

Wiener polarity responses and triangle coverage

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Abstract: The Wiener polarity index $W_p(G)$ counts unordered pairs of vertices at distance three. We study subdivisions in which each edge e of a connected simple graph receives an edge-dependent number $a_e \geq 1$ of new vertices. We obtain an exact formula for the resulting Wiener polarity index and then compare the ordinary subdivision with subdivisions in which a chosen set of edges receives one additional vertex. After removing a linear term, the change in W_p is exactly the number of triangles met by the chosen edges. This identifies the subdivision response with the coverage function of the triangle hypergraph. We use this identity to recover triangle edge sets from small responses, derive a cubic multilinear representation, and formulate fixed subdivision budgets as a maximum-coverage problem. The resulting maximization problem is NP-hard even for graphs with a universal vertex and diameter two, while a greedy rule gives the standard submodular approximation guarantee. Finally, we study a bivariate response polynomial, derive a general union formula, and obtain exact factorizations and sharp budget bounds for edge-disjoint triangles and book graphs.

Keywords: Wiener polarity index, nonuniform subdivision, triangle coverage, combinatorial optimization, response polynomial.

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1. Introduction

The Wiener polarity index is a distance-based graph invariant. For a connected graph G , it is defined by

$$W_p(G) = |\{\{u, v\} \subseteq V(G) : d_G(u, v) = 3\}|.$$

The index originates in Wiener's work on paraffin boiling points [12]. It is also the distance-three coefficient of the Wiener polynomial studied by Hosoya [7]. More generally, the number of pairs of vertices at a prescribed distance has been studied in [8].

Formulas for W_p often combine degree information with corrections for short cycles. A formula involving Zagreb indices and short cycles was given in [2] and later extended to graphs with 4-cycles and more complicated cycle overlap in [11]. The Wiener polarity index has also been studied under standard graph operations [5]. For chemical trees, subdivision and smoothing operations were used in [3]. These works motivate the study of distance-based invariants under edge-dependent subdivision.

Let G be a connected simple graph, and replace every edge e by a path containing $a_e \geq 1$ new vertices. Our first result gives an exact formula for the Wiener polarity index of the resulting graph. If $m = |E(G)|$, $M_1(G)$ is the first Zagreb index, and $t_1(G, \mathbf{a})$ counts triangles whose three edges all satisfy $a_e = 1$, then

$$W_p(S_{\mathbf{a}}(G)) = M_1(G) + \sum_{e \in E(G)} a_e - 3m - 3t_1(G, \mathbf{a}).$$

This formula is compatible with the general short-cycle formula of Tratnik [11]. We include a direct path-counting proof because it isolates the role of the subdivided triangles. We do not treat this specialization alone as the main contribution.

The central part of the paper studies a response obtained by lengthening selected replacement paths. Let $S(G)$ be the ordinary subdivision, in which every edge receives one new vertex. For $F \subseteq E(G)$, let $S_F(G)$ be obtained by giving every edge in F two new vertices and every edge outside F one new vertex. We prove

$$W_p(S_F(G)) - W_p(S(G)) = |F| + 3\tau_G(F),$$

where $\tau_G(F)$ is the number of triangles containing at least one edge of F . Hence the normalized response is exactly a triangle-coverage function.

This identity has three uses. First, responses on sets of at most three edges recover the full collection of triangle edge sets. Second, a fixed budget of extra subdivision vertices becomes a discrete coverage problem on the triangle hypergraph. We prove that the resulting maximization problem is NP-hard even for graphs of diameter two, and then give a simple greedy approximation. Third, we package all binary responses in a bivariate polynomial. The polynomial records how triangle overlap affects subdivision responses. We obtain exact formulas for two opposite overlap patterns: edge-disjoint triangles and book graphs, in which all triangles share one edge.

Although the Wiener polarity index arose in chemical graph theory, the questions studied here are graph-theoretic. The main results concern triangle incidence, set-function responses, and allocation under a fixed edge-subdivision budget. To the best of our knowledge, the connection between selected-edge subdivision responses and the coverage function of the triangle hypergraph has not previously been developed for the Wiener polarity index. The reconstruction, complexity, and enumerative results below are obtained from this connection.

The paper is organized as follows. Section 2 introduces the notation. Section 3 proves the nonuniform subdivision formula. Section 4 develops the response function

and triangle-reconstruction formulas. Section 5 treats fixed budgets, computational complexity, and the greedy rule. Section 6 studies the response polynomial, edge-disjoint triangles, and book graphs.

2. Preliminaries

All graphs in this paper are finite, simple, and connected, and they have at least one edge. For a graph $G = (V, E)$, let $d_G(u, v)$ denote the distance between u and v , and let $d_G(v)$ denote the degree of v . We write $m = |E(G)|$. The first Zagreb index is

$$M_1(G) = \sum_{v \in V(G)} d_G(v)^2.$$

It also satisfies

$$M_1(G) = \sum_{uv \in E(G)} (d_G(u) + d_G(v)).$$

For background on the first Zagreb index, see [6].

Let $\mathcal{T}(G)$ be the set of triangles of G , and let

$$t(G) = |\mathcal{T}(G)|.$$

We identify a triangle with its set of three edges whenever convenient.

