

## On coalition graphs and coalition count of graphs

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**Abstract:** Let  $G$  be graph with vertex set  $V(G)$  and order  $n$ . A set  $S \subseteq V(G)$  is a dominating set of a graph  $G$  if every vertex in  $V(G) \setminus S$  is adjacent to at least one vertex in  $S$ . A coalition in a graph  $G$  consists of two disjoint sets of vertices  $V_1$  and  $V_2$ , neither of which is a dominating set but whose union  $V_1 \cup V_2$  is a dominating set. A coalition partition, abbreviated  $c$ -partition, in a graph  $G$  is a vertex partition  $\pi = \{V_1, V_2, \dots, V_k\}$  such that every set  $V_i$  of  $\pi$  is either a singleton dominating set, or is not a dominating set but forms a coalition with another set  $V_j$  in  $\pi$ . The sets  $V_i$  and  $V_j$  are coalition partners in  $G$ . The coalition number  $C(G)$  equals the maximum order  $k$  of a  $c$ -partition of  $G$ . For any graph  $G$  with a  $c$ -partition  $\pi = \{V_1, V_2, \dots, V_k\}$ , the coalition graph  $CG(G, \pi)$  of  $G$  is a graph with vertex set  $V_1, V_2, \dots, V_k$ , corresponding one-to-one with the set  $\pi$ , and two vertices  $V_i$  and  $V_j$  are adjacent in  $CG(G, \pi)$  if and only if the sets  $V_i$  and  $V_j$  are coalition partners in  $\pi$ . In [T.W. Haynes, J.T. Hedetniemi, S.T. Hedetniemi, A.A. McRae, and R. Mohan, Coalition graphs, Commun. Comb. Optim. 8 (2023), 423-430], authors proved that for every graph  $G$  there exist a graph  $H$  and  $c$ -partition  $\pi$  such that  $CG(H, \pi) \cong G$ , and raised the question: Does there exist a graph  $H^*$  of smaller order  $n^*$  and size  $m^*$  with a  $c$ -partition  $\pi^*$  such that  $CG(H^*, \pi^*) \cong G$ ? In this paper, we constructed a graph  $H^*$  of small order and size and a  $c$ -partition  $\pi^*$  such that  $CG(H^*, \pi^*) \cong G$ . Recently, Haynes et al. [Introduction to coalitions in graphs, AKCE Int. J. Graphs Comb. 17 (2020), 653-659] defined the coalition count  $c(G)$  of a graph  $G$  as the maximum number of different coalition in any  $c$ -partition of  $G$ . We characterize all graphs  $G$  with  $c(G) = 1$ . Further, imposing some suitable conditions on coalition number, we study the properties of coalition count of graph.

**Keywords:** dominating set, coalition partition, coalition number.

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## 1. Introduction

Let  $G = (V(G), E(G))$  be a graph with vertex set  $V(G)$ , edge set  $E(G)$ , order  $n(G) = |V(G)|$ , and size  $m(G) = |E(G)|$ . A set  $S \subseteq V(G)$  is a *dominating set* of a graph  $G$  if every vertex in  $V(G) \setminus S$  is adjacent to at least one vertex in  $S$ . A detailed account of domination in graphs is presented in the recent books on domination theory [8, 9]. In a graph  $G$  of order  $n$ , a vertex of degree one is a *pendant vertex* and a vertex of degree  $n - 1$  is called a *full vertex*. A subset  $V_i$  is called a *singleton set* if  $|V_i| = 1$ . Note that any full vertex forms a singleton dominating set. The open neighborhood  $N(v)$  of a vertex  $v$  in  $G$  is the set of vertices adjacent to  $v$ , while the closed neighborhood of  $v$  is the set  $N[v] = \{v\} \cup N(v)$ . The *domatic partition* is a partition of the vertex set into dominating sets. The *domatic number*  $d(G)$  is equal to the maximum order  $k$  of the vertex partition, called domatic partition,  $P = \{V_1, V_2, \dots, V_k\}$ , such that every set  $V_i$  is a dominating set in  $G$ . For more details on the domatic number refer [13, 14].

