|} = 0 \text{ if and only if } n(C_b) \text{ is a multiple of 4.}$$

*Proof.* By Remark 3, we have  $\mathcal{S}(T) = \{\mathcal{M}, H_1, H_2, H_3\}$ , where  $|C_{H_1}| = |C_{H_2}| = 2, |C_{H_3}| = 3$ . Furthermore, we have  $C_{H_1} = C_{H_2} = \{C_a, C_c\}$  and  $C_{H_3} = \{C_a, C_b, C_c\}$ , where  $C_a$  and  $C_c$  are odd cycles whereas  $C_b$  is an even cycle. Clearly,  $|E(H_1)| = |E(H_2)|$ . We have

$$\begin{aligned} \sum_{H_0 \in \mathcal{S} \setminus \{\mathcal{M}\}} 2^{|C_{H_0}|} (-1)^{|C_{H_0}| + |E(H_0)|} &= 2^{|C_{H_1}|} (-1)^{|C_{H_1}| + |E(H_1)|} \\ &\quad + 2^{|C_{H_2}|} (-1)^{|C_{H_2}| + |E(H_2)|} + 2^{|C_{H_3}|} (-1)^{|C_{H_3}| + |E(H_3)|} \\ &= 4(-1)^{|E(H_1)|} + 4(-1)^{|E(H_2)|} - 8(-1)^{|E(H_3)|}. \end{aligned} \quad (3.8)$$

Since  $|E(H_1)| = |E(H_2)|$ , (3.8) reduces to

$$\begin{aligned} \sum_{H_0 \in \mathcal{S} \setminus \{\mathcal{M}\}} 2^{|C_{H_0}|} (-1)^{|C_{H_0}| + |E(H_0)|} &= 8 \left( (-1)^{|E(H_1)|} - (-1)^{|E(H_3)|} \right) \\ &= 8 \left( (-1)^{|M_{H_1}| + |E(C_a)| + |E(C_c)|} + (-1)^{|M_{H_3}| + |E(C_a)| + |E(C_b)| + |E(C_c)|} \right). \end{aligned} \quad (3.9)$$

Since  $C_a$  and  $C_c$  are odd cycles and  $C_b$  is an even cycle, (3.9) reduces to

$$\sum_{H_0 \in \mathcal{S} \setminus \{\mathcal{M}\}} 2^{|C_{H_0}|} (-1)^{|C_{H_0}| + |E(H_0)|} = 8 \left( (-1)^{|M_{H_1}|} + (-1)^{|M_{H_3}|} \right). \quad (3.10)$$

We have,

$$\begin{aligned} M_{H_1} &= (M_{H_1} \setminus E(\mathcal{B}_3^3)) \cup (M_{H_1} \cap E(\mathcal{B}_3^3)) \\ &= (M_{H_1} \setminus E(\mathcal{B}_3^3)) \cup (M_{H_1} \cap E(C_b)) \cup (M_{H_1} \cap (E(\mathcal{B}_3^3) \setminus E(C_b))). \end{aligned} \quad (3.11)$$

and

$$M_{H_3} = (M_{H_3} \setminus E(\mathcal{B}_3^3)) \cup (M_{H_3} \cap E(\mathcal{B}_3^3)). \quad (3.12)$$

From Fig. 7, it can be easily verified that

$$|M_{H_1} \setminus E(\mathcal{B}_3^3)| = |M_{H_3} \setminus E(\mathcal{B}_3^3)|, \quad |M_{H_1} \cap (E(\mathcal{B}_3^3) \setminus E(C_b))| = |M_{H_3} \cap E(\mathcal{B}_3^3)|,$$

and  $|M_{H_1} \cap E(C_b)| = \frac{n(C_b)}{2}$ . Thus, from (3.11) and (3.12), we have

$$|M_{H_1}| + |M_{H_3}| = 2 |M_{H_1} \setminus E(\mathcal{B}_3^3)| + 2 |M_{H_1} \cap (E(\mathcal{B}_3^3) \setminus E(C_b))| + \frac{n(C_b)}{2}. \quad (3.13)$$