Definition 1 (Subdivision vector). Let

$$\mathbf{a} = (a_e)_{e \in E(G)} \in \mathbb{Z}_{\geq 1}^{E(G)},$$

so that $a_e \geq 1$ for every $e \in E(G)$. The subdivision $S_{\mathbf{a}}(G)$ is obtained by replacing every edge $e = uv$ with a path

$$u = s_0^e - s_1^e - \cdots - s_{a_e}^e - s_{a_e+1}^e = v.$$

Thus a_e new vertices are inserted into e , and the replacement path has length $a_e + 1$.

We use

$$q(\mathbf{a}) = \sum_{e \in E(G)} a_e$$

for the total number of inserted vertices.

Definition 2 (Short triangle). For a subdivision vector $\mathbf{a}$, define

$$t_1(G, \mathbf{a}) = |\{T \in \mathcal{T}(G) : a_e = 1 \text{ for every } e \in E(T)\}|.$$

Such triangles are called short with respect to $\mathbf{a}$.

Let $\mathbf{1}$ be the vector with $a_e = 1$ for every edge. We write

$$S(G) = S_{\mathbf{1}}(G)$$

for the ordinary subdivision graph.

For $F \subseteq E(G)$, define the binary subdivision vector

$$a_e^F = \begin{cases} 2, & e \in F, \\ 1, & e \notin F, \end{cases}$$

and write

$$S_F(G) = S_{\mathbf{a}^F}(G).$$

Thus $S_F(G)$ adds one extra subdivision vertex to every edge of F , starting from the ordinary subdivision.

Definition 3 (Triangle coverage). For $F \subseteq E(G)$, let

$$\tau_G(F) = |\{T \in \mathcal{T}(G) : E(T) \cap F \neq \emptyset\}|.$$

Thus $\tau_G(F)$ is the number of triangles met by F .

For an edge e , we also write $t_e = |\{T \in \mathcal{T}(G) : e \in E(T)\}|$. The *triangle hypergraph* of G is

$$\mathcal{H}_\Delta(G) = (E(G), \{E(T) : T \in \mathcal{T}(G)\}).$$

Its vertices are the edges of G , and its hyperedges are the three-edge sets of the triangles of G . The function $\tau_G(F)$ counts the hyperedges of $\mathcal{H}_\Delta(G)$ met by F .

Edge sets that meet triangles have also been studied in the classical theory of triangle covering and packing; see [4]. In that setting, the objective is typically to find a small edge set that meets every triangle. In contrast, our fixed-budget problem asks for the largest number of triangles that can be met by an edge set of prescribed size.

3. The nonuniform subdivision formula

We first describe the cycles and the paths of length three in $S_{\mathbf{a}}(G)$.

Lemma 1 (Cycles in a subdivision). *Every cycle of $S_{\mathbf{a}}(G)$ is obtained from a cycle C of G , and its length is*

$$\sum_{e \in E(C)} (a_e + 1).$$

In particular, $S_{\mathbf{a}}(G)$ has girth at least 6. Its 6-cycles are exactly the subdivisions of triangles T for which $a_e = 1$ on all three edges.

Proof. Every inserted vertex has degree 2 and belongs to one replacement path. Once a cycle enters a replacement path, it must continue along that path until it reaches the other original endpoint. Contracting every replacement path therefore turns a cycle of $S_{\mathbf{a}}(G)$ into a cycle of G . The replacement of edge e contributes $a_e + 1$ edges, giving the stated length.

A cycle of G has at least three edges, and every replacement path has length at least 2. Hence every cycle in the subdivision has length at least 6. Equality is possible only when the original cycle is a triangle and all three replacement paths have length 2. $\square$

Lemma 2 (Number of three-edge paths). *The number of unoriented simple paths of length three in $S_{\mathbf{a}}(G)$ is*

$$M_1(G) + q(\mathbf{a}) - 3m.$$

Proof. We split the paths into two types.

First, consider paths that lie inside one replacement path. The replacement of edge e has length $a_e + 1$, so it contains

$$(a_e + 1) - 3 + 1 = a_e - 1$$

subpaths of length three. Summing over all edges gives

$$\sum_{e \in E(G)} (a_e - 1) = q(\mathbf{a}) - m.$$

Second, consider paths that use two different replacement paths incident with an original vertex v . Choose two distinct original edges incident with v . There are two length-three paths using one edge on one side of v and two edges on the other side. The contribution at v is therefore

$$2 \binom{d_G(v)}{2} = d_G(v)(d_G(v) - 1).$$

Summing over the original vertices gives

$$\sum_{v \in V(G)} d_G(v)(d_G(v) - 1) = M_1(G) - 2m.$$

A length-three path cannot use three different replacement paths, since passing from one original vertex to another already requires at least two edges. Hence every length-three path not contained in a single replacement path changes from one replacement path to another at exactly one original vertex. Adding the contributions gives

$$q(\mathbf{a}) - m + M_1(G) - 2m = M_1(G) + q(\mathbf{a}) - 3m.$$

$\square$

Theorem 1 (Nonuniform subdivision formula). *Let G be a connected simple graph with at least one edge, and let $\mathbf{a} \in \mathbb{Z}_{\geq 1}^{E(G)}$. Then*

$$W_p(S_{\mathbf{a}}(G)) = M_1(G) + q(\mathbf{a}) - 3m - 3t_1(G, \mathbf{a}).$$

Proof. By Lemma 1, the graph $S_{\mathbf{a}}(G)$ has no cycle of length less than 6. Hence the endpoints of every three-edge path are at distance three. Indeed, a shorter path between the same endpoints, together with the three-edge path, would contain a cycle of length at most 5.