The concept of coalitions in graphs was introduced by Haynes et.al. [4] in 2020 as follows. A *coalition* in a graph  $G$  consists of two disjoint sets of vertices  $V_1$  and  $V_2$ , neither of which is a dominating set but whose union  $V_1 \cup V_2$  is a dominating set. A *coalition partition*, henceforth called a *c-partition*, in a graph  $G$  is a vertex partition  $\pi = \{V_1, V_2, \dots, V_k\}$  such that every set  $V_i$  of  $\pi$  is either a singleton dominating set, or is not a dominating set but forms a coalition with another set  $V_j$  in  $\pi$ . The *coalition number*  $C(G)$  equals the maximum order  $k$  of a *c-partition* of  $G$ , and a *c-partition* of  $G$  having order  $C(G)$  is called a  *$C(G)$ -partition*. In [5], Haynes et al. established upper bounds on the coalition number in terms of the minimum and maximum degree. In [6], they proved that every graph is the coalition graph of some graph, while in [7], they investigated coalition graphs of trees, paths, and cycles. Furthermore, in [4], the authors posed the open problem of characterizing all graphs  $G$  of order  $n$  with coalition number  $C(G) = n$ . This problem was partially addressed by Bakhshesh et al. [3], where the authors characterized all graphs of order  $n$  with  $\delta(G) \leq 1$  that satisfy  $C(G) = n$ , and identified all trees whose coalition number equals  $n - 1$ , which was later fully solved by Yang et al. in [12]. In addition, several variants of coalition have been introduced and studied in [1, 2, 10, 11].

In the paper [4] while introducing coalitions, the authors suggest several related areas for future study, one of which is coalition count of  $G$  defined as follows. The coalition count  $c(G)$  of a graph  $G$  is equal to the maximum number of different coalition in any *c-partition* of  $G$ .

Let us give an example to illustrate the coalition count.

**Example 1.** Consider the cycle  $C_4 = (v_1, v_2, v_3, v_4)$ .

The partition  $\pi_1 = \{\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}\}$  is a *c-partition* of  $C_4$ . No individual set in  $\pi_1$  is a dominating set. However the following different coalition exist:

- The set  $\{v_1\}$  forms coalition with  $\{v_2\}$ ,  $\{v_3\}$  and  $\{v_4\}$ .
- The set  $\{v_2\}$  forms coalition with  $\{v_3\}$  and  $\{v_4\}$ .

- The set  $\{v_3\}$  form coalition with  $\{v_4\}$ .

Thus, every set in  $\pi_1$  forms a coalition with at least one other set. Hence,  $C(C_4) = 4$ , and  $\pi_1$  is a  $C(C_4)$ -coalition partition. But, the total number of different possible coalitions is 6, thus  $c(C_4) = 6$ .

## 2. Coalition graphs

In [6], Haynes et al. defined coalition graph of  $G$  as follows. Let  $G$  be a graph with a  $c$ -partition  $\pi = \{V_1, V_2, \dots, V_k\}$ . The coalition graph  $CG(G, \pi)$  of  $G$  is a graph with vertex set  $V_1, V_2, \dots, V_k$ , and two vertices  $V_i$  and  $V_j$  are adjacent in  $CG(G, \pi)$  if and only if the sets  $V_i$  and  $V_j$  are coalition partners in  $\pi$ . In [6], Haynes et.al proved that, for every graph  $G$ , there is a graph  $H$  and some  $c$ -partition  $\pi$  of  $H$ , such that  $CG(H, \pi) \cong G$ , that is, for every graph  $G = (V, E)$  having  $n$  non-isolated vertices and  $t$  isolated vertices with  $|E| = m$ , they constructed a graph  $H$  of order  $n + m + \binom{n}{2} + t$  and size  $m = \binom{n}{2} + 2m(n-1) + \overline{m}(n-2) + t(n(H)-1)$ , where  $n(H)$  is order of  $H$ , and the  $c$ -partition  $\pi$  such that  $CG(H, \pi) \cong G$ .

Further, they raised the following question: Does there exist a graph  $H^*$  of smaller order  $n^*$  and size  $m^*$  with a  $c$ -partition  $\pi^*$  such that  $CG(H^*, \pi^*) \cong G$ ?