From (3.13), we can conclude that  $|M_{H_1}| + |M_{H_3}|$  is even if and only if  $\frac{n(C_b)}{2}$  is even. Thus, it follows that  $|M_{H_1}|$  and  $|M_{H_3}|$  have the same parity if and only if  $n(C_b)$  is a multiple of 4. Consequently, from (3.10), we deduce that

$$\sum_{H_0 \in \mathcal{S} \setminus \{\mathcal{M}\}} 2^{|C_{H_0}|} (-1)^{|C_{H_0}| + |E(H_0)|} = 0 \text{ if and only if } n(C_b) \text{ is a multiple of 4.} \quad \square$$

By Remark 3 and Lemma 11, we obtain the following result.

**Lemma 12.** *Consider a tricyclic graph  $T \in \mathcal{T}$ . Let  $H_1, H_2$  and  $H_3$  denote the spanning elementary subgraphs of  $T$  such that  $|C_{H_1}| = |C_{H_2}| = 2$  and  $|C_{H_3}| = 3$ . Then the following statements are equivalent.*

- (i) *Base of  $T$  is  $\mathcal{B}_3^3$  (Fig. 7) such that  $n(C_b)$  is a multiple of 4, where  $C_b$  is a non-modular cycle containing exactly two pegs.*

$$(ii) \mathcal{S} = \{\mathcal{M}, H_1, H_2, H_3\}, \text{ where } |C_{H_1}| = |C_{H_2}| = 2, |C_{H_3}| = 3 \text{ and} \\ \sum_{H_0 \in \mathcal{S} \setminus \{\mathcal{M}\}} 2^{|C_{H_0}|} (-1)^{|C_{H_0}| + |E(H_0)|} = 0.$$

The following result provides the necessary and sufficient condition for a tricyclic graph  $T \in \mathcal{T}$  to be unimodular.

**Theorem 4.** *Consider a tricyclic graph  $T \in \mathcal{T}$ . Then  $T$  is unimodular if and only if any one of the following properties is satisfied:*

1. every cycle of  $T$  is a modular cycle,
2.  $T$  contains exactly two non-modular cycles having precisely one peg each, say  $C_\alpha$  and  $C_\beta$ , such that  $n(C_\alpha) + n(C_\beta)$  is a multiple of 4.
3. the base of  $T$  is  $\mathcal{B}_3^3$  (Fig. 7) such that  $n(C_b)$  is a multiple of 4, where  $C_b$  is a non-modular cycle containing exactly two pegs.

*Proof.* By Lemma 6, the set  $\mathcal{S}$  of spanning elementary subgraphs of  $T$  takes one of the following forms:

1.  $\mathcal{S} = \{\mathcal{M}\}$ ,
2.  $\mathcal{S} = \{\mathcal{M}, H_1\}$ , where  $|C_{H_1}| = 2$ ,
3.  $\mathcal{S} = \{\mathcal{M}, H_1, H_2\}$ , where  $|C_{H_1}| = |C_{H_2}| = 2$ ,
4.  $\mathcal{S} = \{\mathcal{M}, H_1, H_2, H_3\}$ , where  $|C_{H_1}| = |C_{H_2}| = 2$  and  $|C_{H_3}| = 3$ .

Observe that if  $\mathcal{S} = \{\mathcal{M}, H_1\}$ , where  $|C_{H_1}| = 2$  then by Lemma 2,  $T$  cannot be unimodular under any conditions. Thus, we eliminate this case from the possible forms of  $\mathcal{S}$  for the unimodularity of  $T$ . Now, by applying Theorem 2, we conclude that  $T$  is unimodular if and only if  $\mathcal{S}$  fulfills one of the following conditions:

1.  $\mathcal{S} = \{\mathcal{M}\}$ ,
2.  $\mathcal{S} = \{\mathcal{M}, H_1, H_2\}$ , where  $|C_{H_1}| = |C_{H_2}| = 2$  and \\  $\sum_{H_0 \in \mathcal{S} \setminus \{\mathcal{M}\}} 2^{|C_{H_0}|} (-1)^{|C_{H_0}| + |E(H_0)|} = 0$ ,
3.  $\mathcal{S} = \{\mathcal{M}, H_1, H_2, H_3\}$ , where  $|C_{H_1}| = |C_{H_2}| = 2, |C_{H_3}| = 3$  and \\  $\sum_{H_0 \in \mathcal{S} \setminus \{\mathcal{M}\}} 2^{|C_{H_0}|} (-1)^{|C_{H_0}| + |E(H_0)|} = 0$ .

Finally, the proof is completed by combining the above cases with observations from Lemma 7, Lemma 10 and Lemma 12.  $\square$

We define the following classes of tricyclic graphs based on their bases and specific properties:

1. **Class  $\mathcal{T}_1$ :** This class consists of tricyclic graphs  $T$  whose base  $T^B$  does not contain  $\mathcal{Q}^*$  (see Fig. 6) as an induced subgraph.
2. **Class  $\mathcal{T}_2$ :** This class consists of tricyclic graphs  $T$  whose base  $T^B$  is any one of  $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4, \mathcal{H}_5, \mathcal{H}_6, \mathcal{H}_7$  or  $\mathcal{H}_8$  (see Figs. 8- 15) under the condition that  $n(C_\alpha) + n(C_\beta)$  is a multiple of 4, where each of  $C_\alpha$  and  $C_\beta$  is a non-modular cycle containing exactly one peg.
3. **Class  $\mathcal{T}_3$ :** This class consists of tricyclic graphs  $T$  whose base  $T^B$  is  $\mathcal{B}_3^3$  (see Fig. 7) provided  $n(C_b)$  is a multiple of 4, where  $C_b$  is a non-modular cycle containing exactly two pegs.

A complete characterization of all unimodular tricyclic graphs  $T \in \mathcal{T}$  is presented in the following result.

**Theorem 5.** *Consider a tricyclic graph  $T \in \mathcal{T}$ . Then  $T$  is unimodular if and only if  $T \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$ .*

*Proof.* By Lemma 3, it follows that every cycle of  $T$  is modular if and only if its base  $T^B$  does not contain  $\mathcal{Q}^*$  as an induced subgraph. Furthermore, by Lemma 8, it follows that  $T$  contains exactly two non-modular cycles having precisely one peg each, say  $C_\alpha$  and  $C_\beta$ , such that  $n(C_\alpha) + n(C_\beta)$  is a multiple of 4 if and only if the base of  $T$  is one of the forms  $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4, \mathcal{H}_5, \mathcal{H}_6, \mathcal{H}_7$  or  $\mathcal{H}_8$  (Figs.8- 15), under the condition that  $n(C_\alpha) + n(C_\beta)$  is a multiple of 4, where each of  $C_\alpha$  and  $C_\beta$  is a non-modular cycle containing exactly one peg. Now the theorem follows directly from Theorem 4.  $\square$

*Algorithm to Check the Unimodularity of a Tricyclic Graph  $T \in \mathcal{T}$ .*

**Step 1. Check  $\mathcal{Q}^*$ :** Determine whether  $\mathcal{Q}^*$  (Fig. 6) is an induced subgraph of  $T$ .

- If  $\mathcal{Q}^*$  is *not* an induced subgraph of  $T$ , then  $T$  is unimodular.

**Step 2. Check the base of  $T$ :** If  $\mathcal{Q}^*$  is an induced subgraph of  $T$ , check whether the base of  $T$  is of the form  $\mathcal{B}_3^3$  (Fig. 7).

- If the base is of the form  $\mathcal{B}_3^3$ , identify the cycle that contains two pegs. Let this cycle be  $C_\alpha$ .
- If  $n(C_\alpha)$  is divisible by 4, then  $T$  is unimodular.
- Otherwise,  $T$  is not unimodular.