We next determine when two different three-edge paths have the same endpoints. If the two paths shared an internal vertex, their union would contain a cycle of length at most 4, which is impossible. Thus their internal vertices are disjoint, and their union is a 6-cycle. Conversely, the two arcs between opposite vertices of a 6-cycle are distinct three-edge paths with the same endpoints.

By Lemma 1, every 6-cycle comes from one short triangle. Each such cycle has three opposite pairs, so it creates three extra path counts. These extra counts do not overlap between two short triangles. To see this, an opposite pair on such a 6-cycle consists of an original vertex x and the unique subdivision vertex on the original edge uv opposite x . The pair determines the original triangle xuv .

Lemma 2 counts all three-edge paths. Subtracting the three extra counts from each short triangle proves the formula. $\square$

Remark 1 (Relation to an earlier general formula). A general formula for $W_p(H)$ in terms of Zagreb indices, short cycles, and overlaps among short cycles is given in [11]. Lemma 1 shows that the only short cycles in $S_{\mathbf{a}}(G)$ are the 6-cycles coming from short triangles. Substituting the degree and cycle data of $S_{\mathbf{a}}(G)$ into that formula also yields Theorem 1. The proof above is independent and uses only path counting.

If every edge receives r new vertices, we write the graph as $S_r(G)$.

Corollary 1 (Uniform subdivisions). *For every integer $r \geq 1$,*

$$W_p(S_r(G)) = \begin{cases} M_1(G) - 2m - 3t(G), & r = 1, \\ M_1(G) + (r - 3)m, & r \geq 2. \end{cases}$$

Proof. If $a_e = r$ for all e , then $q(\mathbf{a}) = rm$. When $r = 1$, every triangle is short. When $r \geq 2$, no triangle is short. Apply Theorem 1. $\square$

Corollary 2 (Triangle-free graphs). *If G is triangle-free, then*

$$W_p(S_{\mathbf{a}}(G)) = M_1(G) + q(\mathbf{a}) - 3m.$$

In particular, for a fixed total number of inserted vertices, the value does not depend on how the vertices are distributed among the edges.

Proof. A triangle-free graph has $t_1(G, \mathbf{a}) = 0$ for every subdivision vector. Apply Theorem 1. $\square$

4. Subdivision responses and triangle reconstruction

We now compare $S_F(G)$ with the ordinary subdivision $S(G)$.

Theorem 2 (Subdivision response formula). *For every $F \subseteq E(G)$,*

$$\boxed{W_p(S_F(G)) = W_p(S(G)) + |F| + 3\tau_G(F).}$$

Proof. The binary vector $\mathbf{a}^F$ has total $q(\mathbf{a}^F) = m + |F|$. A triangle remains short exactly when none of its edges belongs to F . Hence

$$t_1(G, \mathbf{a}^F) = t(G) - \tau_G(F).$$

Theorem 1 gives

$$\begin{aligned} W_p(S_F(G)) &= M_1(G) + (m + |F|) - 3m - 3(t(G) - \tau_G(F)) \\ &= (M_1(G) - 2m - 3t(G)) + |F| + 3\tau_G(F). \end{aligned}$$

The expression in parentheses is $W_p(S(G))$ by Corollary 1. $\square$

Definition 4 (Normalized response). For $F \subseteq E(G)$, define

$$\rho_G(F) = \frac{W_p(S_F(G)) - W_p(S(G)) - |F|}{3}.$$

Theorem 2 states that

$$\boxed{\rho_G(F) = \tau_G(F).}$$

Thus the normalized Wiener-polarity response is the hyperedge-coverage function of $\mathcal{H}_\Delta(G)$.

4.1. Two small examples

Let $G = C_3$. Its ordinary subdivision is C_6 , while lengthening one replacement path gives C_7 . Since $W_p(C_6) = 3$ and $W_p(C_7) = 7$, one extra subdivision vertex changes the Wiener polarity index by 4. The response formula splits this change into the linear term 1 and the triangle term $3\tau_G(F) = 3$. This change from C_6 to C_7 is illustrated in Figure 1.

Let D be the diamond graph, obtained from K_4 by deleting one edge. It has two triangles sharing the edge e_0 in Figure 2.

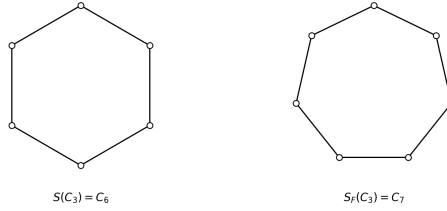


Figure 1. Lengthening one replacement path of the ordinary subdivision of a triangle changes a 6-cycle into a 7-cycle.

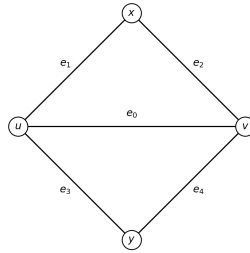


Figure 2. The diamond graph. The shared edge e_0 belongs to two triangles, while every other edge belongs to one triangle.

The degrees are 3, 3, 2, 2, so $M_1(D) = 26$. Also, $m = 5$ and $t(D) = 2$. Hence

$$W_p(S(D)) = 26 - 2(5) - 3(2) = 10.$$

For example,

$$W_p(S_{\{e_0\}}(D)) = 10 + 1 + 3(2) = 17,$$

whereas

$$W_p(S_{\{e_1\}}(D)) = 10 + 1 + 3(1) = 14.$$

Thus the location of an extra subdivision vertex can matter even when the total number of inserted vertices is fixed.