**Theorem 1.** *For every graph  $G$ , there exist a graph  $H^*$  and  $c$ -partition  $\pi^*$  such that  $CG(H^*, \pi^*) \cong G$ .*

*Proof.* Let  $G$  be a graph with vertex set  $V(G) = \{v_1, v_2, \dots, v_n\} \cup \{w_1, w_2, \dots, w_t\}$ , where each  $v_i$  has degree at least one and each  $w_i$  is an isolate. Let  $G' = G - \{w_1, \dots, w_t\}$ . Then  $|E(G)| = |E(G')| = m$  and  $|E(G')| = \overline{m}_{G'}$ , where  $m + \overline{m}_{G'} = \binom{n}{2}$ . To construct  $H^*$ , we begin with a complete graph  $K_n$  on vertices  $\{v_1, \dots, v_n\}$ , corresponding to the non-isolates of  $G$ . If  $n$  is even, partition these vertices into  $\frac{n}{2}$  disjoint pairs and delete the edge within each pair; if  $n$  is odd, form  $\frac{n-1}{2}$  such pairs and delete the edge within each pair and retain all edges incident with  $v_n$ . These  $n$  vertices are referred to as the base vertices of  $H^*$ . The partition  $\pi^*$  is initially taken as the singleton partitions  $V_i = \{v_i\}$ ,  $1 \leq i \leq n$ . We will add to the sets in the partition as we build  $H^*$ .

Now we consider the following cases:

**Case 1:** Let  $G \cong K_n \cup tK_1$ . If  $n$  is even and  $t \neq 0$ , we construct  $H^*$  by adding each isolate vertex  $w_i \in V(G)$  to  $H^*$  and connecting  $w_i$  to every other vertex of  $H^*$ . Then, the partition  $\pi^*$  is extended by adding singleton sets  $W_i = \{w_i\}$  for  $n \leq i \leq t$ . In this construction, no set  $V_i$  in  $\pi^*$  is a dominating set, but every pair of sets  $V_i$  in  $\pi^*$  forms a coalition. Therefore, the coalition graph satisfies  $CG(H^*, \pi^*) \cong K_n \cup tK_1$ , where  $n$  is even.

If  $n$  is odd and  $t \neq 0$ , we first extend  $H^*$  by introducing a new vertex  $u_n$  associated with vertex  $v_n$ , such that  $u_n$  is adjacent to all base vertices except  $v_n$ . The vertex  $u_n$  is assigned to a set  $V_k$  in  $\pi^*$ , where  $k \neq n$ , and the edge  $u_n v_k$  is removed from

$H^*$ . Next, for every isolate  $w_i \in V(G)$ , we add  $w_i$  to  $H^*$  and connect it to every other vertex of  $H^*$ . The partition  $\pi^*$  is then extended by singleton sets  $W_i = \{w_i\}$  for  $n \leq i \leq t$ . Further, none of the sets  $V_i$  is a dominating set, but every pair of sets  $V_i$  in  $\pi^*$  forms a coalition. Thus,  $CG(H^*, \pi^*) \cong K_n \cup tK_1$ , where  $n$  is odd.

**Case 2:** If  $G \cong G' \cup tK_1$ , where  $G' \not\cong K_n$ , then without loss of generality, we assume that  $v_n$  is not a full vertex in  $G$ . For each edge  $v_j v_k \in E(\overline{G'})$ , introduce a new vertex  $v_{jk}$  adjacent to every base vertex except  $v_j$  and  $v_k$ , and place  $v_{jk}$  into some set of  $\pi^*$  other than  $V_j$  or  $V_k$  and then remove edges between the vertices in the same set  $V_i$  in  $\pi^*$ . Here it is important to note that none of the base vertices of  $H^*$  is a full vertex. Finally, add each isolate  $w_i \in V(G)$  to  $H^*$  and insert an edge from  $w_i$  to every other vertices of  $H^*$ . Now, extend  $\pi^*$  by a singleton sets  $W_i = \{w_i\}$ ,  $1 \leq i \leq n$ .

In the resulting graph, every edge of  $H^*$  is incident with either a base vertex or a dominating vertex, and no set  $V_j$  of  $\pi^*$  is itself a dominating set. For any edge  $v_j v_k \in E(G)$ . Since  $v_j$  and  $v_k$  collectively dominate every vertex in  $H^*$ ,  $V_j \cup V_k$  is a coalition in  $H^*$ . Further, if  $v_j v_k \notin E(G)$ , then  $V_j \cup V_k$  is not a dominating set as there is no vertex in  $V_j \cup V_k$  that will dominate the vertex  $v_{jk}$ . Each  $w_i$  is dominating vertex in  $H^*$ , so  $w_i$  is an isolate in  $G$ . Hence the partition  $\pi^* = \{V_1, \dots, V_n, W_1, \dots, W_t\}$  is a  $c$ -partition of  $H^*$  satisfying  $CG(H^*, \pi^*) \cong G$ .  $\square$