**Step 3. Three-cycle arrangement:** If  $\mathcal{Q}^*$  is an induced subgraph and the base is not of the form  $\mathcal{B}_3^3$ , then proceed as follows:

- **Edge-disjoint cycles:** If all three cycles of  $T$  are edge-disjoint, there exist exactly two non-modular cycles. Let these be  $C_\alpha$  and  $C_\beta$ .
  - If  $n(C_\alpha) + n(C_\beta)$  is divisible by 4, then  $T$  is unimodular.

- Otherwise,  $T$  is not unimodular.
- **Cycles sharing edges:** If two cycles share some edges, then  $T$  contains exactly three non-modular cycles: one cycle formed by the overlap of the other two, two cycles with exactly one peg each, and one cycle with two pegs. Consider the two cycles that have exactly one peg each. Let these be  $C_\alpha$  and  $C_\beta$ .
  - If  $n(C_\alpha) + n(C_\beta)$  is divisible by 4, then  $T$  is unimodular.
  - Otherwise,  $T$  is not unimodular.

In the appendix, we present some examples of unimodular tricyclic graphs with a unique perfect matching belonging to each of the classes  $\mathcal{T}_1$ ,  $\mathcal{T}_2$ , and  $\mathcal{T}_3$ .

**Conflict of Interest:** The authors declare that they have no conflict of interest.

**Data Availability:** Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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## Appendix

Here, we present some examples of unimodular tricyclic graphs with a unique perfect matching belonging to the classes  $\mathcal{T}_1$ ,  $\mathcal{T}_2$  and  $\mathcal{T}_3$ .

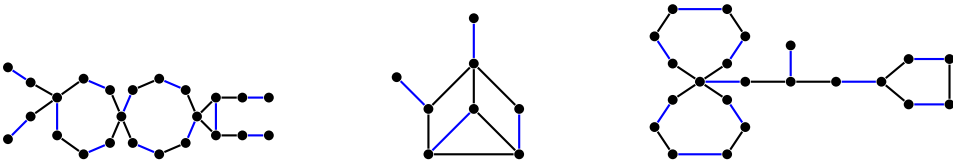


Figure 16: Examples of graphs belonging to the class  $\mathcal{T}_1$ .

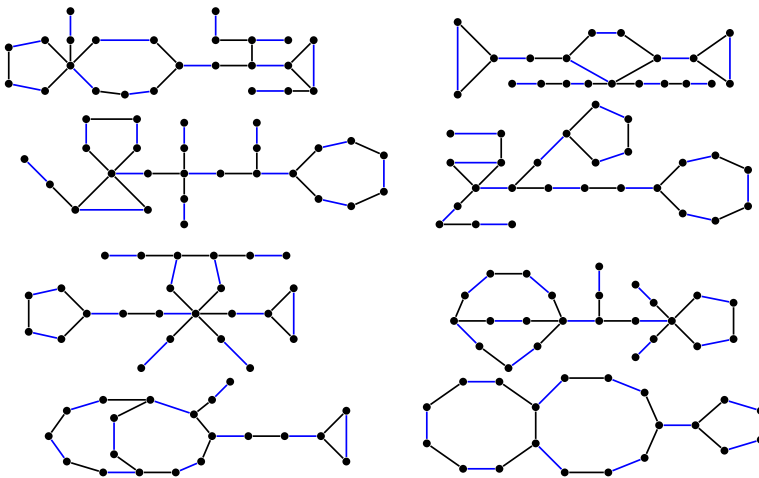


Figure 17: Examples of graphs belonging to the class  $\mathcal{T}_2$ .

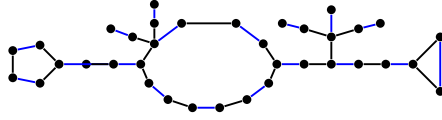


Figure 18: An example of a graph belonging to the class  $\mathcal{T}_3$ .