4.2. Marginal and local responses

For $e \notin F$, define

$$\Delta_e \rho_G(F) = \rho_G(F \cup \{e\}) - \rho_G(F).$$

Proposition 1 (Marginal response). For $e \notin F$,

$$\Delta_e \rho_G(F) = |\{T \in \mathcal{T}(G) : e \in E(T), E(T) \cap F = \emptyset\}|.$$

Therefore ρ_G is normalized, monotone, and submodular. More precisely, if $A \subseteq B \subseteq E(G)$ and $e \notin B$, then

$$\Delta_e \rho_G(A) \geq \Delta_e \rho_G(B).$$

Proof. Since $\rho_G(F) = \tau_G(F)$, adding e counts exactly the triangles that contain e and were not already met by F . This proves the marginal formula.

We have $\rho_G(\emptyset) = 0$, so the function is normalized. Adding edges cannot reduce the number of covered triangles, so it is monotone. If $A \subseteq B$, every triangle missed by B is also missed by A . Thus the set counted by $\Delta_e \rho_G(B)$ is contained in the set counted by $\Delta_e \rho_G(A)$. $\square$

Corollary 3 (Single-edge and two-edge data). *Let e and f be distinct edges of G . Then*

$$\rho_G(\{e\}) = t_e,$$

and

$$\rho_G(\{e\}) + \rho_G(\{f\}) - \rho_G(\{e, f\}) = |\{T \in \mathcal{T}(G) : e, f \in E(T)\}|.$$

Because G is simple, the second quantity is 1 if e and f belong to a common triangle and 0 otherwise. Also,

$$\sum_{e \in E(G)} \rho_G(\{e\}) = 3t(G).$$

Proof. The first formula follows because the triangles met by $\{e\}$ are exactly the triangles containing e .

For the second formula, $\rho_G(\{e\}) + \rho_G(\{f\})$ counts triangles containing both edges twice, while $\rho_G(\{e, f\})$ counts every covered triangle once. Their difference is the number of triangles containing both e and f . Two distinct edges of a simple graph can lie together in at most one triangle. Finally, every triangle contributes 1 to exactly three terms in the last sum. $\square$

The two-edge formula can be written directly in terms of Wiener polarity indices:

$$\begin{aligned} & \rho_G(\{e\}) + \rho_G(\{f\}) - \rho_G(\{e, f\}) \\ &= \frac{W_p(S_{\{e\}}(G)) + W_p(S_{\{f\}}(G)) - W_p(S_{\{e, f\}}(G)) - W_p(S(G))}{3}. \end{aligned}$$

Here, by a “second difference” we mean the alternating combination

$$W_p(S_{\{e\}}(G)) + W_p(S_{\{f\}}(G)) - W_p(S_{\{e, f\}}(G)) - W_p(S(G)).$$

By the earlier formula, dividing this quantity by 3 gives 1 exactly when e and f lie in a common triangle, and 0 otherwise. Thus this second difference detects whether two edges occur in a common triangle.

4.3. A multilinear response formula

For an edge e , let $z_e(F)$ be 1 when $e \in F$ and 0 otherwise. For two distinct edges e and f , let

$$c_{ef} = \begin{cases} 1, & e \text{ and } f \text{ lie in a common triangle,} \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 3 (Multilinear response formula). *For every $F \subseteq E(G)$,*

$$\rho_G(F) = \sum_{e \in E(G)} t_e z_e(F) - \sum_{\substack{\{e,f\} \subseteq E(G) \\ e \neq f}} c_{ef} z_e(F) z_f(F) + \sum_{T \in \mathcal{T}(G)} \prod_{e \in E(T)} z_e(F).$$

Thus the unique multilinear polynomial representing ρ_G has degree at most 3, and its cubic monomials are exactly the triangles of G .

Proof. A triangle T is met by F exactly when at least one of its three edges is selected. Its indicator is

$$1 - \prod_{e \in E(T)} (1 - z_e(F)).$$

Expanding this product gives the sum of the three one-edge terms, minus the three two-edge terms, plus the three-edge term. Summing over all triangles gives the formula. The coefficient of z_e is t_e . Two distinct edges of a simple graph lie in at most one common triangle, so the coefficient of $z_e z_f$ is $-c_{ef}$. The cubic term from T is $\prod_{e \in E(T)} z_e(F)$. $\square$

Corollary 4 (Recovering the triangle edge sets). *For three distinct edges e, f, g , define*

$$\begin{aligned} \kappa_G(e, f, g) &= \rho_G(\{e\}) + \rho_G(\{f\}) + \rho_G(\{g\}) \\ &\quad - \rho_G(\{e, f\}) - \rho_G(\{e, g\}) - \rho_G(\{f, g\}) \\ &\quad + \rho_G(\{e, f, g\}). \end{aligned}$$

Then

$$\kappa_G(e, f, g) = \begin{cases} 1, & \{e, f, g\} = E(T) \text{ for a triangle } T, \\ 0, & \text{otherwise.} \end{cases}$$

Consequently, the response values on edge sets of size at most 3 determine the complete collection of triangle edge sets of G .