Properties of graph  $H^*$  constructed above

Graph $G$	$n(H^*)$	$m(H^*)$
$G \cong K_n \cup tK_1$ , $n$ is even	$n + t$	$\binom{n}{2} - \frac{n}{2} + t(n(H^*) - 1)$
$G \cong K_n \cup tK_1$ , $n$ is odd	$n + t + 1$	$\binom{n}{2} - \frac{n-1}{2} + n - 2 + t(n(H^*) - 1)$
$G \cong G' \cup tK_1$ , where $G' \not\cong K_n$	$n + \overline{m}_{G'} + t$	$\binom{n}{2} - \lfloor \frac{n}{2} \rfloor + \overline{m}_{G'}(n - 3) + t(n(H^*) - 1)$

### 3. Properties of coalition count $c(G)$ of graph $G$

The following observations on coalition count  $c(G)$  of graph  $G$  motivated us to study the properties of  $c(G)$ .

1. The coalition number  $C(G)$  and coalition count  $c(G)$  of graph  $G$  are not comparable. For example, Consider complete graph  $K_n$ ,  $C(K_n) = n > 0 = c(K_n)$ , but for  $C_4$ ,  $c(C_4) = 6 > C(C_4) = 4$  and for  $P_4$ ,  $c(P_4) = C(P_4) = 4$ .
2. The  $c$ -partition used to count the coalition number and coalition count need not be unique. For example, consider  $P_6 = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ . The partition  $\pi_2 = \{\{v_2\}, \{v_4\}, \{v_1, v_6\}, \{v_3\}, \{v_5\}\}$  is a  $c$ -partition of  $P_6$ . Thus,  $C(P_6) = 5$  but the total number of different possible coalitions is 3. Now, consider  $\pi_3 = \{\{v_2, v_4\}, \{v_6\}, \{v_1, v_3\}, \{v_5\}\}$  is a  $c$ -partition of  $P_6$ , where  $c(G) = 5$ .
3. The maximum number of edges among all the graphs  $CG(G, \pi)$  is the coalition count of  $G$ .

We next establish a relationship between the coalition count and the domatic number.

**Theorem 2.** *For any graph  $G$  with no isolated vertices and  $f$  full vertices,  $c(G) \geq d(G) - f$ .*

*Proof.* Assume  $P = \{V_1, V_2, \dots, V_k\}$  is a domatic partition of  $G$  with  $k = d(G)$ . Without loss of generality, suppose that  $V_1, V_2, \dots, V_{k-1}$  are minimal dominating sets. If any  $V_i$  (for  $1 \leq i \leq k-1$ ) is not minimal, we replace it with a minimal dominating set  $V'_i \subseteq V_i$ , and add the vertices in  $V_i \setminus V'_i$  to  $V_k$ . Since  $G$  has  $f$  full vertices, there will be  $f$  singleton dominating sets. Now, consider a domatic-partition

$$P_1 = \left\{ \underbrace{V'_1, \dots, V'_f}_{f \text{ full vertices}}, \underbrace{V'_{f+1}, \dots, V'_{k-1}}_{\text{minimal dominating sets}}, V'_k \right\}. \quad \text{Then each of } k - f - 1 \text{ minimal}$$

dominating sets can be partitioned into two non-empty, non-dominating subsets  $V_{i,1}$  and  $V_{i,2}$ , where  $i = f+1, \dots, k-1$ , whose union is dominating. Thus, these minimal dominating sets contribute at least  $k - f - 1$  coalitions.

Consider the vertex partition  $P_2 = \{V'_1, \dots, V'_f, V_{f+1,1}, V_{f+2,2}, \dots, V_{k-1,1}, V_{k-2,2}, V'_k\}$ . Further, if  $V'_k$  is also a minimal dominating set, then  $V'_k$  can be partitioned into two non-empty, non-dominating subsets whose union is dominating. Therefore, by replacing  $\{V'_k\}$  in  $P_2$  using these two sets, we get a  $c$ -partition of  $G$ . Thus,  $c(G) \geq k - f - 1 + 1 = k - f$ .

Suppose  $V'_k$  is not a minimal dominating set. Let  $V''_k \subseteq V'_k$  be a minimal dominating set, and let  $V''_k = V_{k,1} \cup V_{k,2}$ , where  $V_{k,1}$  and  $V_{k,2}$  are non-empty, non-dominating sets that together form a coalition. Define  $W = V'_k \setminus V''_k$ . If  $W$  is a dominating set, then there exist at least  $k+1$  disjoint dominating sets in  $G$ , contradicting  $d(G) = k$ . Thus,  $W$  is not a dominating set in  $G$ . Now, by replacing  $V'_k$  in  $P_2$  by  $V_{k,1}, V_{k,2}$  and  $W$ , we obtain a  $c$ -partition of  $G$ . If  $W$  can form a coalition with another non-dominating set then  $c(G) \geq k - f + 1$ . If  $W$  does not form a coalition with any set, then consider the union  $V_{k,2} \cup W$  as a set in  $c$ -partition. This set remains non-dominating but forms a coalition with  $V_{k,1}$ . Thus,  $c(G) \geq k - f$ . Therefore,  $c(G) \geq k - f = d(G) - f$ .  $\square$

The bound of Theorem 2 is sharp for stars  $K_{1,n-1}$ , for  $n \geq 3$ , as  $c(K_{1,n-1}) = 1$ ,  $d(G) = 2$  and  $f = 1$ . Also, the bound is sharp for complete graph, where  $c(K_n) = 0$  and  $d(K_n) = f = n$ .

An *independent set* is a set of vertices in  $G$  such that no two vertices in the set are adjacent. The *independence number*  $\alpha(G)$  of a graph  $G$  is the cardinality of the largest independent set in  $G$ .

In next theorem, we characterize the graphs whose coalition count is exactly one.

**Theorem 3.** *Let  $G$  be a graph with  $f$  full vertices. Then  $c(G) = 1$  if and only if  $\alpha(G) = n - f$ .*

*Proof.* Consider a graph  $G$  with  $f$  full vertices  $U = \{v_1, v_2, \dots, v_f\}$  and  $c(G) = 1$ . Then there exist exactly two sets  $V_i$  and  $V_j$  which are not dominating sets, but  $V_i \cup V_j$  is a dominating set. Since  $V_i$  is not a dominating set, there exists a vertex  $x \in V_j$ , such that  $(N[x] \setminus U) \cap V_i = \phi$ .

**Claim:**  $N(x) \setminus U = \phi$ .

Suppose  $N(x) \setminus U \neq \phi$ . Then by adding the vertices in  $V_j \setminus (N[x])$  to  $V_i$ , it follows that the set  $V_i$  is not a dominating set and  $V_j = N[x]$ . Now consider the sets  $V_{j_1} = \{x\}$  and  $V_{j_2} = N(x)$ . We note that the set  $V_i$  form coalition with the sets  $\{x\}$  and  $N(x)$ . Thus,  $c(G) \geq 2$ , a contradiction. In similar way one can prove that if there exist a vertex in  $y \in V_i$  such that  $N(y) \setminus U \neq \phi$ , then  $c(G) \geq 2$ . Thus,  $N(x) \setminus U = \phi$ . Therefore,  $\alpha(G) = n - f$ .

Conversely, suppose  $G$  has  $f$  full vertices and  $\alpha(G) = n - f$ . Then, there are  $n - f$  vertices which are not adjacent to each other, and all these  $n - f$  vertices is of degree  $f$ . Thus, in order to form a coalition partition, these  $n - f$  vertices must be divided into exactly two sets  $V_i$  and  $V_j$  which are not dominating sets but  $V_i \cup V_j$  is a dominating set. If we partition these  $n - f$  vertices into at least three sets  $V_i, V_j$  and  $V_k$ . Then clearly,  $V_i$  do not form coalition with  $V_j$  as there is no vertices either in  $V_i$  or in  $V_j$  which can dominate the vertices of  $V_k$ . Similarly,  $V_i$  cannot form coalition with  $V_k$ . Thus,  $c(G) = 1$ .  $\square$

The union of two graphs  $G_1$  and  $G_2$  is the graph with vertex set  $V(G_1) \cup V(G_2)$  and edge set  $E(G_1) \cup E(G_2)$ . It is denoted by  $G_1 \cup G_2$ . The join of two graphs  $G_1$  and  $G_2$  is the graph obtained by taking the union of  $G_1$  and  $G_2$ , and adding edges between every vertex of  $G_1$  and every vertex of  $G_2$ . It is denoted by  $G_1 + G_2$ .