Proof. For each edge e , let $\mathcal{T}_e$ be the set of triangles containing e . Since

$$\rho_G(F) = \left| \bigcup_{e \in F} \mathcal{T}_e \right|,$$

the displayed alternating sum is the inclusion–exclusion formula for

$$|\mathcal{T}_e \cap \mathcal{T}_f \cap \mathcal{T}_g|.$$

A triangle belongs to this intersection exactly when its edge set is $\{e, f, g\}$. $\square$

5. Subdivision budgets as a triangle-coverage problem

We now allow more than one extra vertex on an edge. Let $B \geq 0$ be the number of inserted vertices beyond the ordinary subdivision, and define

$$\mathcal{A}_B(G) = \left\{ \mathbf{a} \in \mathbb{Z}_{\geq 1}^{E(G)} : \sum_{e \in E(G)} (a_e - 1) = B \right\}.$$

Thus every $\mathbf{a} \in \mathcal{A}_B(G)$ satisfies $a_e \geq 1$ for every $e \in E(G)$. For $\mathbf{a} \in \mathcal{A}_B(G)$, let

$$F(\mathbf{a}) = \{e \in E(G) : a_e \geq 2\}$$

be the support of the extra subdivisions.

Theorem 4 (Budget response formula). *For every $\mathbf{a} \in \mathcal{A}_B(G)$,*

$$\boxed{W_p(S_{\mathbf{a}}(G)) = W_p(S(G)) + B + 3\tau_G(F(\mathbf{a}))}.$$

Proof. We have $q(\mathbf{a}) = m + B$. A triangle is short exactly when it avoids $F(\mathbf{a})$, so

$$t_1(G, \mathbf{a}) = t(G) - \tau_G(F(\mathbf{a})).$$

Substitute these expressions into Theorem 1. $\square$

The exact values $a_e \geq 2$ do not matter after an edge becomes active. Extra vertices beyond the first extra vertex on the same edge contribute only through the total budget B . The triangle term depends only on the support $F(\mathbf{a})$.

Define

$$\delta_{\Delta}(G) = \min_{e \in E(G)} t_e \quad \text{and} \quad \Delta_{\Delta}(G) = \max_{e \in E(G)} t_e.$$

For $B \geq 0$, define the triangle-coverage number

$$C_B(G) = \max \{ \tau_G(F) : F \subseteq E(G), |F| \leq B \}.$$

Equivalently, $C_B(G)$ is the maximum number of hyperedges of $\mathcal{H}_{\Delta}(G)$ that can be met by at most B vertices of that hypergraph.

Theorem 5 (Sharp budget extrema). *Let $B \geq 1$. Then*

$$\min_{\mathbf{a} \in \mathcal{A}_B(G)} W_p(S_{\mathbf{a}}(G)) = W_p(S(G)) + B + 3\delta_{\Delta}(G)$$

and

$$\max_{\mathbf{a} \in \mathcal{A}_B(G)} W_p(S_{\mathbf{a}}(G)) = W_p(S(G)) + B + 3C_B(G).$$

Both values are attained.

Proof. Every vector in $\mathcal{A}_B(G)$ has a nonempty support $F(\mathbf{a})$. Pick $e \in F(\mathbf{a})$. Since all triangles containing e are covered,

$$\tau_G(F(\mathbf{a})) \geq t_e \geq \delta_{\Delta}(G).$$

Equality is attained by choosing an edge e with $t_e = \delta_{\Delta}(G)$ and placing all B extra vertices on that edge.

For the maximum, every support has at most B edges, so

$$\tau_G(F(\mathbf{a})) \leq C_B(G).$$

Conversely, let F attain $C_B(G)$. If F is nonempty, give one extra vertex to each edge of F and place every remaining extra vertex on one edge of F . If $C_B(G) = 0$, then G is triangle-free and any one-edge support has coverage zero. Thus the upper bound is attained. $\square$

For $B = 1$, Theorem 5 gives

$$\min_{\mathbf{a} \in \mathcal{A}_1(G)} W_p(S_{\mathbf{a}}(G)) = W_p(S(G)) + 1 + 3\delta_{\Delta}(G)$$

and

$$\max_{\mathbf{a} \in \mathcal{A}_1(G)} W_p(S_{\mathbf{a}}(G)) = W_p(S(G)) + 1 + 3\Delta_{\Delta}(G).$$

5.1. Computational difficulty

The maximum in Theorem 5 is a genuine optimization problem. The *Maximum Vertex Coverage* problem asks for exactly B vertices of a graph H that are incident with as many edges as possible. Since adding a vertex cannot decrease the number of covered edges, the optimum is unchanged if “exactly B ” is replaced by “at most B ”, provided $B \leq |V(H)|$. The problem is NP-hard even for bipartite graphs [1].

Theorem 6 (NP-hardness of the budget maximum). *Computing $C_B(G)$ is NP-hard, even when G has a universal vertex and diameter two. Consequently, maximizing $W_p(S_{\mathbf{a}}(G))$ over $\mathbf{a} \in \mathcal{A}_B(G)$ is NP-hard on the same graph class.*

Proof. We reduce from Maximum Vertex Coverage on bipartite graphs. Let H be a bipartite graph and let B be the budget. Add one isolated vertex z to H , and call the resulting bipartite graph H' . The isolated vertex does not change the maximum number of edges covered by at most B vertices.

Now add a new vertex c adjacent to every vertex of H' . Let

$$G = K_1 \vee H'.$$

The vertex c is universal. We may restrict to instances in which H has at least one edge, since the remaining instances are trivial. Thus H' contains the isolated vertex z and at least two other vertices. Consequently, G is not complete and has diameter two.

Because H' is bipartite, it has no triangles. Therefore, every triangle of G has the form cuv , where $uv \in E(H') = E(H)$. Hence the edges of H are in one-to-one correspondence with the triangles of G .