Using Theorem 3, we deduce the following result:

**Corollary 1.** *Let  $G$  be a graph of order  $n$  with  $f$  full vertices. Then  $c(G) = 1$  if and only if  $G \cong (K_f + pK_1) \cup qK_1$ , where  $f + p + q = n$ .*

**Theorem 4.** *Let  $G$  be a graph with exactly one full vertex and  $\delta(G) = 1$ . If  $C(G) = s (s \geq 2)$ , then  $c(G) = s - 2$ . Further,  $CG(G, \pi) \cong K_1 \cup K_{1, s-2}$ .*

*Proof.* Consider a graph  $G$  with  $\delta(G) = 1$  and a full vertex  $u_1$ . For  $s = 2, 3$ , clearly result holds. For  $s \geq 4$ , let  $u_2, u_3, \dots, u_p$  are pendant vertices attached to  $u_1$ . Then  $\pi = \{\{u_1\}, W\}$ , where  $W = \{V_1, V_2, \dots, V_{s-1}\}$  is the partition of the  $V(G) \setminus \{u_1\}$  such that every  $V_i$  is not a dominating set but forms coalition with some other set  $V_j$  in  $\pi$ . Here it is important to note that the pendant vertices must form a single set otherwise, it is impossible to form coalition with other sets as they are independent to each other. Without loss of generality let  $\{u_2, u_3, \dots, u_p\} \subseteq V_1$ . Therefore, all other sets in  $W$  must form coalition with the unique set  $V_1$ . If not, then there exist no vertex other than  $u_1$  that will dominate the vertices  $u_2, u_3, \dots, u_p$ . Since  $\{u_1\}$  is a dominating set, it do not form coalition with any set in  $W$ . Thus,  $c(G) = s - 2$ . Since  $G$  has only one full vertex and  $\delta(G) = 1$ , the coalition graph  $CG(G, \pi)$  will have

exactly two components. Further, as  $CG(G, \pi)$  has  $s$  vertices out of which the vertex corresponding to the set  $\{u_1\}$  do not form coalition with any other vertices, it will be an isolated vertex in  $CG(G, \pi)$ . Let  $H$  be the another component of  $CG(G, \pi)$ . Then the remaining  $s - 1$  vertices out of which one vertex corresponding to the set  $\{u_2, u_3, \dots, u_p\}$  form coalition with all  $s - 2$  vertices and as  $c(G) = s - 2$ ,  $H \cong K_{1, s-2}$ . Thus,  $CG(G, \pi) \cong K_1 \cup K_{1, s-2}$ .  $\square$

A coalition partition  $\pi$  of  $G$  is a singleton coalition partition if every set in  $\pi$  consists of a single vertex. If a graph  $G$  has a singleton coalition partition, then  $G$  is referred to as a singleton-partition graph (SP-graph).

Next, we relate coalition number  $C(G)$  of graph  $G$  with coalition count of  $G$ . We omit the proof as it is straight forward.

**Remark 1.** For any graph  $G$  with  $f$  full vertices having coalition number  $C(G)$  and coalition count  $c(G)$ ,  $c(G) \geq \lceil \frac{C(G) - f}{2} \rceil$ .

**Theorem 5.** For any SP-graph  $G$  of order  $n$  with no full vertices,  $c(G) \geq \alpha(G)$ .

*Proof.* Let  $G$  is a singleton-partition graph. Then  $\pi = \{\{v_1\}, \{v_2\}, \dots, \{v_n\}\}$  is a  $c$ -partition. Since  $G$  has no full vertex, each set  $V_i$  must form a coalition with some set  $V_j$ ,  $1 \leq i < j \leq n$ . By remark 1,  $c(G) \geq \lceil \frac{n}{2} \rceil$ . If  $\alpha(G) \leq \lceil \frac{n}{2} \rceil$ , then clearly result holds. Suppose  $\alpha(G) \geq \lceil \frac{n}{2} \rceil + 1$ . Then there are at least  $\lceil \frac{n}{2} \rceil + 1$  vertices in  $G$  are independent. Let  $I = \{v_1, v_2, \dots, v_\alpha\}$  is the maximum independent set. Further, it is easy to observe that any set in  $\pi$  corresponding to the vertices of  $I$  will form coalition with the set corresponding to the vertex not in  $I$ . Thus, there will be at least  $\alpha$  coalitions. Therefore,  $\alpha(G) \leq c(G)$ .  $\square$

**Conflict of Interest:** The authors declare that they have no conflict of interest.

**Data Availability:** Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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