The spoke cv meets exactly the triangles corresponding to the edges of H incident with v . An original edge $uv \in E(H)$ meets only the one triangle cuv . If a set $F \subseteq E(G)$ contains an original edge uv , replacing uv by the spoke cu does not increase the size of F and does not decrease the number of covered triangles. Repeating this replacement shows that an optimal set for $C_B(G)$ can be chosen entirely from the spokes incident with c .

For $X \subseteq V(H)$, the spoke set

$$F_X = \{cv : v \in X\}$$

covers exactly the triangles corresponding to edges of H having at least one endpoint in X . Therefore,

$$C_B(G) = \max_{\substack{X \subseteq V(H) \\ |X| \leq B}} |\{uv \in E(H) : u \in X \text{ or } v \in X\}|.$$

The right-hand side is the Maximum Vertex Coverage optimum for H . Thus computing $C_B(G)$ is NP-hard.

The second statement follows from Theorem 5, because $W_p(S(G))$ and B are fixed and the maximum differs from $3C_B(G)$ only by an additive constant. $\square$

5.2. A greedy choice rule

Theorem 6 motivates approximation methods. A simple greedy rule chooses an edge that covers the largest number of still-uncovered triangles at each step. This is the standard greedy method for a monotone submodular function. Its general estimate is due to Nemhauser, Wolsey, and Fisher [9]; we include the short proof in our setting.

Let $1 \leq B \leq m$. Start with $F_0 = \emptyset$. After choosing F_i , choose an edge $e_{i+1} \notin F_i$ that maximizes

$$\tau_G(F_i \cup \{e\}) - \tau_G(F_i),$$

and set $F_{i+1} = F_i \cup \{e_{i+1}\}$.

Theorem 7 (Greedy coverage bound). *The greedy set F_B satisfies*

$$\tau_G(F_B) \geq \left(1 - \left(1 - \frac{1}{B}\right)^B\right) C_B(G) \geq \left(1 - \frac{1}{e}\right) C_B(G).$$

Proof. Let F^* be a set of at most B edges with $\tau_G(F^*) = C_B(G)$. Suppose the greedy method has chosen F_i . By monotonicity,

$$C_B(G) - \tau_G(F_i) = \tau_G(F^*) - \tau_G(F_i) \leq \tau_G(F_i \cup F^*) - \tau_G(F_i).$$

Write

$$F^* \setminus F_i = \{e_1, \dots, e_r\}.$$

Here, $\{e_1, \dots, e_0\}$ is understood to be the empty set. Adding these edges one at a time and telescoping gives

$$\begin{aligned} \tau_G(F_i \cup F^*) - \tau_G(F_i) &= \sum_{j=1}^r \left(\tau_G(F_i \cup \{e_1, \dots, e_j\}) \right. \\ &\quad \left. - \tau_G(F_i \cup \{e_1, \dots, e_{j-1}\}) \right). \end{aligned}$$

By submodularity, the marginal gain of e_j can only decrease as the selected set grows. Hence

$$\tau_G(F_i \cup F^*) - \tau_G(F_i) \leq \sum_{e \in F^* \setminus F_i} (\tau_G(F_i \cup \{e\}) - \tau_G(F_i)).$$

Therefore,

$$C_B(G) - \tau_G(F_i) \leq \sum_{e \in F^* \setminus F_i} (\tau_G(F_i \cup \{e\}) - \tau_G(F_i)).$$

If $\tau_G(F_i) = C_B(G)$, then the desired estimate already holds. Otherwise, $F^* \setminus F_i$ is nonempty. There are at most B terms, so at least one edge has marginal gain at least

$$\frac{C_B(G) - \tau_G(F_i)}{B}.$$

The greedy choice has at least this gain, so

$$C_B(G) - \tau_G(F_{i+1}) \leq \left(1 - \frac{1}{B}\right) (C_B(G) - \tau_G(F_i)).$$

Starting from $\tau_G(F_0) = 0$ and applying the inequality B times gives

$$C_B(G) - \tau_G(F_B) \leq \left(1 - \frac{1}{B}\right)^B C_B(G).$$

Rearranging proves the first bound. The second follows from

$$\left(1 - \frac{1}{B}\right)^B \leq e^{-1}.$$

□

By Theorem 2, the greedy set satisfies

$$W_p(S_{F_B}(G)) = W_p(S(G)) + B + 3\tau_G(F_B).$$

Thus Theorem 7 approximates the triangle-dependent part of the maximum Wiener-polarity response.

6. The response polynomial and triangle overlap

We now record all binary subdivision responses in one polynomial. Bivariate graph polynomials have previously been used to record the size of a selected vertex set together with the size of an associated covered set. For example, the bivariate domination polynomial records the size of a vertex set and the size of its external neighborhood [10]. Our polynomial instead selects edges of the original graph and records the number of triangles met by those edges.

Definition 5 (Response polynomial). Define

$$\Phi_G(x, y) = \sum_{F \subseteq E(G)} x^{|F|} y^{\tau_G(F)}.$$

The power of x records the number of activated edges, and the power of y records the number of covered triangles.

For $\mathcal{A} \subseteq \mathcal{T}(G)$, let $U(\mathcal{A}) = \bigcup_{T \in \mathcal{A}} E(T)$, with $U(\emptyset) = \emptyset$.

Theorem 8 (General response-polynomial formula). *For every graph G ,*

$$\Phi_G(x, y) = \sum_{\mathcal{A} \subseteq \mathcal{T}(G)} y^{t(G) - |\mathcal{A}|} (1 - y)^{|\mathcal{A}|} (1 + x)^{m - |U(\mathcal{A})|}.$$

Proof. For a fixed edge set F , a triangle T contributes a factor y when it is covered and a factor 1 when it is not covered. Therefore,

$$y^{\tau_G(F)} = \prod_{T \in \mathcal{T}(G)} (y + (1 - y)\mathbf{1}_{\{E(T) \cap F = \emptyset\}}),$$

where $\mathbf{1}$ is the indicator function.

Expand the product. Fix $\mathcal{A} \subseteq \mathcal{T}(G)$ and choose the second term for exactly the triangles in $\mathcal{A}$. Then each of the $t(G) - |\mathcal{A}|$ triangles outside $\mathcal{A}$ contributes a factor y , while each of the $|\mathcal{A}|$ triangles in $\mathcal{A}$ contributes a factor

$$(1 - y)\mathbf{1}_{\{E(T) \cap F = \emptyset\}}.$$

Thus the corresponding term in the expansion is

$$y^{t(G) - |\mathcal{A}|} (1 - y)^{|\mathcal{A}|} \prod_{T \in \mathcal{A}} \mathbf{1}_{\{E(T) \cap F = \emptyset\}}.$$

The product of indicators equals 1 exactly when F avoids every edge belonging to a triangle in $\mathcal{A}$, that is, exactly when

$$F \cap U(\mathcal{A}) = \emptyset.$$

Hence

$$\sum_{\substack{F \subseteq E(G) \\ F \cap U(\mathcal{A}) = \emptyset}} x^{|F|} = (1 + x)^{m - |U(\mathcal{A})|}.$$

Summing over all $\mathcal{A} \subseteq \mathcal{T}(G)$ proves the formula. $\square$

The formula depends on how triangle edge sets overlap. We first treat edge-disjoint triangles and then book graphs, where all triangles share one edge.

6.1. Edge-disjoint triangles

Assume that no edge lies in two different triangles. Let $t = t(G)$, and let h be the number of edges that lie in no triangle. Then

$$m = h + 3t.$$

Corollary 5 (Factorization for edge-disjoint triangles). *If the triangles of G are edge-disjoint, then*

$$\Phi_G(x, y) = (1 + x)^h (1 + y((1 + x)^3 - 1))^t.$$

Proof. For one triangle, selecting none of its edges contributes 1. Selecting a nonempty subset covers the triangle and contributes

$$y(3x + 3x^2 + x^3) = y((1 + x)^3 - 1).$$

Each edge outside all triangles contributes a factor $1 + x$. Since the triangle edge sets are disjoint, all choices are independent. $\square$

Corollary 6 (Counting binary subdivision values). *If the triangles of G are edge-disjoint, then the number of sets $F \subseteq E(G)$ with $|F| = B$ and $\tau_G(F) = j$ is*

$$[x^B y^j] \Phi_G(x, y) = \binom{t}{j} [x^B] (1 + x)^h ((1 + x)^3 - 1)^j.$$

This is also the number of binary subdivisions with B extra vertices and Wiener polarity index

$$W_p(S(G)) + B + 3j.$$

Proof. The coefficient of $x^B y^j$ counts exactly the sets with the stated size and coverage. In Corollary 5, choosing the y term from exactly j of the t triangle factors gives the formula. The Wiener-polarity statement follows from Theorem 2. $\square$

Theorem 9 (Sharp binary bounds for edge-disjoint triangles). *Suppose the triangles of G are edge-disjoint. For every integer B with $0 \leq B \leq m$ and every $F \subseteq E(G)$ with $|F| = B$,*

$$\max \left\{ 0, \left\lceil \frac{B - h}{3} \right\rceil \right\} \leq \tau_G(F) \leq \min\{B, t\}.$$

Both bounds are sharp.

Proof. At most h selected edges can lie outside all triangles. Thus at least $\max\{0, B - h\}$ selected edges lie in triangles. Since one covered triangle contains at most three selected edges, these selected triangle edges must meet at least

$$\max \left\{ 0, \left\lceil \frac{B - h}{3} \right\rceil \right\}$$

triangles. To attain the lower bound, first choose as many nontriangle edges as possible, and then choose all three edges from one triangle before moving to another.

For the upper bound, each selected edge lies in at most one triangle, so B selected edges cover at most B triangles and at most the total number t . To attain the bound, choose one edge from each of $\min\{B, t\}$ different triangles; if $B > t$, choose the remaining edges arbitrarily. $\square$

Combining Theorems 2 and 9 gives sharp Wiener-polarity bounds for every exact binary budget.

6.2. Book graphs

The book graph B_t consists of t triangles sharing one common edge. Let the common edge be uv , and let the remaining vertices be $w_1, \dots, w_t$, where the i th page is the triangle uvw_i . Thus

$$E(B_t) = \{uv\} \cup \{uw_i, vw_i : 1 \leq i \leq t\}.$$

The graph has $2t + 1$ edges and t triangles.

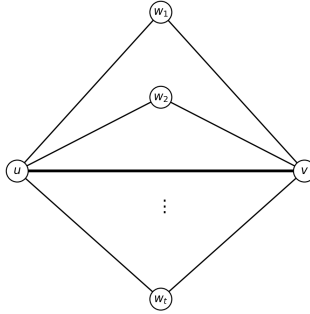


Figure 3. The book graph B_t . All t triangles share the edge uv .

Theorem 10 (Response polynomial of a book graph). *For every $t \geq 1$,*

$$\Phi_{B_t}(x, y) = xy^t(1+x)^{2t} + (1+y(2x+x^2))^t.$$

Proof. Separate the edge sets $F \subseteq E(B_t)$ according to whether the common edge uv is selected.

If $uv \in F$, then all t triangles are covered. The common edge contributes x , the covered triangles contribute y^t , and the remaining $2t$ private edges may be chosen arbitrarily. This gives

$$xy^t(1+x)^{2t}.$$

Now assume $uv \notin F$. For each page, selecting neither private edge contributes 1. Selecting exactly one private edge contributes $2xy$, and selecting both contributes x^2y . Therefore, one page contributes

$$1 + y(2x + x^2).$$

The private edges of distinct pages are disjoint, so the t page choices multiply. Adding the two cases gives the result. $\square$

Corollary 7 (Exact coefficients for book graphs). For $0 \leq B \leq 2t + 1$ and $0 \leq j \leq t$,

$$[x^B y^j] \Phi_{B_t}(x, y) = \mathbf{1}_{\{j \leq B \leq 2j\}} \binom{t}{j} \binom{j}{B-j} 2^{2j-B} + \mathbf{1}_{\{j=t\}} \binom{2t}{B-1},$$

where an invalid binomial coefficient is interpreted as zero.

Proof. First suppose the common edge is not selected. Such a choice is possible only when $j \leq B \leq 2j$. Choose the j covered pages in $\binom{t}{j}$ ways. If r of these pages contribute both private edges and the other $j - r$ pages contribute one private edge, then

$$B = 2r + (j - r) = j + r,$$

so $r = B - j$. Choose these r pages in $\binom{j}{B-j}$ ways. Each of the remaining $j - r = 2j - B$ covered pages contributes one of two private edges, giving the factor 2^{2j-B} .

If the common edge is selected, then $j = t$, and the remaining $B - 1$ edges are chosen from the $2t$ private edges. This gives the second term. $\square$

Theorem 11 (Sharp binary bounds for book graphs). Let $1 \leq B \leq 2t + 1$, and let $F \subseteq E(B_t)$ with $|F| = B$. Then

$$\min \left\{ t, \left\lceil \frac{B}{2} \right\rceil \right\} \leq \tau_{B_t}(F) \leq t.$$

Both bounds are sharp.

Proof. The upper bound is immediate, and it is attained by selecting the common edge.

For the lower bound, if the common edge is selected, then all t triangles are covered, so the stated bound holds. If the common edge is not selected, all B selected edges are private. Since the common edge is absent, necessarily $B \leq 2t$. Each page contains only two private edges, so the selected edges meet at least $\lceil B/2 \rceil$ pages, and this quantity is at most t .

To attain it, avoid the common edge and select both private edges from one page before moving to the next page. This uses B private edges and covers $\lceil B/2 \rceil$ pages whenever $B \leq 2t$. For $B = 2t + 1$, every edge is selected and all t pages are covered. $\square$

The two common vertices of B_t have degree $t + 1$, and every page vertex has degree 2. Hence

$$M_1(B_t) = 2(t + 1)^2 + 4t = 2t^2 + 8t + 2.$$

Since $|E(B_t)| = 2t + 1$ and $t(B_t) = t$, Corollary 1 gives

$$W_p(S(B_t)) = 2t^2 + t.$$

Therefore, Theorems 2 and 11 yield the sharp bounds

$$2t^2 + t + B + 3 \min \left\{ t, \left\lceil \frac{B}{2} \right\rceil \right\} \leq W_p(S_F(B_t)) \leq 2t^2 + 4t + B.$$

The edge-disjoint and book cases illustrate opposite effects of triangle overlap. With edge-disjoint triangles, spreading selected edges across different triangles maximizes coverage. In a book graph, one common edge covers every triangle at once.

7. Conclusion and further questions

We studied the Wiener polarity index after every edge of a graph is subdivided at least once, but not necessarily by the same amount. The exact formula

$$W_p(S_{\mathbf{a}}(G)) = M_1(G) + q(\mathbf{a}) - 3m - 3t_1(G, \mathbf{a})$$

comes from counting three-edge paths and correcting the repeated counts created by the 6-cycles corresponding to short triangles.

The main viewpoint is the normalized response

$$\rho_G(F) = \frac{W_p(S_F(G)) - W_p(S(G)) - |F|}{3}.$$

We proved that this response is exactly the number of triangles met by F . It therefore reveals the coverage function of the triangle hypergraph. Responses on sets of at most three edges recover every triangle edge set, while the multilinear representation records the same information algebraically.

Under a fixed subdivision budget, the minimum response has a simple exact formula, while the maximum becomes a triangle-coverage problem. We showed that this maximum is NP-hard even for diameter-two graphs with a universal vertex. The submodular viewpoint then gives a natural greedy approximation. Finally, the response polynomial records all binary subdivision responses and distinguishes different patterns of triangle overlap. Edge-disjoint triangles and book graphs admit explicit formulas, exact coefficient counts, and sharp budget bounds.

Several questions remain. It would be useful to obtain closed forms for the response polynomial on further overlapping-triangle families, such as chains or trees of triangles. One could also study algorithms for $C_B(G)$ on restricted graph classes. Finally, for vertex pairs at distance $k > 3$, longer paths and cycles should produce analogous but more complicated response functions.

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